Statistical generalizations of the optical cross-section theorem with application to inverse scattering

P. Scott Carney and Emil Wolf

Department of Physics and Astronomy and Rochester Theory Center for Optical Science and Engineering, University of Rochester, Rochester, New York 14627

Girish S. Agarwal

Physical Research Laboratory, Navrangpura, Ahmedabad, India 38009

Received February 13, 1997; revised manuscript received July 10, 1997; accepted July 21, 1997

A fundamental result of scattering theory, the so-called optical theorem, applies to situations where the field incident on the scatterer is a monochromatic plane wave and the scatterer is deterministic. We present generalizations of the theorem to situations where either the incident field or the scatterer or both are spatially random. By using these generalizations we demonstrate the possibility of determining the structure of some random scatterers from the knowledge of the power absorbed from two plane waves incident on it. © 1997 Optical Society of America [S0740-3232(97)02912-8]

1. INTRODUCTION

One of the central results of scattering theory is the so-called optical theorem.[1] It relates the rate at which energy is scattered by a finite-range potential or by a finite object and the amplitude of the scattered wave in the forward direction (the direction of incidence). The theorem has a long and interesting history, which has been reviewed by Newton.[2] In its quantum mechanical context it appears that the theorem was first formulated by Feenberg.[3] In the context of classical theory it was derived by van der Hulst[4] for scalar waves and by Jones[5] (see also Born and Wolf[6]) for electromagnetic waves.

In the usual formulation of the optical theorem, the field incident on the scatterer is assumed to be a monochromatic plane wave and the scatterer to be strictly deterministic. Neither of these two assumptions is appropriate in many practical situations. The incident field is sometimes a partially coherent wave, and the scattering potential that characterizes the response of the scatterer to the incident field is often a random function of position and sometimes also of time. It is, therefore, desirable to obtain generalizations of the optical cross-section theorem to situations where either the incident field or the scatterer or both are described statistically. In this paper we present such generalizations,[7] and we show how the results can be used for determining structure of some media from measurements of the power extinguished by the scatterer when two plane waves are incident on it.

We begin with a brief account of the usual formulation of the theorem. Let us consider scattering of a monochromatic plane wave $\Psi^{(i)}(\mathbf{r}, t)$ by a medium characterized by a time-independent finite-range potential assumed, to begin with, to be deterministic. We take the plane wave to be of unit amplitude, propagating in a direction specified by a unit vector $\mathbf{u}_0$, so that

$$\Psi^{(i)}(\mathbf{r}, t) = \psi^{(i)}(\mathbf{r}, \omega)\exp(-i\omega t), \tag{1.1}$$

$$\psi^{(i)}(\mathbf{r}, \omega) = \exp(ik\mathbf{u}_0 \cdot \mathbf{r}). \tag{1.2}$$

Here $\mathbf{r}$ denotes the position vector of a point in space, t denotes the time, ω denotes the frequency, and

$$k = \frac{\omega}{c} \tag{1.3}$$

is the free-space wave number, c being the speed of light in vacuum. In the framework of classical theory the scattering potential, $F(\mathbf{r}, \omega)$, say, is defined by the expression

$$F(\mathbf{r}, \omega) = k^2\eta(\mathbf{r}, \omega) \tag{1.4a}$$

$$= \frac{k^2}{4\pi}[n^2(\mathbf{r}, \omega) - 1], \tag{1.4b}$$

where $\eta(\mathbf{r}, \omega)$ is the dielectric susceptibility and $n(\mathbf{r}, \omega)$ is the refractive index of the scattering medium.

We will assume that the scattering is elastic (no change of frequency on scattering), and we denote by $\psi(\mathbf{r}, \omega)$ the time-independent part of the total field, generated by the interaction of the incident wave with the scatterer. At a point P in the far zone, at distance r from an origin O that is chosen in the region D occupied by the scatterer, in the direction specified by a unit vector $\mathbf{u}$ (see Fig. 1), the field may be expressed as

$$\psi(r\mathbf{u}, \mathbf{u}_0; \omega) = \psi^{(i)}(\mathbf{u}_0; \omega) + \psi^{(s)}(r\mathbf{u}, \mathbf{u}_0, \omega), \tag{1.5}$$

where the scattered field $\psi^{(s)}$ has the form

0740-3232/97/123366-06$10.00

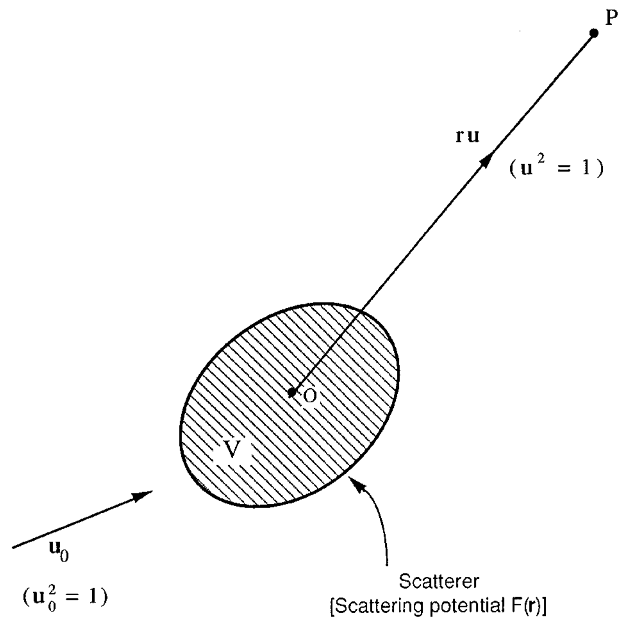

Fig. 1. Illustration of the notation.

