Quantum theory of interferometers with phase-conjugate mirrors

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The quantum-statistical characteristics of the field at the output port of an interferometer with a phase-conjugate mirror are evaluated. The explicit form of the density matrix of the field at the output port is given. The effect of an attenuator in the other arm of the interferometer is also discussed.

1. INTRODUCTION

Phase-conjugate mirrors (PCM's) have unusual properties[1]; for example, these devices enable one to correct for the distortions introduced by a medium. Recently, characteristics of the interference pattern produced by a PCM have been studied both experimentally[2] and theoretically.[3–5] For example, one finds that the fringe pattern depends on the phase of the field that is incident upon the PCM's and on the phase of the reflection coefficient of the PCM's. Since the interference pattern depends on the phase, it is obviously expected to be sensitive to the phase fluctuations and to the statistics of the incident fields. The PCM's are also known to have remarkable quantum features[6]; for example, the conjugate field can be generated even in the absence of the incident field, i.e., when the incident field is in the vacuum state. It is thus important to study quantum-mechanically the interference phenomena produced by a PCM. We therefore formulate in this paper the quantum theory of interferometers[7] in which one of the mirrors is replaced by a PCM.[13] The classical results are obviously contained in our quantum theory. We show in Section 2 the relation between the input and the output fields. The output contains a noise term that arises because of a PCM. The explicit form of the photon-number distribution is given when the input field is in a coherent state. Our analysis reveals that a Michelson interferometer with a PCM has more noise than the conventional interferometer. In Section 3 we demonstrate that the interference pattern disappears if the input field is a chaotic field unless part of the input field is used to pump the PCM. Finally, we discuss in Section 4 the effect of an attenuator that one would typically use to improve the visibility.

2. RELATION BETWEEN THE FIELDS AT THE OUTPUT AND INPUT PORTS—QUANTUM STATISTICS OF THE OUTPUT FIELD

In order to obtain the quantum statistics of the field at the output port, we first derive the relation between the field operators at the input and output ports. Figure 1 represents schematically a Michelson interferometer in which one of the mirrors has been replaced by a PCM. Let the fields at the input ports be denoted by $\hat{a}$ and $\hat{b}$, respectively, which satisfy boson commutation relations. In general, $\hat{b}$ corresponds to a field in the vacuum state. The fields in the two arms of the interferometer are characterized by the annihilation operators $\hat{f}$ and $\hat{g}$. At the beam splitter, these are given by

$$\hat{f} = t\hat{a} + r\hat{b}, \qquad \hat{g} = r\hat{a} + t\hat{b}. \tag{1}$$

The reflection and transmission coefficients are assumed to satisfy

$$|r|^2 + |t|^2 = 1, \qquad rt^* + r^*t = 0. \tag{2}$$

The field incident at the PCM is $\hat{f}e^{ikL}$. The PCM changes[6,14] the input field into a field $\hat{d}$,

$$\hat{d} = A\hat{f}^+e^{-ikL} + B\hat{c}. \tag{3}$$

Here the coefficients A and B depend on the characteristics of the PCM. These also depend on the way in which the PCM is pumped. In Eq. (3) $\hat{c}$ is the annihilation operator for the conjugate field. The field at the beam splitter in the arm (2) of the interferometer is $\hat{d}e^{-ikL}$. The field at the beam splitter in arm 1 of the interferometer is $\hat{g}e^{-2ikl}$. Therefore the total field at the detector is given by the operator $\hat{h}$,

$$\hat{h} = r\hat{d}e^{-ikL} + t\hat{g}e^{-2ikl}, \tag{4}$$

which, when Eqs. (1) and (3) are used, reduces to

$$\hat{h} = rAe^{-2ikL}(t^*\hat{a}^+ + r^*\hat{b}^+) + te^{-2ikl}(r\hat{a} + t\hat{b}) + B\hat{c}re^{-ikL}. \tag{5}$$

The field statistics at the detector can be obtained from the statistics of the fields $\hat{a}$, $\hat{b}$, and $\hat{c}$. Fields $\hat{b}$ and $\hat{c}$ are in the vacuum state. We assume that input field $\hat{a}$ is in a coherent state $|z\rangle$. Clearly, the mean amplitude of the field at the detector is

$$\langle\hat{h}\rangle = rAe^{-2ikL}t^*z^* + rtze^{-2ikl}. \tag{6}$$

This expression is the basis[2,3] for studying the classical interference effects produced by reflection from a PCM.

