Effect of nonlinear boundary conditions on nonlinear phenomena in optical resonators

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Received April 15, 1987; accepted July 14, 1987

The question of proper boundary conditions at the surface of a nonlinear medium is discussed. It is shown that the nonlinearity of the boundary conditions could have a significant effect for large fields. Thus the nonlinear boundary conditions significantly affect the transmission behavior near the upper bistability threshold.

In this Letter we examine the effects of the nonlinearity of the boundary conditions at the surface of a nonlinear medium on nonlinear phenomena such as optical transmission. In particular, we study the significance of the nonlinear nature of the boundary conditions. It should be noted that the correct boundary conditions that follow from Maxwell's equations are the continuity of the tangential components of the electric and magnetic fields at the interface. However, the usual treatments[1] of nonlinear transmission through a Fabry–Perot resonator are based on the linear reflection and transmission coefficients of the interface. The two approaches can lead to results in rough agreement with each other if the nonlinearity of the medium is small. In this Letter we demonstrate the significance of the nonlinear boundary conditions by examining the dispersive bistability using the two approaches. We are using the term nonlinear boundary condition in the sense that the boundary conditions involve magnetic fields that are nonlinear functionals of the electric fields in the medium.

Consider the geometrical arrangement of Fig. 1. On each side of the nonlinear medium there is a linear layered medium that serves to define the linear reflection and transmission coefficients of the interface. Moreover, the linear layered media, for suitably chosen parameters, enable one to have sharp Airy resonances, which are essential for low-threshold nonlinear optical phenomena. The electric field in the nonlinear medium can be expressed as

$$E(x, t) = e^{-i\omega t}(\epsilon_f e^{ikx} + \epsilon_b e^{-ikx}),$$
$$k = k_0 n, \qquad k_0 = \omega/c, \tag{1}$$

where n is the linear refractive index of the nonlinear medium and ϵ_f and ϵ_b are the slowly varying envelopes of the fields propagating in forward and backward directions, respectively. For a medium with nonlinearity of the form

$$\mathbf{D}^{\mathrm{NL}} = n^2\chi[A\mathbf{E}(\mathbf{E}\cdot\mathbf{E}^*) + B\mathbf{E}^*(\mathbf{E}\cdot\mathbf{E})], \tag{2}$$

and for a TE-polarized incident wave, the slowly varying amplitudes ϵ_f and ϵ_b are given approximately by

$$\epsilon_{f,b} = \epsilon_\pm \exp(iq_\pm x); \qquad q_\pm = \frac{k\alpha}{2}(|\epsilon_\pm|^2 + 2|\epsilon_\mp|^2);$$
$$\alpha = \chi(A + B). \tag{3}$$

In using the slowly varying amplitude approximation in obtaining Eqs. (3) we have ignored the contribution of fast phase factors such as $\exp(\pm i3kx)$. Such terms are important for media with strong nonlinearities. It may be further noted here that for a lossless medium (real n) and real values of χ, A, and B the nonlinearity of the medium contributes only to the phase, and there is no exchange between the amplitudes of the forward and backward waves.

The transmission and reflection coefficients for the composite medium of Fig. 1 are given in terms of the elements $\bar{m}_{ij}$ of the characteristic matrix[2] by

$$T = \left|\frac{2n_i}{(\bar{m}_{11} + \bar{m}_{12}n_i)n_i + (\bar{m}_{21} + \bar{m}_{22}n_i)}\right|^2, \tag{4}$$

$$R = \left|\frac{(\bar{m}_{11} + \bar{m}_{12}n_i)n_i - (\bar{m}_{21} + \bar{m}_{22}n_i)}{(\bar{m}_{11} + \bar{m}_{12}n_i)n_i + (\bar{m}_{21} + \bar{m}_{22}n_i)}\right|^2, \tag{5}$$

where n_i is the refractive index of the medium from which the wave is incident and of the medium in which the wave is transmitted. The characteristic matrix $\bar{M}$ is given in terms of the characteristic matrix of each layer $M(L_j)$ as