$$\psi^{(s)}(r\mathbf{u}, \mathbf{u}_0, \omega) = f(\mathbf{u}, \mathbf{u}_0; \omega) \frac{\exp(ikr)}{r}, \tag{1.6}$$

$\mathbf{f}(\mathbf{u}, \mathbf{u}_0; \omega)$ being the scattering amplitude.

The optical cross-section theorem asserts that the total cross section $\sigma(\mathbf{u}_0; \omega)$ (also called the extinction cross section), which is the sum of the scattering cross section $\sigma^{(s)}(\mathbf{u}_0, \omega)$ and the absorption cross-section $\sigma^{(a)}(\mathbf{u}_0, \omega)$, is given by the expression

$$\sigma(\mathbf{u}_0; \omega) = \frac{4\pi}{k} \operatorname{Im} f(\mathbf{u}_0, \mathbf{u}_0; \omega), \tag{1.7}$$

where Im denotes the imaginary part.

2. OPTICAL CROSS-SECTION THEOREM WITH RANDOM SCATTERERS

Let us now consider the generalization of the optical cross-section theorem to situations where the medium is spatially random but the incident field is, as before, a monochromatic plane wave. More specifically, we assume that the dielectric susceptibility $\eta(\mathbf{r}, \omega)$ and, consequently, the scattering potential $F(\mathbf{r}, \omega)$, are, for each frequency ω, random functions of position, characterized by appropriate statistical ensembles. The scattering amplitude $f(\mathbf{u}, \mathbf{u}_0; \omega)$ and the total cross section $\sigma(\mathbf{u}_0; \omega)$ will then be random variables. The optical cross-section theorem (1.7) will hold for each realization of the ensembles, and one obtains, on taking the expectation value of Eq. (1.7), the following natural generalization of the optical cross-section theorem for scattering on a spatially random medium:

$$\langle \sigma(\mathbf{u}_0; \omega) \rangle = \frac{4\pi}{k} \operatorname{Im} \langle f(\mathbf{u}_0, \mathbf{u}_0; \omega) \rangle. \tag{2.1}$$

The angular brackets denote, of course, the averaging over the ensemble of the scatterer.

As in the deterministic case, it is clear that for any realistic situation the scattering cross section for scattering on random media cannot be determined exactly. We will sketch out a perturbation procedure for calculating it approximately.

One readily finds from the basic integral equation of potential scattering[8] that for each realization of the statistical ensemble of the scatterer, the scattering amplitude is expressible in the form (if, for the sake of simplicity we now suppress the explicit dependence of the various quantities on the frequency ω)

$$f(\mathbf{u}, \mathbf{u}_0) = \sum_{n=1}^{\infty} f_n(\mathbf{u}, \mathbf{u}_0), \tag{2.2}$$

with

$$f_1(\mathbf{u}, \mathbf{u}_0) = k^2 \int_V \eta(\mathbf{r}_1) \exp[-ik(\mathbf{u} - \mathbf{u}_0) \cdot \mathbf{r}_1] \mathrm{d}^3 r_1, \tag{2.3a}$$

$$f_2(\mathbf{u}, \mathbf{u}_0) = k^4 \int_V \int_V \eta(\mathbf{r}_1) \eta(\mathbf{r}_2) G(\mathbf{r}_1, \mathbf{r}_2) \times \exp[-ik(\mathbf{u} \cdot \mathbf{r}_1 - \mathbf{u}_0 \cdot \mathbf{r}_2)] \mathrm{d}^3 r_1 \mathrm{d}^3 r_2, \tag{2.3b}$$

$$\vdots$$

$$\begin{aligned} f_n(\mathbf{u}, \mathbf{u}_0) = k^{2n} \int_V \int_V \cdots \int_V & \eta(\mathbf{r}_1) \eta(\mathbf{r}_2) \cdots \eta(\mathbf{r}_n) \\ & \times G(\mathbf{r}_1, \mathbf{r}_2) G(\mathbf{r}_2, \mathbf{r}_3) \cdots G(\mathbf{r}_{n-1}, \mathbf{r}_n) \\ & \times \exp[-ik(\mathbf{u} \cdot \mathbf{r}_1 - \mathbf{u}_0 \cdot \mathbf{r}_n)] \\ & \times \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 \cdots \mathrm{d}^3 r_n, \end{aligned} \tag{2.3c}$$

where

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{\exp(ik|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|} \tag{2.4}$$

is the outgoing free-space Green's function at frequency ω and V denotes the scattering volume.