It should be borne in mind that the interference pattern depends on the phase of $A(t^*/t)(z^*/z)\exp[-2ik(L-l)]$. The phase of A depends on the way in which the PCM is pumped, i.e., on the phase of the square of the pumping field. Thus the relative phase between the pumping field and input field must be fixed. If this relative phase is a random quantity, then the interference pattern will be washed away.

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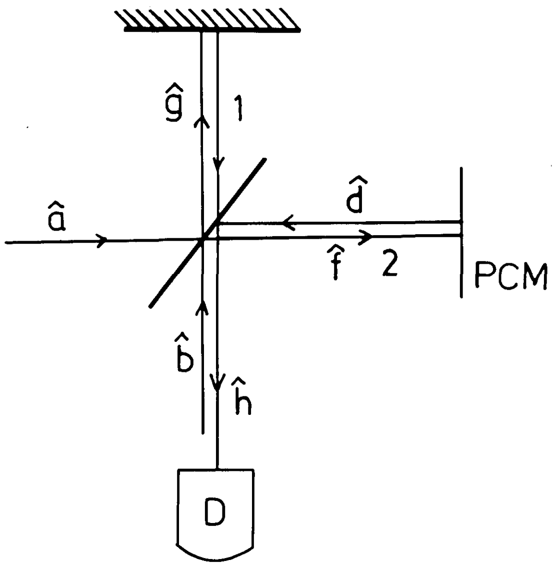

Fig. 1. Schematic illustration of Michelson interferometer with one mirror replaced by a PCM.

The fluctuations in the fields are easily computed by using the properties of the coherent states and vacuum state. Our calculations show that

$$\langle \hat{h}^2\rangle - \langle \hat{h}\rangle^2 = (rt^* + r^*t)rtA\,\exp[-2ik(L+l)] = 0, \quad (7)$$

$$\langle \hat{h}^+\hat{h}\rangle - \langle \hat{h}^+\rangle\langle \hat{h}\rangle = |A|^2|r|^2, \quad (8)$$

where Eqs. (2) have also been used. The fringe visibility v can be computed by using Eqs. (6) and (8):

$$v = 2|t|^2|A| \Big/ \left[|t|^2(1+|A|^2) + \frac{|A|^2}{|z|^2}\right], \quad (9)$$

which, when the quantum correction is ignored, reduces to the classical result

$$v = 2|A|/(1+|A|^2). \quad (10)$$

Clearly, the fringe visibility can be significantly affected by input fields with very low photon number. It is interesting to note that the field at the detector has no phase-sensitive fluctuations, e.g., $\langle(\Delta\hat{h})^2\rangle = 0$, which is due to the unitary property of the beam splitter used to divide the input beam and to combine the beams. In contrast, the total field in arm 2 of the interferometer has phase-sensitive fluctuations. The quantum fluctuation $\langle\Delta\hat{h}^+\Delta\hat{h}\rangle$ is dependent on the properties of the PCM.

The higher-order mean values can be calculated in terms of the Wigner function[15,16] associated with the field h. The Wigner function $\Phi(z_h, z_h^*)$ is defined by

$$\Phi(z_h, z_h^*) = (1/\pi^2)\int \mathrm{d}^2\alpha\,\exp[(\alpha z_h^* - \alpha^* z_h)] \times \mathrm{Tr}\{\rho\,\exp[-(\alpha\hat{h}^+ - \alpha^*\hat{h})]\}. \quad (11)$$

The Wigner function associated with the input field a in a coherent state $|z\rangle$ is

$$\Phi(z_a, z_a^*) = (2/\pi)\exp(-2|z_a - z|^2). \quad (12)$$

Using Eqs. (5) and (12) in Eq. (11), we have proved that

$$\Phi(z_h, z_h^*) = \frac{1}{\pi(\tfrac{1}{2} + |A|^2|r|^2)}\exp\left[-\frac{|z_h - \langle\hat{h}\rangle|^2}{(\tfrac{1}{2} + |A|^2|r|^2)}\right]. \quad (13)$$

Thus the Wigner function for the field at the detector is Gaussian centered at the mean value $\langle\hat{h}\rangle$ of the field. Higher-order fluctuations in the field follow from the Gaussian property of the Wigner function. The field at the detector can be viewed as the one obtained by superposing a coherent field with amplitude $\langle\hat{h}\rangle$ and a chaotic field with mean photon number $|A|^2|r|^2$. The photon-number distribution for such a field is well known[17]:

$$p(n) = \frac{(|Ar|^2)^n}{(1+|Ar|^2)^{n+1}}\exp\left(-\frac{|\langle\hat{h}\rangle|^2}{|Ar|^2+1}\right) \times L_n\left[-\frac{|\langle\hat{h}\rangle|^2}{|Ar|^2(|Ar|^2+1)}\right], \quad (14)$$

where L_n is the Laguerre polynomial of degree n. The fluctuations in the photon number are given by

$$\langle(\hat{h}^+\hat{h})^2\rangle - \langle\hat{h}^+\hat{h}\rangle^2 = |Ar|^2(|Ar|^2+1) + |\langle\hat{h}\rangle|^2(2|Ar|^2+1). \quad (15)$$

