Quantum analysis of optical bistability and spectrum of fluctuations

L. M. Narducci, R. Gilmore, and Da Hsuan Feng

Department of Physics and Atmospheric Science, Drexel University, Philadelphia, Pennsylvania 19104

G. S. Agarwal

Physics Department, University of Hyderabad, Hyderabad, India 500 001

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We discuss the approach to equilibrium and the fluctuations of a bistable system under dynamical conditions such that the field variables can be eliminated adiabatically. The atomic system evolves under the action of the coherent pumping of an external field and of collective and incoherent relaxation processes. The competition between pumping and relaxation effects causes the atomic steady-state configurations to depend discontinuously on the strength of the driving field. We derive an explicit expression for the spectrum of the forward-scattered light, which exhibits hysteresis and a discontinuous dependence on the driving-field amplitude.

The occurrence of instabilities and discontinuous transitions in optical systems has been under discussion for some time, most notably in connection with the behavior of saturable absorbers and dye lasers.[1] Of special current interest are the transmission properties of a Fabry–Perot cavity filled with an absorptive or dispersive medium and driven by an external optical field.[2–4] Under appropriate conditions, the light intensity transmitted by this system has been predicted to undergo discontinuous variations and to exhibit hysteresis effects. Successful switching of an optical cavity from a state of low to one of high transmission and vice versa has been demonstrated with sodium vapor[5] and ruby[6]; other approaches have also been considered with an eye toward numerous potential applications.[7–9]

In addition, the bistable operation of a passive system has raised numerous theoretical questions: in the language of statistical mechanics, a bistable device is an open system driven away from thermal equilibrium by the action of an external field. It is of interest to investigate the steady-state properties of this system, its fluctuations and relaxation around steady state, and the possible existence of cooperative effects.

A simple quantum-mechanical model of absorptive bistability has been discussed extensively by Bonifacio and Lugiato.[10] These authors have considered the case of perfect matching between the excited cavity mode, the atomic absorption line, and the carrier frequency of the driving field. They have derived analytical conditions for the observation of bistability and pointed to the existence of remarkable fluctuation properties in the vicinity of the switching thresholds.

In this Letter we present additional features of the quantum-mechanical model of bistability.[11] We analyze the approach to equilibrium for different values of the driving-field amplitude and propose a calculation of the steady-state spectrum of the transmitted light. As anticipated in Ref. 10, the spectrum consists of a single peak centered at the resonant atomic frequency along the cooperative (highly absorbing) atomic steady-state branch. Along the single-atom branch instead, the spectrum develops a pair of sidebands that are displaced from the central peak by an amount proportional to the Rabi frequency of the driving field. The abrupt occurrence of this dynamic Stark shift above the upper bistability threshold corresponds to the atomic system's becoming highly transparent. Along the high-transmission branch and for decreasing driving-field amplitudes, the two sidebands merge continuously into the central peak. In addition, in the neighborhood of both upper and lower bistability thresholds, the atomic system exhibits critical slowing down, as evidenced by the spectral narrowing of the transmitted light.

The atomic evolution is described by the reduced density operator W solution of the following master equation:

$$\frac{dW}{dt} = -iLW + \Lambda_S W + \Lambda_A W, \tag{1a}$$

where

$$iLW = i\Omega_I(S^+ + S^-, W), \tag{1b}$$

$$\Lambda_S W = \frac{2g^2}{\kappa}\left(S^- W S^+ - \frac{1}{2} W S^+ S^- - \frac{1}{2} S^+ S^- W\right), \tag{1c}$$

$$\Lambda_A W = \sum_{i=1}^{N} \gamma_\perp \left[(S_i^-, W S_i^+) + (S_i^- W, S_i^+)\right]. \tag{1d}$$

Equation (1a) has been derived under the Born–Markoff approximations and after adiabatic elimination of the cavity field operators. The term iLW describes the reversible coherent interaction of the atoms with the external classical field, while $\Lambda_S W$ and $\Lambda_A W$ describe

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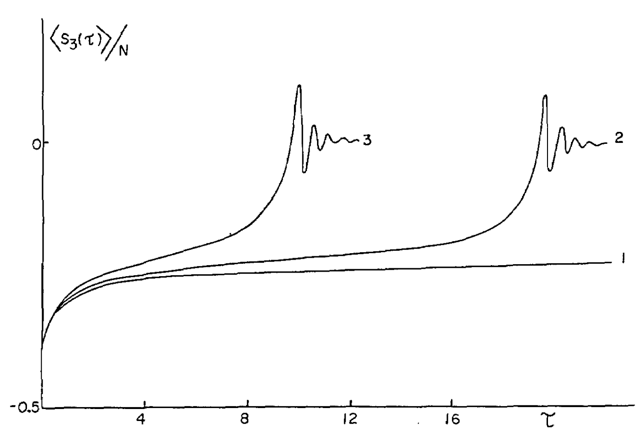