On substituting from Eq. (2.2) into the optical cross-section theorem (2.1) for scattering on spatially random media, we readily find that

$$\langle \sigma(\mathbf{u}_0) \rangle = \sum_{n=1}^{\infty} \langle \sigma_n(\mathbf{u}_0) \rangle, \tag{2.5}$$

where

$$\langle \sigma_1(\mathbf{u}_0) \rangle = 4\pi k \operatorname{Im} \int_V C_1(\mathbf{r}_1) \mathrm{d}^3 r_1, \tag{2.6a}$$

$$\langle \sigma_2(\mathbf{u}_0) \rangle = 4\pi k^3 \operatorname{Im} \int_V C_2(\mathbf{r}_1, \mathbf{r}_2) G(\mathbf{r}_1 - \mathbf{r}_2) \times \exp[-ik\mathbf{u}_0 \cdot (\mathbf{r}_1 - \mathbf{r}_2)] \mathrm{d}^3 r_1 \mathrm{d}^3 r_2, \tag{2.6b}$$

$$\vdots$$

$$\langle \sigma_n(\mathbf{u}_0)\rangle = 4\pi k^{2n-1}\ \mathrm{Im}\int_V \cdots \int_V C_n(\mathbf{r}_1,\ \mathbf{r}_2,\cdots \mathbf{r}_n) \times G(\mathbf{r}_1 - \mathbf{r}_2)G(\mathbf{r}_2 - \mathbf{r}_3)\cdots G(\mathbf{r}_{n-1} - \mathbf{r}_n) \times \exp[-ik\mathbf{u}_0 \cdot (\mathbf{r}_1 - \mathbf{r}_n)]\mathrm{d}^3r_1\mathrm{d}^3r_2\cdots \mathrm{d}^3r_n, \tag{2.6c}$$

where

$$C_1(r_1) = \langle \eta(r_1)\rangle \tag{2.7}$$

is the mean of η and

$$C_n(\mathbf{r}_1,\ \mathbf{r}_2,\cdots \mathbf{r}_n) = \langle\ \eta(\mathbf{r}_1)\,\eta(\mathbf{r}_2)\cdots\ \eta(\mathbf{r}_n)\rangle \tag{2.8}$$

is the n-point spatial correlation function of the dielectric susceptibility of the scattering medium. We stress that the susceptibility η and consequently the correlation functions C_n depend not only on spatial variables but also on the frequency, which, for the sake of simplicity, we do not show explicitly.

The formula (2.5), together with expressions (2.6), makes it, in principle, possible to calculate the total cross section for scattering of a monochromatic plane wave on a spatially random medium from the knowledge of the mean of the dielectric susceptibility of the scatterer and its n-point spatial correlation functions C_n.

3. GENERALIZATION TO STOCHASTIC INCIDENT FIELD

Suppose next that the scattering medium is deterministic but that the incident field is random; more specifically, suppose that the incident field is statistically stationary and has arbitrary state of coherence. Under these circumstances the concept of cross section no longer applies, for the cross section expresses the rate at which energy is scattered or extinguished in terms of (i.e., normalized by) the rate at which energy is incident on the scatterer per unit area perpendicular to the direction of propagation of the incident wave. If the incident wave is not a plane wave, such normalization is not possible, because there is no longer a unique direction of incidence.[9] In such situations it is more appropriate to consider the (nonnormalized) rate at which energy is scattered or extinguished. We will denote these quantities by $P^{(s)}$ and $P^{(e)}$, respectively.

Suppose first that the field incident on the scatterer is a deterministic free field,[10] which propagates into the half-space $z > 0$. It may be represented as linear superposition of plane wave modes, *viz.*,

$$\psi^{(i)}(\mathbf{r}) = \int_{|\mathbf{u}_\perp'|^2\leq 1} a(\mathbf{u}')\exp(ik\mathbf{u}' \cdot \mathbf{r})\mathrm{d}^2u_\perp', \tag{3.1}$$

where $\mathbf{u}' \equiv (u_x',\ u_y',\ u_z')$ is a real unit vector, $\mathbf{u}_\perp' \equiv (u_x',\ u_y',\ 0)$, and $u_z' = +\sqrt{1 - \mathbf{u}_\perp^2}$. Each plane wave, labeled by a particular value of the vector $\mathbf{u}'$, will give rise to a scattered field whose (weighted) scattering amplitude is $a(\mathbf{u}')f(\mathbf{u},\ \mathbf{u}')$, and, consequently, the weighted scattering amplitude of the total field is

$$F(\mathbf{u}) = \int_{|\mathbf{u}_\perp'|\leq 1} a(\mathbf{u}')f(\mathbf{u},\ \mathbf{u}')\mathrm{d}^2u_\perp'. \tag{3.2}$$

More explicitly, the scattered field in the far zone generated by scattering of the incident field (3.1) is given by the expression

$$\psi^{(s)}(r\mathbf{u}) = F(\mathbf{u})\,\frac{\exp(ikr)}{r}, \tag{3.3}$$

where $F(\mathbf{u})$ is given by Eq. (3.2).

