Coherence theory of laser resonator modes

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An analysis of transverse laser resonator modes is presented, based on a recently developed coherence theory in the space-frequency domain. The modes are introduced by means of solutions of an integral equation that expresses a steady-state condition for a second-order correlation function of the field across a mirror of the laser cavity. All solutions of this integral equation are found to be expressible as quadratic forms involving the Fox–Li modes of the conventional theory. If there is no degeneracy, each mode is shown to be necessarily completely spatially coherent, at each frequency, within the framework of second-order correlation theory. It is also shown that, if several transverse modes are excited, the output cannot be completely spatially coherent.

1. INTRODUCTION

Since the publication of a fundamental paper by Fox and Li,[1] studies of laser resonator modes have generally been based on an integral equation that expresses a steady-state condition of the optical field at a mirror of an open resonator. Because these studies idealize the situation by treating the light as being strictly monochromatic, they cannot yield any information about the coherence properties of the output.

An early attempt to elucidate coherence properties of laser resonator modes was made by Wolf.[2] He considered the "transmission line analogue" of a laser resonator that is due to Fox and Li, and he showed that quasi-monochromatic light that is initially partially spatially coherent, or even incoherent, may become completely spatially coherent after a sufficient number of transits. This result implies that spatial coherence of the light in the laser resonator may be generated in the process of propagation and by diffraction at the mirrors of the resonant cavity. Various extensions and generalizations of this analysis were later made by Streifer,[3] Allen *et al.*,[4] and by Gori.[5]

In this paper we present a new theory of laser resonator modes based on an integral equation that expresses a steady-state condition for the cross-spectral density of the field of any spectral composition. It is shown that all the solutions of this equation are expressible as quadratic forms involving the Fox–Li modes of the conventional theory. It is found that if there is no degeneracy, each mode is, within the framework of second-order correlation theory, necessarily completely spatially coherent at each temporal frequency. It is also shown that if several (transverse) modes are excited, the output cannot be spatially completely coherent. These results are in agreement with results of experiments carried out by Bertolotti *et al.* many years ago.[6]

2. COHERENT-MODE REPRESENTATION OF FIELDS OF ANY STATE OF COHERENCE

In the analysis of the main problem that we discuss in this paper we make use of the mode representation of fields of any state of coherence developed in several recent papers. It will therefore be useful to begin by summarizing the main results pertaining to this representation.

Let $V(\mathbf{r}, t)$ be the complex analytic signal representation[7] of a stationary field in some finite closed domain D of free space. Here $\mathbf{r}$ denotes the position vector of a typical point in D and t denotes the time. Let

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \langle V^*(\mathbf{r}_1, t) V(\mathbf{r}_2, t+\tau)\rangle \tag{2.1}$$

(with the asterisk denoting the complex conjugate) be the mutual coherence function of the field and let

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) e^{i\omega\tau} \mathrm{d}\tau \tag{2.2}$$

be the cross-spectral density.

It was shown in Ref. 8 that, for a large class of stationary fields, the cross-spectral density is a continuous, Hermitian, nonnegative definite, Hilbert–Schmidt kernel and hence by a three-dimensional generalization of a well-known theorem it admits of a Mercer expansion[11]

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \sum_n \lambda_n(\omega)\psi_n^*(\mathbf{r}_1, \omega)\psi_n(\mathbf{r}_2, \omega). \tag{2.3}$$

Here the ψ_n are the eigenfunctions and the λ_n are the eigenvalues of the Fredholm integral equation

$$\int_D W(\mathbf{r}_1, \mathbf{r}_2, \omega)\psi_n(\mathbf{r}_1, \omega)\mathrm{d}^3 r_1 = \lambda_n(\omega)\psi_n(\mathbf{r}_2, \omega). \tag{2.4}$$

The eigenfunctions are orthonormalized over the domain D, i.e.,

$$\int_D \psi_n^*(\mathbf{r}, \omega)\psi_m(\mathbf{r}, \omega)\mathrm{d}^3 r = \delta_{nm}, \tag{2.5}$$

where δ_{nm} is the Kronecker symbol. It is known that the integral Eq. (2.4) has at least one nonvanishing eigenvalue and that the expansion (2.3) is absolutely and uniformly convergent.

In the present context the Mercer expansion (2.3) has an interesting physical meaning. It expresses the cross-spectral