Fig. 1. Evolution of $S_3(\tau)$ as a function of the scaled time, $\tau = \gamma_\perp t$, for $c = 10$ and different values of the driving field, y. The three curves correspond to: (1) $y = 11.05$, (2) $y = 11.1$, (3) $y = 13.0$. The threshold value for optical bistability is close to $y = 11.055$ for $c = 10$.

the collective and the single-atom relaxation, respectively.[12] The operators $S^{\pm} = \sum_i S_i^{\pm}$ are the collective polarization operators, and Ω_I is the Rabi frequency. The parameters g, $\gamma_\perp$, and κ are the atom-field coupling constant and the atomic and field decay rates, respectively.

The exact evolution equations for the expectation values of the polarization and atomic energy operators can easily be derived from Eq. (1a). The result is the well-known infinite hierarchy of coupled-moment equations. On introduction of the usual factorization ansatz (e.g., $\langle S^+S_3\rangle = \langle S^+\rangle\langle S_3\rangle$), the equations of motion for the atomic expectation values become[10]

$$\frac{dS}{d\tau} = -S - \sqrt{2}\,yS_3 + \frac{4c}{N}SS_3, \qquad (2a)$$

$$\frac{dS_3}{d\tau} = -2\left(S_3 + \frac{N}{2}\right) + \sqrt{2}\,yS - \frac{4c}{N}S^2, \qquad (2b)$$

where $S = -i\langle S^+\rangle = i\langle S^-\rangle$, $S_3 = \langle S_3\rangle$, and $\tau = \gamma_\perp t$. The parameter $y = \sqrt{2}\,\Omega_{I/\gamma_\perp}$ is proportional to the incident-field amplitude, and $c = g^2N/2\gamma_\perp\kappa$ provides a measure of the density of atoms in the cavity.

Equations (2a) and (2b) have been solved numerically. An interesting feature of the time-dependent solutions is their discontinuous dependence on the incident-field amplitude for sufficiently large values of c, a feature that has already been discussed in Ref. 10. An example is given in Fig. 1, where $S_3(\tau)$ is plotted as a function of time for different incident-field amplitudes. A small change in y around its threshold value can alter the dynamics of the atomic population quite drastically by forcing nonmonotonic behavior and by altering its steady-state value.

The steady-state solutions for S and S_3 take the form

$$S = \frac{N}{\sqrt{2}}\frac{x}{1+x^2},$$
$$S_3 = -\frac{N}{2}\frac{1}{1+x^2}, \qquad (3)$$

where the transmitted-field amplitude, $x = \sqrt{2}\,(\Omega_I - g^2S/\kappa)/\gamma_\perp$, is related to the incident-field amplitude, y, by the cubic equation

$$y = x + \frac{2cx}{1+x^2}. \qquad (4)$$

For $c < 4$, Eq. (4) predicts a single-valued relation between x and y for all values of the incident amplitude, y. For $c > 4$ and $y_{\min} < y < y_{\max}$, the output-field amplitude is a multivalued function of the output field. The parameters $y_{\min}$ and $y_{\max}$ give the upper and lower bistability thresholds.

The discontinuous behavior of the transmitted field and its hysteresis properties as a function of the input, y, are related to the multivalued nature of x. The behavior can also be understood in terms of elementary catastrophe theory where it is seen that optical bistability is just another example of cusp catastrophe and that the hysteresis of the transmitted field is a consequence of the so-called delay convention.[13]