Now the rate at which energy is extinguished by scattering and absorption of the plane-wave mode labeled by a particular value $\mathbf{u}'$ is, according to Eq. (2.1), equal to $(4\pi/k)\mathrm{Im}\,a^*(\mathbf{u}')F(\mathbf{u})$, and hence the power extinguished by scattering and absorption of the incident field [Eq. (3.1)] is given by the expression

$$P^{(e)} = \frac{4\pi}{k}\ \mathrm{Im}\int_{|\mathbf{u}_\perp'|\leq 1} a^*(\mathbf{u}')F(\mathbf{u}')\mathrm{d}^2u_\perp. \tag{3.4}$$

On substituting from Eq. (3.2) into Eq. (3.4) we obtain for $P^{(e)}$ the expression

$$P^{(e)} = \frac{4\pi}{k}\ \mathrm{Im}\int_{|\mathbf{u}_\perp'|\leq 1}\int_{|\mathbf{u}_\perp''|\leq 1} a^*(\mathbf{u}')a(\mathbf{u}'')f(\mathbf{u}',\ \mathbf{u}'')\mathrm{d}^2u_\perp'\mathrm{d}^2u_\perp''. \tag{3.5}$$

Suppose now that the incident field is not deterministic but varies randomly with position. Then the spectral amplitude $a(\mathbf{u})$ is a random function, and we obtain, on taking the ensemble average of Eq. (3.5), the following expression for the averaged power extinguished (i.e., scattered and absorbed) by interaction of the incident random field with the scatterer:

$$\langle P^{(e)}\rangle = \frac{4\pi}{k}\ \mathrm{Im}\int_{|\mathbf{u}_\perp'|\leq 1}\int_{|\mathbf{u}_\perp''|\leq 1} \mathcal{A}(\mathbf{u}',\ \mathbf{u}'')f(\mathbf{u}',\ \mathbf{u}'')\mathrm{d}^2u_\perp'\mathrm{d}^2u_\perp''. \tag{3.6}$$

Here

$$\mathcal{A}(\mathbf{u}',\ \mathbf{u}'') = \langle a^*(\mathbf{u}')a(\mathbf{u}'')\rangle \tag{3.7}$$

is the angular correlation function that characterizes the second-order coherence properties of the incident field (Ref. 11, Sec. 5.6.3). The average in Eq. (3.7) is now taken over an ensemble of space-frequency realizations in the sense of coherence theory in the space-frequency domain.[12]

The formula (3.6) is the required generalized analog of the usual optical cross-section theorem when the field incident on the scatterer is random.

In the special case when the incident field is a (deterministic) plane wave of unit amplitude, which propagates in the direction specified by a unit vector $\mathbf{u}_0$, the angular correlation function has the form

$$\mathcal{A}(\mathbf{u}', \mathbf{u}'') = |a(\mathbf{u}_0)|^2[\delta^{(2)}(\mathbf{u}' - \mathbf{u}_0)\delta^{(2)}(\mathbf{u}'' - \mathbf{u}_0)], \tag{3.8}$$

where $\delta^{(2)}$ is the two-dimensional Dirac delta function. On substituting from Eq. (3.8) into Eq. (3.6) we obtain at once the formula

$$\langle P^{(e)} \rangle = |a(\mathbf{u}_0)|^2 \frac{4\pi}{k} \operatorname{Im} f(\mathbf{u}_0, \mathbf{u}_0). \tag{3.9}$$

In this case $\langle P^{(e)} \rangle$ may be identified with the extinction cross section $\sigma(\mathbf{u}_0)$, and we see that except for a slight change in notation, the formula (3.9) is just the ordinary optical cross-section theorem [Eq. (2.1)].

Finally we note that if not only the incident field but also the scatterer is spatially random, the expression for the averaged power extinguished on scattering can readily be obtained from Eq. (3.6). It is necessary only to average that expression over the statistical ensemble of the scatterer. Denoting this average by an overbar, we obtain the required formula:

$$\overline{\langle P^{(e)} \rangle} = \frac{4\pi}{k} \operatorname{Im} \int\limits_{|\mathbf{u}'_\perp| \leq 1} \int\limits_{|\mathbf{u}''_\perp| \leq 1} \mathcal{A}(\mathbf{u}', \mathbf{u}'') \times \overline{f(\mathbf{u}', \mathbf{u}'')} \mathrm{d}^2 u'_\perp \mathrm{d}^2 u''_\perp . \tag{3.10}$$

It is of interest to note that the two averaging procedures are taken independently, over the ensemble of the incident field and over the ensemble of the scatterer.

We will refer to the formulas (3.5), (3.6) and (3.10) as power extinction theorems.

4. USE OF THE POWER EXTINCTION THEOREM FOR STRUCTURE DETERMINATION OF RANDOM SCATTERERS

We will now show that the power extinction theorem (3.5) may be used to obtain, in a novel way, information about the structure of some random media.