These results, viz. Eqs. (13)–(15), are to be compared with the corresponding results for the usual Michelson interferometer:

$$\Phi(z_h, z_h^*) = (2/\pi)\exp(-2|z_h - \langle\hat{h}\rangle|^2), \quad (16)$$

$$\langle\hat{h}\rangle = rtz(e^{-2ikl} + e^{-2ikL}), \quad (17)$$

$$\langle(\hat{h}^+\hat{h})^2\rangle - \langle\hat{h}^+\hat{h}\rangle^2 = \langle\hat{h}^+\hat{h}\rangle, \quad (18)$$

$$p(n) = \frac{\bar{n}^n e^{-\bar{n}}}{n!}, \qquad \bar{n} = |\langle\hat{h}\rangle|^2. \quad (19)$$

Thus the field at the detector is in a coherent state with amplitude $\langle\hat{h}\rangle$, if the input field is in a coherent state. The phase-conjugating device in the interferometer not only changes the interference pattern but also adds quantum fluctuations to the field.

3. INCOHERENT LIGHT AS INPUT TO THE INTERFEROMETER WITH PHASE-CONJUGATE MIRRORS

We next discuss some unusual properties of an interferometer with a PCM. We show that the output field shows no interference effects if the input field is an incoherent field, for example, the input field may be a chaotic field or a field in Fock state $|n\rangle$. This is in contrast to the usual Michelson interferometer. To see this, let us consider a chaotic field, with an average number of photons $\bar{n}$, incident upon the *conventional* interferometer. The field operator at the detector is

$$\hat{h} = rt(e^{-2ikl} + e^{-2ikL})a + (r^2e^{-2ikL} + t^2e^{-2ikl})b. \quad (20)$$

From Eq. (20) and the properties of the chaotic field, it is straightforward to show that

$$\langle\hat{h}\rangle = 0, \qquad \langle\hat{h}^2\rangle = 0,$$

$$\langle\hat{h}^+\hat{h}\rangle = |r|^2|t|^2\bar{n}|e^{-2ikl} + e^{-2ikL}|^2. \quad (21)$$

Expression (21) leads to the usual interference pattern, although the input field is incoherent. The interference pattern is preserved because the two parts of the field in the two arms of the interferometer arise from the *same* source.

Let us next consider the interferometer in which one of the mirrors has been replaced by a PCM. The field operator at the detector is now given by Eq. (5). Using Eq. (5) and the properties of the chaotic field, one can show that

$$\langle \hat{h} \rangle = 0, \qquad \langle \hat{h}^2 \rangle = 2A\bar{n}|t|^2 r^2 \exp[-2ik(L+l)],$$
$$\langle \hat{h}^+ \hat{h} \rangle = |r|^2|A|^2 + \bar{n}|t|^2|r|^2(1+|A|^2). \tag{22}$$

The intensity distribution [Eqs. (22)] does not exhibit any interference pattern. Thus the interference pattern of the type predicted by Eq. (6) is washed out because of the incoherence of the source and the phase-conjugation process. This suggests that the PCM makes the field in arm 2 of the interferometer statistically uncorrelated with the field in arm 1 of the interferometer for the case of an incoherent field. The fields became statistically uncorrelated in a restricted sense since $\langle \hat{h}^2 \rangle \neq 0$, i.e., if we write

$$\hat{h} = \hat{h}^{(1)} + \hat{h}^{(2)}, \qquad \hat{h}^{(1)} = t\hat{g}e^{-2ikl}, \qquad \hat{h}^{(2)} = r\hat{d}e^{-ikL}, \tag{23}$$

then

$$\langle \hat{h}^{(1)+}\hat{h}^{(2)} \rangle = 0, \qquad \langle \hat{h}^{(1)}\hat{h}^{(2)} \rangle \neq 0. \tag{24}$$

This can also be understood at the classical level. The phase of the field, apart from factors such as kl, in arm 1 (arm 2) is φ ($-\varphi$ owing to the phase-conjugation process). Thus the relative phase is 2φ, which is a random (deterministic) quantity for an incoherent (coherent) field. This is why the interference pattern disappears. For the conventional interferometer the relative phase is independent of φ, and hence the interference pattern survives even for an incoherent input field. This analysis presumes that the phase of the reflection coefficient of the PCM is a deterministic quantity. The phase of A is partly determined by the field used to pump the PCM. Our analysis suggests that interference effects with PCM can be observed only if there is some correlation between the input field and the field used to pump the PCM. Thus the interference pattern can be restored if the same field is used both as input and as the pump for the PCM, as in the case when the combination $[A(z^*/z)]$ has no random phase.