We now discuss the spectrum of the transmitted light, which is proportional to the Fourier transform of the atomic correlation function, $\langle S^+(t'+t)S^-(t')\rangle$. In particular, we are interested in the steady-state value, $\lim_{t'\to\infty}\langle S^+(t'+t)S^-(t')\rangle \equiv \langle S^+(t)S^-\rangle$. To this end we define the set of correlation functions,[14] $\chi_1 = \langle S^+(t)S^-\rangle - \langle S^+\rangle\langle S^-\rangle$, $\chi_2 = \langle S^-(t)S^-\rangle - \langle S^-\rangle^2$, and $\chi_3 = \langle S_3(t)S^-\rangle - \langle S_3\rangle\langle S^-\rangle$. With the help of the regression theorem,[15] and assuming that the atomic fluctuations in steady state behave as random Gaussian fluctuations, the above correlation functions satisfy the set of linear equations,

$$\frac{d}{d\tau}\chi = M\chi, \qquad (5)$$

where

$$M = \begin{bmatrix} -\left(1+\frac{2c}{1+x^2}\right) & 0 & -i\sqrt{2}x \\ 0 & -\left(1+\frac{2c}{1+x^2}\right) & i\sqrt{2}x \\ \frac{i}{\sqrt{2}}(y-2x) & -\frac{i}{\sqrt{2}}(y-2x) & -2 \end{bmatrix}, \qquad (6)$$

and $\chi \equiv (\chi_1,\chi_2,\chi_3)$.

The solution of Eq. (5) requires knowledge of the steady-state fluctuations, $\chi_i^{(0)}$ ($i = 1, 2, 3$). This information must be provided independently of the above development. To this end, we have constructed a set of coupled equations for the first- and second-order moments and factorized the third-order moments according to the *same factorization ansatz* that was used in deriving Eq. (5). After solving the coupled-moment equations in steady state, we have derived the following initial conditions:

$$\chi_1^{(0)} = \frac{N}{2A}\frac{x^2}{1+x^2}\frac{x(2xA+2x-y)}{(2+A)[A+x(2x-y)]},$$
$$\chi_2^{(0)} = \frac{N}{2A}\frac{x^2}{1+x^2}\frac{x(2x-y)+A(2+A)-Axy}{(2+A)[A+x(2x-y)]},$$
$$\chi_3^{(0)} = i\frac{N}{2\sqrt{2}}\frac{x^2}{1+x^2}\frac{2xA+2x-y}{(2+A)[A+x(2x-y)]}, \qquad (7)$$

where $A = 1 + 2c/(1+x^2)$.

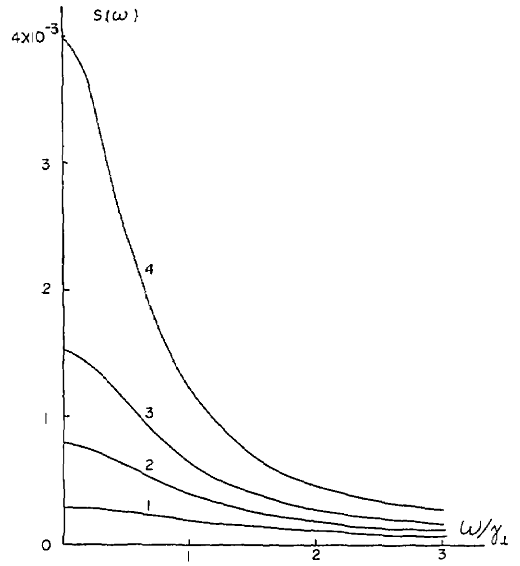

Fig. 2. The spectrum of the atomic fluctuations [Eq. (9)] for $c = 10$ and several values of the driving field: (1) $y = 10.0$, (2) $y = 10.4$, (3) $y = 10.6$, (4) $y = 10.8$. The threshold value for optical bistability is $y = 11.055$. The stationary states of the atomic system belong to the cooperative branch.

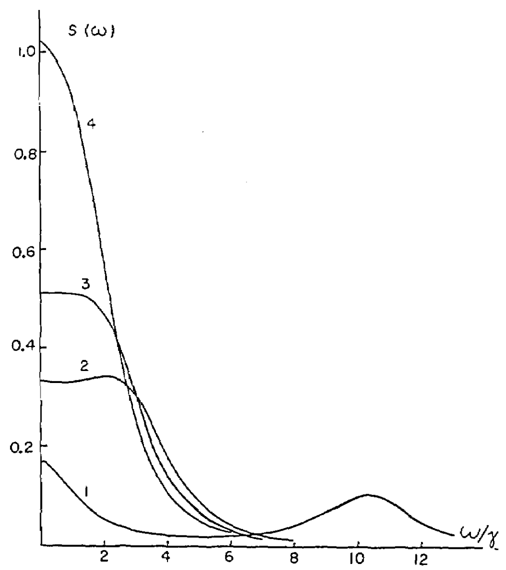

Fig. 3. The spectrum of the atomic fluctuations [Eq. (9)] for $c = 10$ and several values of the driving field: (1) $y = 11.0$, (2) $y = 8.82$, (3) $y = 8.77$, (4) $y = 8.73$. The stationary states of the system belong to the single atom branch.