Suppose that the scatterer is nonabsorbing, so that

$$\operatorname{Im} \eta(\mathbf{r}) = 0 \tag{4.1}$$

and that it is δ correlated:

$$\langle \eta(\mathbf{r}_1)\eta(\mathbf{r}_2) \rangle = \Gamma_\eta(\mathbf{r}_1)\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2). \tag{4.2}$$

The (real) function $\Gamma_\eta(\mathbf{r})$ evidently characterizes the strength of the spatial fluctuations of the dielectric susceptibility. We will carry out the calculations within the accuracy of the first Born approximation. The scattering amplitude for each realization of the ensemble of the scatterer is then given by Eq. (2.3b). Hence the average of Eq. (3.5) becomes, if we take the amplitude of the incident wave to have any prescribed value (denoted by a, rather than being unity),

$$\langle P^{(e)} \rangle = 4\pi k^3 |a|^2 \operatorname{Im} \int_V \int_V \langle \eta(\mathbf{r}_1)\eta(\mathbf{r}_2) \rangle G(\mathbf{r}_1, \mathbf{r}_2) \times \exp[-ik\mathbf{u}_0 \cdot (\mathbf{r}_1 - \mathbf{r}_2)] \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 . \tag{4.3}$$

On interchanging the variables of integration and making use of the symmetry property of the integrand in Eq. (4.3), we obtain for $\langle P^{(e)} \rangle$ the expression

$$\langle P^{(e)} \rangle = 4\pi k^4 |a|^2 \int_V \int_V \langle \eta(\mathbf{r}_1)\eta(\mathbf{r}_2) \rangle \frac{\sin k|\mathbf{r}_1 - \mathbf{r}_2|}{k|\mathbf{r}_1 - \mathbf{r}_2|} \times \cos[k\mathbf{u}_0 \cdot (\mathbf{r}_1 - \mathbf{r}_2)] \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 . \tag{4.4}$$

Using assumed expression (4.2) for the correlation of the dielectric susceptibility, Eq. (4.4) reduces to

$$\langle P^{(e)} \rangle = 4\pi k^4 |a|^2 \tilde{\Gamma}_\eta(0), \tag{4.5}$$

where

$$\tilde{\Gamma}_\eta(\mathbf{K}) = \int_V \Gamma_\eta(\mathbf{r}) \exp(i\mathbf{K} \cdot \mathbf{r}) \mathrm{d}^3 r \tag{4.6}$$

is the spatial Fourier transform of $\Gamma_\eta(\mathbf{r})$.

The formula (4.5) shows that, in the present case, the total extinguished power is proportional to a particular spatial frequency component of the strength $\Gamma_\eta(\mathbf{r})$ of the correlation function of the scatterer, namely the zero-frequency component.

Next let us consider an incident field that consists of two monochromatic plane waves of the same frequency ω and of complex amplitudes A_1 and A_2, propagating in directions specified by unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ (Fig. 2). The amplitude function in Eq. (3.1) is then given by the expression

$$a(\mathbf{u}') = A_1 \delta^{(2)}(\mathbf{u}' - \mathbf{u}_1) + A_2 \delta^{(2)}(\mathbf{u}' - \mathbf{u}_2), \tag{4.7}$$

and Eq. (3.5) becomes, after taking the average over the ensemble of the scatterer,

$$\langle P^{(e)} \rangle = \langle P_{11}^{(e)} \rangle + \langle P_{22}^{(e)} \rangle + \langle P_{12}^{(e)} \rangle + \langle P_{21}^{(e)} \rangle, \tag{4.8}$$

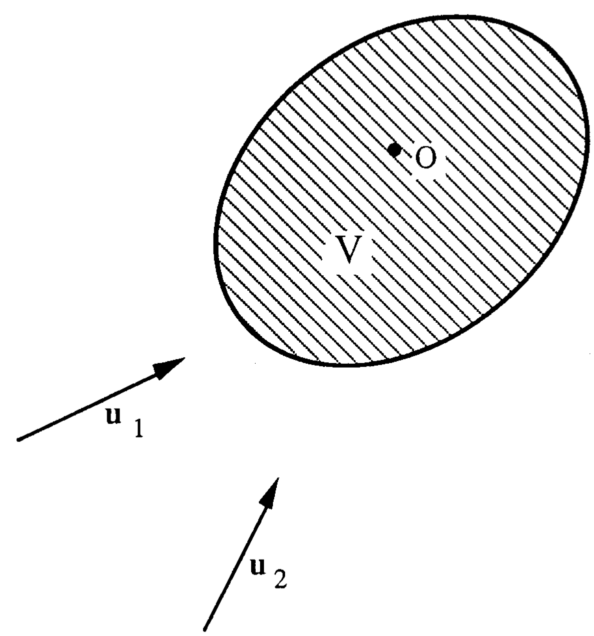

Fig. 2. Illustration of the notation used in Section 4 in the analysis relating to structure determination of random scatterers from measurements of the power extinguished on scattering. $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors in directions of incidence of two monochromatic plane waves of amplitudes A_1 and A_2.