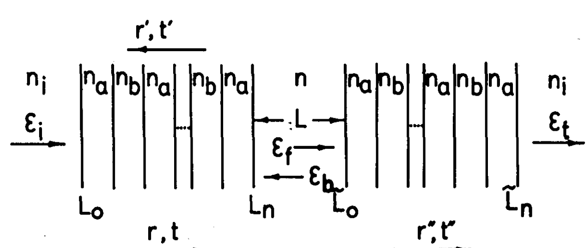

Fig. 1. Schematic diagram of the Fabry–Perot resonator with reflection coatings on each face of the nonlinear medium.

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$$\tilde{M} = M(L_1 - L_0)M(L_2 - L_1)\ldots M(\tilde{L}_0 - L_n)\ldots M(\tilde{L}_n - \tilde{L}_{n-1}),$$

$$M(L) = \frac{k_0}{k_+ + k_-}\begin{bmatrix} \frac{k_-}{k_0}\exp(-ik_+L) + \frac{k_+}{k_0}\exp(ik_-L) & \exp(-ik_+L) - \exp(ik_-L) \\ \frac{k_-k_+}{k_0^2}[\exp(+ik_+L) - \exp(ik_-L)] & \frac{k_+}{k_0}\exp(-ik_+L) + \frac{k_-}{k_0}\exp(ik_-L) \end{bmatrix},$$

$$k_\pm = k + q_\pm. \quad (6)$$

If one of the layers is linear, then the corresponding characteristic matrix is well known[3] and can be obtained by setting $q_\pm = 0$. It should be borne in mind that the characteristic matrix for each nonlinear medium depends on the intensities of the waves in that medium. In our earlier work[2] we used a fixed-point iteration method to calculate T.

Next, we consider the usual approach to the problem of optical transmission through a nonlinear Fabry–Perot resonator. Let r, r', r'' and t, t', t'' be the linear reflection and transmission coefficients, respectively, of the structures loaded on each side of the nonlinear medium. The usual boundary conditions at the two faces of the nonlinear medium are

$$\begin{aligned} \epsilon_t &= t'' \exp[ik_0(n\tilde{L}_0 - n_i\tilde{L}_n)]\epsilon_f(\tilde{L}_0), \\ \epsilon_b(\tilde{L}_0) &= r'' \exp(2ik\tilde{L}_0)\epsilon_f(\tilde{L}_0), \\ \epsilon_f(L_n) &= t \exp[ik_0(n_0L_0 - nL_n)]\epsilon_i \\ &\quad + r' \exp(-2ikL_n)\epsilon_b(L_n). \end{aligned} \quad (7)$$

On using Eqs. (3) and (7) we find the transmitted field in terms of the incident field as

$$\epsilon_t = tt'' \exp[ik_0n_i(\tilde{L}_n - L_0)][\exp(-ik_+L) - r'r'' \exp(ik_+L)]^{-1}\epsilon_i. \quad (8)$$

Here L is the length of the nonlinear medium. The linear reflection and transmission coefficients can be expressed in terms of the elements of the characteristic matrix of each layered structure:

$$r = \frac{1}{D_1}[m_{11} + m_{12}n)n_i - (m_{21} + m_{22}n)], \quad t = 2n_i/D_1,$$

$$r' = \frac{1}{D_1}[(m_{22} + m_{12}n_i)n - (m_{21} + m_{11}n_i)], \quad t' = 2n/D_1,$$

$$r'' = \frac{1}{D_2}[(m_{11} + m_{12}n_i)n - (m_{21} + m_{22}n_i)], \quad t'' = 2n/D_2,$$

$$\begin{aligned} D_1 &= (m_{11} + m_{12}n)n_i + (m_{21} + m_{22}n), \\ D_2 &= (m_{11} + m_{12}n_i)n + (m_{21} + m_{22}n_i), \end{aligned} \quad (9)$$

where m_{ij} are the elements of the characteristic matrix $\tilde{M}$, defined as follows:

$$\begin{aligned} \tilde{M} &= M(L_1 - L_0)\ldots M(L_n - L_{n-1}) \\ &= M(\tilde{L}_1 - \tilde{L}_0)\ldots M(\tilde{L}_n - \tilde{L}_{n-1}). \end{aligned} \quad (10)$$

Here we have assumed that the nonlinear medium is loaded on both sides by the same structures.