4. EFFECT OF AN ATTENUATOR IN ARM 1 OF THE INTERFEROMETER

We saw that the fringe visibility [Eq. (10)] depends on the reflectivity of the PCM. Since the reflectivity of a PCM is generally quite small, the visibility is very low. The visibility can be improved by inserting an attenuator into arm 1 of the interferometer. The attenuator not only attenuates the field but also adds noise terms to the field in arm 1. The transformation of the field after it passes through the attenuator is well known.[18] The values of a and a^+a before and after the transformation are related by

$$\langle \hat{g} \rangle = \sqrt{G}\langle \hat{g} \rangle_0,$$
$$\langle \hat{g}^+\hat{g} \rangle = G\langle \hat{g}^+\hat{g} \rangle_0 + \beta(1-G). \tag{25}$$

Here G is the attenuation coefficient, and β is related to the attenuator characteristics. The Wigner function of the field before and after the transformation is also simply related. If the Wigner function of the input is

$$\Phi(z, z^*) = \frac{1}{\pi\sqrt{{\tau_0}^2 - 4\mu\mu^*}} \times \exp\left[-\frac{\mu(z-z_0)^2 + \mu^*(z^*-z_0^*)^2 + \tau_0|z-z_0|^2}{{\tau_0}^2 - 4\mu\mu^*}\right],$$

$$\langle \Delta\hat{g}^2 \rangle = -2\mu^*, \qquad \langle \Delta\hat{g}^+\Delta\hat{g} \rangle = \tau_0 - 1/2, \tag{26}$$

then the Wigner function of the output is

$$\Phi(z, z^*) = \frac{1}{\pi\sqrt{\tau^2 - 4\mu\mu^* G^2}} \times \exp\{-[\mu G(z - z_0\sqrt{G})^2 + \mu^* G(z^* - z_0^*\sqrt{G})^2 + \tau|z - z_0\sqrt{G}|^2]/(\tau^2 - 4\mu\mu^* G^2)\}, \tag{27}$$

where

$$\tau \equiv G(\tau_0 - 1/2) + \beta(1-G) + 1/2. \tag{28}$$

Note that the field in arm 1 passes through the attenuator twice, and hence the transformations [Eqs. (25) and (27)] are to be applied twice. From Eqs. (25) we also have the following result for the deviation from the mean value:

$$\langle \Delta\hat{g}^+\Delta\hat{g} \rangle = G\langle \Delta\hat{g}^+\Delta\hat{g} \rangle_0 + \beta(1-G) = \beta(1-G) \tag{29}$$

if initially $\langle \Delta\hat{g}^+\Delta\hat{g} \rangle_0 = 0$. So if such a field passes twice through the attenuator, then

$$\langle \Delta\hat{g}^+\Delta\hat{g} \rangle = G\beta(1-G) + \beta(1-G)$$
$$= \beta(1-G^2). \tag{30}$$

The mean value of $\hat{h}$ and the fluctuation in $\hat{h}$ can be computed by using Eq. (4) and Eqs. (26)–(30). We cite the result of such a calculation:

$$\langle \hat{h} \rangle = rAe^{-2ikL}t^*z^* + rtzG^2e^{-2ikl}, \tag{31}$$

$$\langle \Delta\hat{h}^2 \rangle = 0,$$
$$\langle \Delta\hat{h}^+\Delta\hat{h} \rangle = |A|^2|r|^2 + |t|^2\beta(1-G^2). \tag{32}$$

The Wigner function of the output is now given by [cf. Eq. (13)]:

$$\Phi(z_h, {z_h}^*) = \frac{1}{\pi[1/2 + |A|^2|r|^2 + \beta|t|^2(1-G^2)]} \times \exp\left\{-\frac{|z_h - \langle \hat{h} \rangle|^2}{[1/2 + |A|^2|r|^2 + \beta|t|^2(1-G^2)]}\right\}. \tag{33}$$

It is clear from Eq. (31) that the fringe visibility can be improved by choosing G suitably.

In conclusion, we have presented a first-principle quantum theory of an interferometer with a PCM, and we have answered practical questions concerning the effects of the attenuator and the phase fluctuations of the input field.

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