where

$$\langle P_{ij}^{(e)}\rangle = \frac{4\pi}{k}\,\mathrm{Im}\int_V\int_V A_i^* A_j \delta^{(2)}(\mathbf{u}' - \mathbf{u}_i)\delta^{(2)}(\mathbf{u}'' - \mathbf{u}_j) \times \langle f(\mathbf{u}', \mathbf{u}'')\rangle \mathrm{d}^2u'_\perp \mathrm{d}^2u''_\perp = \frac{4\pi}{k}\,\mathrm{Im}\, A_i^* A_j f(\mathbf{u}_i, \mathbf{u}_j). \tag{4.9}$$

We see at once from Eqs. (4.9) and (3.9) that $\langle P_{11}^{(e)}\rangle$ and $\langle P_{22}^{(e)}\rangle$ are just the extinction power for scattering of each of the two incident plane waves. The other two terms $\langle P_{12}^{(e)}\rangle$ and $\langle P_{21}^{(e)}\rangle$ represent contributions to the total extinction power $\langle P^{(e)}\rangle$ arising from interference of the two scattered waves. We can readily evaluate this contribution. We have, on taking the average of Eq. (2.3b) over the ensemble of the scatterer,

$$\mathrm{Im}\, A_i^* A_j f(\mathbf{u}_i, \mathbf{u}_j) = k^4\,\mathrm{Im}\int_V\int_V A_i^* A_j \langle \eta(\mathbf{r}_1)\eta(\mathbf{r}_2)\rangle G(\mathbf{r}_1, \mathbf{r}_2) \times \exp[-ik(\mathbf{u}_i\cdot\mathbf{r}_1 - \mathbf{u}_j\cdot\mathbf{r}_2)]\mathrm{d}^3r_1\mathrm{d}^3r_2, \tag{4.10}$$

and hence, using also the symmetry properties of the integrand,

$$\begin{aligned}\mathrm{Im}[A_1^* A_2 f(\mathbf{u}_1, \mathbf{u}_2) + A_1 A_2^* f(\mathbf{u}_2, \mathbf{u}_1)] &= 2k^4\,\mathrm{Im}\int_V\int_V \langle \eta(\mathbf{r}_1)\eta(\mathbf{r}_2)\rangle G(\mathbf{r}_1, \mathbf{r}_2) \\ &\quad\times \mathrm{Re}\{A_1^* A_2 \exp[ik(\mathbf{u}_1\cdot\mathbf{r}_1 - \mathbf{u}_2\cdot\mathbf{r}_2)]\}\mathrm{d}^3r_1\mathrm{d}^3r_2 \\ &= 2k^5\int_V\int_V |A_1A_2|\langle \eta(\mathbf{r}_1)\eta(\mathbf{r}_2)\rangle \frac{\sin k|\mathbf{r}_1 - \mathbf{r}_2|}{k|\mathbf{r}_1 - \mathbf{r}_2|} \\ &\quad\times \cos[k(\mathbf{u}_1\cdot\mathbf{r}_1 - \mathbf{u}_2\cdot\mathbf{r}_2) - \theta]\mathrm{d}^3r_1\mathrm{d}^3r_2,\end{aligned} \tag{4.11}$$

where

$$\theta = \arg(A_1^* A_2). \tag{4.12}$$

On substituting for the correlation function $\langle \eta(\mathbf{r}_1)\eta(\mathbf{r}_2)\rangle$ the assumed expression (4.2), Eq. (4.11) reduces to

$$\mathrm{Im}[A_1^* A_2 f(\mathbf{u}_1, \mathbf{u}_2) + A_1^* A_2 f(\mathbf{u}_2, \mathbf{u}_1)] = 2|A_1A_2|k^5\int_V \Gamma_\eta(\mathbf{r}_1)\cos[k(\mathbf{u}_1 - \mathbf{u}_2)\cdot\mathbf{r}_1 - \theta]\mathrm{d}^3r_1. \tag{4.13}$$

Using this expression in Eq. (4.9), we obtain for the extinction power arising from the interference of the two scattered waves the expression

$$\langle P_{12}^{(e)}\rangle + \langle P_{21}^{(e)}\rangle = 8\pi k^4|A_1A_2|\int_V \Gamma_\eta(\mathbf{r}_1) \times \cos[k(\mathbf{u}_1 - \mathbf{u}_2)\cdot\mathbf{r}_1 - \theta]\mathrm{d}^3r_1, \tag{4.14}$$

or, in terms of the spatial Fourier transform, defined by Eq. (4.6), of the function $\Gamma_\eta(\mathbf{r})$,

$$\langle P_{12}^{(e)}\rangle + \langle P_{21}^{(e)}\rangle = 4\pi k^4|A_1A_2|\{\tilde{\Gamma}_\eta[k(\mathbf{u}_1 - \mathbf{u}_2)]\exp(-i\theta) + \tilde{\Gamma}_\eta[k(\mathbf{u}_2 - \mathbf{u}_1)]\exp(i\theta)\}. \tag{4.15}$$

It will be convenient to indicate the dependence of the interference term on the angle θ, defined by Eq. (4.12). We therefore set

$$\langle P_{12}^{(e)}\rangle + \langle P_{21}^{(e)}\rangle = P_{\mathrm{int}}(\theta). \tag{4.16}$$