In what follows we discuss the differences in the optical transmission that are due to the use of the approximate Eq. (8). Before we present our numerical results we would like to point out an important property, viz., Eq. (8) is obtained from our result [Eq. (4), which has been derived by using the nonlinearity of the boundary conditions] if we replace the characteristic matrix M [Eq. (6)] for the nonlinear medium by

$$M = \frac{1}{2}\begin{bmatrix} [\exp(-ik_+L) + \exp(ik_-L)] & \frac{1}{n}[\exp(-ik_+L) - \exp(ik_-L)] \\ n[\exp(-ik_+L) - \exp(ik_-L)] & [\exp(-ik_+L) + \exp(ik_-L)] \end{bmatrix}. \quad (11)$$

The use of Eq. (11) is equivalent to using the following approximate expression for the magnetic field in

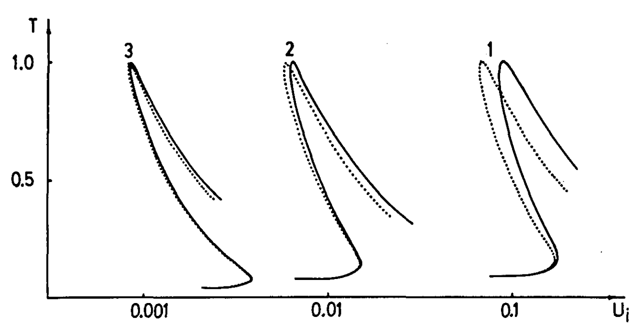

Fig. 2. The transmission coefficient T as a function of the incident intensity U_i. The solid curves are the exact results, whereas the dotted curves are obtained by ignoring the nonlinearity of the boundary conditions. Different curves are labeled by the values of N that in turn refer to the number of coatings. The other parameters have been chosen as $n_a = 2.3$, $n_b = 1.3077$, $n_i = 1$, $n = 1.7149$, and $k_0nL = (2 - 2\Delta)\pi$. $\Delta = 0.113$ for $N = 1$, $\Delta = 0.04$ for $N = 2$, and $\Delta = 0.018$ for $N = 3$. Here Δ essentially gives the half-width of the linear transmission resonances, i.e., the width of the resonance obtained by ignoring the nonlinearity of the medium.

the nonlinear medium:

$$H_Y = -\frac{k_+}{k_0}\exp(ik_+x)\,\epsilon_+ + \frac{k_-}{k_0}\exp(-ik_-x)\epsilon_-$$
$$\approx -n[\exp(ik_+x)\epsilon_+ - \exp(-ik_-x)\epsilon_-]. \qquad (12)$$

To illustrate the difference, we consider a nonlinear slab coated on both sides by alternating N low-index (n_b) and $N+1$ high-index (n_a) $\lambda/4$ plates. Let the coating materials be linear. We show the bistability in transmission in Fig. 2, with the incident intensity defined as

$$U_i = \alpha|\epsilon_i|^2. \qquad (13)$$

We present the results obtained by using both approximate theory based on Eq. (8) and our exact theory. It is clear from Fig. 2 that a higher bistability threshold leads to higher deviations from the exact theory. The corrections are almost insignificant for $N = 3$, when the threshold is rather low. Note further that for a given N the deviations are more prominent near the upper bistability threshold.

In conclusion, the full nonlinearity of the boundary conditions is important whenever one deals with relatively large intensities in the medium. This conclusion obviously also holds not only for a medium with cubic nonlinearity but also for other types of nonlinearity of the medium.

G. S. Agarwal is grateful to the University Grants Commission, Government of India, for supporting this work.

References

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