The solution of Eq. (5) is now carried out. In particular, the Laplace transform of $\chi_1(\tau)$ is given by

$$\tilde{\chi}_1(z) = \frac{N}{2A}\frac{x^2}{1+x^2}\frac{1}{A+x(2x-y)} \times \frac{x^2(z+A)(z+2+A)+x(2x-y)[A+x(2x-y)]}{(z+A)D(z)}, \tag{8}$$

where $D(z) = (z+A)(z+2) + 2x(2x-y)$.

The required spectrum of the atomic fluctuations is directly given by

$$S(\omega) = \mathrm{Re}\,\tilde{\chi}_1(z)|_{z=i\omega}. \tag{9}$$

It is easy to see that in the limit of vanishing atomic density, ($c \to 0$), Eq. (9) leads to the well-known single-atom resonance-fluorescence spectrum.[16] For the interesting case $c > 4$, typical spectra are shown in Figs. 2 and 3. As the incident field increases from $y = 0$ to $y = y_{max}$ (Fig. 2), the spectrum consists of a single broadened line with a monotonically decreasing halfwidth, which vanishes in the vicinity of the upper threshold as $(y_{max} - y)^{1/2}$. For $y > y_{max}$, the spectrum develops sidebands discontinuously, and, for increasing fields, it approaches the usual single-atom spectrum of resonance fluorescence. When the driving field is decreased along the single-atom branch (Fig. 3), the sidebands merge continuously into the central peak, which becomes narrower and narrower as the bistability threshold is approached. For $y < y_{min}$, the atomic system jumps back to the cooperative branch, thus exhibiting discontinuity and hysteresis.

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References

1. V. P. Chebotavev, I. M. Beterov, and V. N. Lisitsyn, IEEE J. Quantum Electron. **QE-4,** 788 (1968); J. F. Scott, M. Sargent, and C. D. Cantrell, Opt. Commun. **15,** 13 (1975); J. F. Scott, Opt. Commun. **15,** 343 (1975); R. B. Schaefer, and C. R. Willis, Phys. Lett. **58A,** 53 (1976); L. A. Lugiato, P. Mandel, S. T. Dembinski, and A. Kossakowski, Phys. Rev. A (to be published).
2. A. Szöke, V. Daneu, J. Goldhar, and N. A. Kernit, Appl. Phys. Lett. **15,** 376 (1966).
3. J. W. Austin and L. G. DeShazer, J. Opt. Soc. Am. **61,** 650 (1971); E. Spiller, J. Opt. Soc. Am. **61,** 669 (1971); J. Appl. Phys. **43,** 1673 (1972).
4. S. L. McCall, Phys. Rev. A **9,** 1515 (1974).
5. H. M. Gibbs, S. L. McCall, and T. N. C. Venkatesan, Phys. Rev. Lett. **36,** 1135 (1976).
6. T. N. C. Venkatesan and S. L. McCall, Appl. Phys. Lett. **30,** 282 (1977).
7. P. W. Smith and E. H. Turner, Appl. Phys. Lett. **30,** 280 (1977).
8. F. S. Felberg and J. H. Marburger, Appl. Phys. Lett. **28,** 731 (1976).
9. K. Jain and G. W. Pratt, Jr., Appl. Phys. Lett. **28,** 719 (1976).
10. R. Bonifacio and L. A. Lugiato, Opt. Commun. **19,** 172 (1976); Phys. Rev. Lett. (to be published).
11. G. S. Agarwal, L. M. Narducci, D. H. Feng, and R. Gilmore, in *Proceedings of the Fourth Coherence and Quantum Optics Conference,* L. Mandel and E. Wolf, eds. (to be published).
12. R. Bonifacio, P. Schwendimann, and F. Haake, Phys. Rev. A **4,** 302 (1971).
13. R. Gilmore and L. M. Narducci, Phys. Rev. A (to be published).
14. The expectation values $\langle S^{\pm}\rangle$, $\langle S_3\rangle$ denote the steady-state values.
15. M. Lax, in *Statistical Physics, Phase Transitions and Superfluidity,* M. Chretien, E. P. Gross, and S. Deser, eds. (Gordon and Breach, New York, 1968), Vol. 2, p. 269.
16. B. R. Mollow, Phys. Rev. **188,** 1969 (1969); Phys. Rev. A **5,** 1522 (1972).