Suppose that one performs the scattering experiments twice, once with a pair of plane waves whose phase difference (4.12) has the value θ_1 and once with a pair of plane waves whose phase difference has the value θ_2. It then follows from Eqs. (4.16) and (4.15) that

$$P_{\mathrm{int}}(\theta_1) = 4\pi k^4|A_1A_2|\{\tilde{\Gamma}_\eta[k(\mathbf{u}_1 - \mathbf{u}_2)]\exp(-i\theta_1) + \tilde{\Gamma}_\eta[k(\mathbf{u}_2 - \mathbf{u}_1)]\exp(i\theta_1)\}, \tag{4.17a}$$

and

$$P_{\mathrm{int}}(\theta_2) = 4\pi k^4|A_1A_2|\{\tilde{\Gamma}_\eta[k(\mathbf{u}_1 - \mathbf{u}_2)]\exp(-i\theta_2) + \tilde{\Gamma}_\eta[k(\mathbf{u}_2 - \mathbf{u}_1)]\exp(i\theta_2)\}. \tag{4.17b}$$

We may readily solve these equations for each of the two Fourier components of Γ_η and find that

$$\tilde{\Gamma}_\eta[k(\mathbf{u}_1 - \mathbf{u}_2)] = \frac{-i}{8\pi k^4|A_1A_2|\sin(\theta_2 - \theta_1)} \times [P_{\mathrm{int}}(\theta_1)\exp(i\theta_2) - P_{\mathrm{int}}(\theta_2)\exp(i\theta_1)]. \tag{4.18}$$

In particular, with the choice $\theta_1 = 0$ and $\theta_2 = \pi/2$,

$$\tilde{\Gamma}_\eta[k(\mathbf{u}_1 - \mathbf{u}_2)] = \frac{1}{8\pi k^4|A_1A_2|}[P_{\mathrm{int}}(0) + iP_{\mathrm{int}}(\pi/2)]. \tag{4.19}$$

The formula (4.8), together with the expressions (4.17) has an important physical implication. It shows that from measurements of the total extinction power generated on scattering of both of the incident plane waves and from measurements of the extinction powers $\langle P_{11}^{(e)}\rangle$ and $\langle P_{22}^{(e)}\rangle$ generated by separately scattering each of the two plane waves, one can calculate the interference term (4.16). If the measurements are performed with two different values of the phase difference θ between the two incident plane waves, the interference term provides, according to Eq. (4.18), information about some of the spatial Fourier components $\bar{\Gamma}_\eta(\mathbf{K})$ of the function $\Gamma_\eta(\mathbf{r})$ that characterizes the strength of the spatial fluctuations of the medium. If the direction of incidence of one of the plane waves, say, $\mathbf{u}_1$, is fixed, and measurements are made for different directions of incidence $\mathbf{u}_2$ of the second wave, one can determine Fourier components labeled by the vector $\mathbf{K}$ whose end points lie on a sphere σ of radius $k = \omega/c$ [see Fig. 3(a)]. If further the direction $\mathbf{u}_1$ is also varied, one can determine Fourier components labeled by a vector $\mathbf{K}$ whose end points fill a sphere Σ of radius $2k = 2\omega/c$, centered at the origin in the Fourier space [Fig. 3(b)]. These two spheres are strictly analogous to the Ewald sphere of reflection and the Ewald limiting sphere, well known in the theory of x-ray diffraction by crystals.[13]

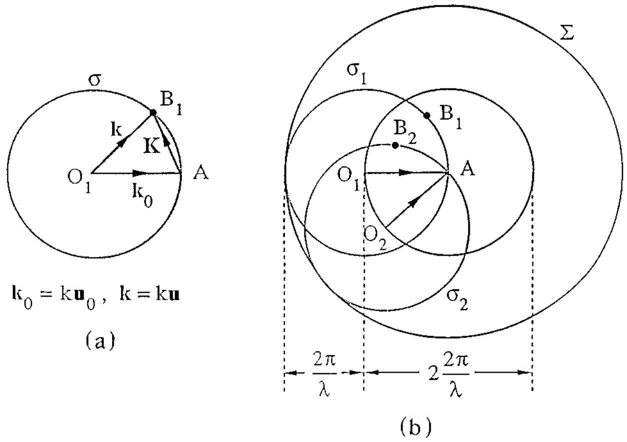

Fig. 3. Illustrating the accessible Fourier components $\tilde{\Gamma}_\eta(\mathbf{K})$ of the function $\Gamma_\eta(\mathbf{r})$ that characterizes the spatial fluctuations of the random medium, from measurements of the power extinguished on scattering of two monochromatic plane waves. (a) Ewald's sphere of reflection (σ), (b) Ewald's limiting sphere, generated by Ewald's spheres of reflection σ_1, σ_2,... for different directions of incidence, represented by vectors O_1A, O_2A,... .

Thus our results suggest a new method for determining the low-pass filtered version,

$$[\Gamma_\eta(r)]_{\mathrm{LP}} = \frac{1}{(2\pi)^3} \int_{|\mathbf{K}|<2k} \tilde{\Gamma}_\eta(\mathbf{K}) \exp(i\mathbf{K}\cdot\mathbf{r}) \mathrm{d}^3K, \tag{4.20}$$

of the function $\Gamma_\eta(\mathbf{r})$, which characterizes the average magnitude of the spatial fluctuations of the scatterer.

In this section we have considered reconstruction of only the simplest type of random media, namely, those whose spatial fluctuations are δ correlated. It is obviously desirable to investigate whether the method can be extended to a broader class of scatterers.

5. CONCLUDING REMARKS

In this paper we have presented several generalizations of the optical cross-section theorem. The theorem is usually formulated for situations where the incident field is a monochromatic plane wave and the scatterer is deterministic. We have obtained a generalization of the theorem which applies to scattering on random media. We have also obtained a generalization to situations where the incident field is stochastic. We further showed that these generalizations may be used to determine the strength of the fluctuations of the dielectric susceptibility of δ-correlated random scatterers. This possibility may be relevant to diffraction tomography with random media, which has been receiving some attention in recent years.[14–17]

ACKNOWLEDGMENT

This research was supported by the U.S. Department of Energy under grant DE-FG02-90ER 14119.

REFERENCES AND NOTES

1. Although this term is frequently used by quantum physicists, it is hardly known to workers in classical optics. We will use instead the somewhat more descriptive term optical cross-section theorem.
2. R. G. Newton, "Optical theorem and beyond," Am. J. Phys. **44**, 639–642 (1976). In the third paragraph on page 641 of this paper, there is an incorrect statement implying that the phase of the scattering amplitude can be determined from intensity correlation experiments pioneered by Hanbury Brown and Twiss.
3. E. Feenberg, "The scattering of slow electrons by neutral atoms," Phys. Rev. **40**, 40–54 (1932).
4. H. C. van der Hulst, "On the attenuation of plane waves by obstacles of arbitrary size and form," Physica (Utrecht) **15**, 740–746 (1949).
5. D. S. Jones, "On the scattering cross section of an obstacle," Phil. Mag. **46**, 957–962 (1955).
6. M. Born and E. Wolf, *Principles of Optics*, 6th ed. (Cambridge U. Press, Cambridge, 1997), p. 659, Eq. (110).
7. To our knowledge, only one paper, by M. Nieto-Vesperinas, G. Ross, and M. A. Fiddy, "The optical theorems: a new interpretation for partially coherent light," Optik **55**, 165–171 (1980), has been published that attempts to generalize the theorem to partially coherent incident light. We are not aware of any publication that deals with its generalization to scattering on random media.
8. See, for example, P. Roman, *Advanced Quantum Theory* (Addision-Wesley, Reading, Mass., 1965), Sec. 3-2.
9. This point seems to have been overlooked by M. C. Li in his paper "Scattering initiated by two coherent beams" [Phys. Rev. A **9**, 1635–1643 (1974)], in which the possibility of determining the imaginary part of the scattering amplitude from scattering of two monochromatic plane waves is discussed.
10. By a free field we mean a field that can be represented as a linear superposition of homogeneous plane-wave modes only. The properties of free fields appear to have been first systematically studied by G. C. Sherman, "Diffracted wave fields expressible by plane-wave expansions containing only homogeneous waves," Phys. Rev. Lett. **21**, 761–764 (1968) and in J. Opt. Soc. Am. **59**, 697–711 (1969). See also Manuel Nieto-Vesperinas, *Scattering and Diffraction in Physical Optics* (Wiley, New York, 1991), Sec. 2.7. General fields that propagate into the half-space $z > 0$ include also inhomogeneous (evanescent) waves, whose amplitudes decay exponentially with the distance of propagation. Free fields are usually excellent approximations to actual fields encountered in practice, except in the immediate neighborhood of scattering bodies. An example is given in Ref. 11, Sec. 3.2.3.
11. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge U. Press, 1995), Secs. 3.2.2 and 3.2.3.
12. This theory applies to random fields that are statistically stationary, such as considered here. For fields of this kind the Fourier representation cannot be used, because the stationary fields do not die out with increasing time. For a discussion of coherence theory in the space-frequency domain, see Sec. 4.7 of Ref. 11.
13. R. W. James, *The Optical Principles of Diffraction of X-rays* (Bell, London, 1948), p. 14.
14. D. Rouseff and R. P. Porter, "Diffraction tomography with random media," J. Acoust. Soc. Am. **89**, 1599–1609 (1991).
15. J. Howard, "Laser probing of weakly scattering media," J. Opt. Soc. Am. A **8**, 1955–1963 (1991).
16. V. E. Kunitstyn and E. D. Tereshchenko, "Radio tomography of the ionosphere," IEEE Antennas Propag. Mag. **34**, 22–32 (1992).
17. D. G. Fischer and E. Wolf, "Theory of diffraction tomography for quasi-homogeneous random objects," Opt. Commun. **133**, 17–21 (1997).