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density $W(\mathbf{r}_1, \mathbf{r}_2, \omega)$ of the field as a linear combination of cross-spectral densities

$$W^{(n)}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \psi_n^*(\mathbf{r}_1, \omega)\psi_n(\mathbf{r}_2, \omega) \tag{2.6}$$

that factorize with respect to the two spatial variables. Such factorized cross-spectral densities represent fields that are completely spatially coherent, because their degree of spectral coherence at frequency ω, viz.,[12]

$$\mu^{(n)}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \frac{W^{(n)}(\mathbf{r}_1,\mathbf{r}_2,\omega)}{[W^{(n)}(\mathbf{r}_1, \mathbf{r}_1, \omega)]^{1/2}[W^{(n)}(\mathbf{r}_2, \mathbf{r}_2, \omega)]^{1/2}}, \tag{2.7}$$

is unimodular:

$$|\mu^{(n)}(\mathbf{r}_1, \mathbf{r}_2, \omega)| = 1. \tag{2.8}$$

Moreover, it follows at once from results of Sec. 3 of Ref. 9 that, throughout D, $W^{(n)}(\mathbf{r}_1, \mathbf{r}_2, \omega)$ satisfies the Helmholtz equations

$$\nabla_j{}^2 W^{(n)}(\mathbf{r}_1, \mathbf{r}_2, \omega) + k^2 W^{(n)}(\mathbf{r}_1, \mathbf{r}_2, \omega) = 0 \qquad (j = 1, 2), \tag{2.9}$$

where $\nabla_j{}^2$ is the Laplacian operator acting with respect to the variable $\mathbf{r}_j$ and

$$k = \omega/c, \tag{2.10}$$

c being the speed of light in vacuum. Because $W^{(n)}$ satisfies the same differential equations as W (cf. Ref. 13), it is clear that the Mercer expansion (2.3) represents the cross-spatial density of the field as a linear superposition of modes that are completely coherent in the space-frequency domain.

It was shown also in Ref. 10 that one may construct an ensemble of monochromatic wave functions $\{U(\mathbf{r}, \omega)\cdot \exp(-i\omega t)\}$, all of the same frequency ω, such that the cross-spatial density of the field is their cross-correlation function, i.e., that

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle U^*(\mathbf{r}_1, \omega)U(\mathbf{r}_2, \omega)\rangle_\omega, \tag{2.11}$$

where the angular brackets with the suffix ω denote that the average is taken over this ensemble. The $U(\mathbf{r}, \omega)$ are expressible in the form

$$U(\mathbf{r}, \omega) = \sum a_n(\omega)\psi_n(\mathbf{r}, \omega), \tag{2.12}$$

where the ψ_n are again the eigenfunctions of the integral Eq. (2.4) and the a_n are random coefficients satisfying the requirement that

$$\langle a^*_n(\omega)a_m(\omega)\rangle_\omega = \lambda_n(\omega)\delta_{nm}. \tag{2.13}$$

Moreover, the random functions $U(\mathbf{r}, \omega)$ satisfy the Helmholtz equation

$$\nabla^2 U(\mathbf{r}, \omega) + k^2 U(\mathbf{r}, \omega) = 0 \tag{2.14}$$

throughout D.

Finally we note that, if we substitute for W from Eq. (2.3) into the Fourier inverse of Eq. (2.2), we obtain the following representation of the mutual coherence function of the field:

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \sum_n \Gamma^{(n)}(\mathbf{r}_1, \mathbf{r}_2, \tau), \tag{2.15}$$

where

$$\Gamma^{(n)}(\mathbf{r}_1, \mathbf{r}_2, \tau) = \int_0^\infty \lambda_n(\omega)\psi_n^*(\mathbf{r}_1, \omega)\psi_n(\mathbf{r}_2, \omega)e^{-i\omega\tau}\mathrm{d}\omega. \tag{2.16}$$

The function $\Gamma^{(n)}(\mathbf{r}_1, \mathbf{r}_2, \tau)$ is the mutual coherence function associated with the mode labeled by the index n.

It should be noted that, unlike the degree of spectral coherence $\mu^{(n)}(\mathbf{r}_1, \mathbf{r}_2, \omega)$, the complex degree of coherence

$$\gamma^{(n)}(\mathbf{r}_1, \mathbf{r}_2, \tau) = \frac{\Gamma^{(n)}(\mathbf{r}_1,\mathbf{r}_2,\tau)}{[\Gamma^{(n)}(\mathbf{r}_1, \mathbf{r}_1, 0)]^{1/2}[\Gamma^{(n)}(\mathbf{r}_2, \mathbf{r}_2, 0]^{1/2}} \tag{2.17}$$

of a mode in the space-time domain is, in general, not unimodular. Hence $\Gamma^{(n)}(\mathbf{r}_1, \mathbf{r}_2, \tau)$ is a mutual coherence function of a field that is not completely coherent in the space-time domain. The distinction between complete coherence in the two domains is rather basic in the context of our subsequent analysis.

3. STEADY-STATE CONDITION FOR SECOND-ORDER COHERENCE OF LIGHT AT A MIRROR OF A LASER RESONATOR

Consider a light distribution across a mirror A. We assume that the distribution is characterized by a statistical ensemble that is stationary, at least in the wide sense. However, we do not impose any restriction on the bandwidth of the light. We denote by $W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ the cross-spectral density of the light at two points on A, specified by position vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$. Suppose that some of the light that propagates from A reaches another mirror B, with the two mirrors forming an open, empty, resonant cavity. Let $W_1(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ be the cross-spectral density of the light returning to A. After reflection at A the process of propagation and reflection continues. We denote by $W_j(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ the cross-spectral density of the light across the mirror A after completion of j complete cycles $A \to B \to A$.

According to the two-dimensional version of Eq. (2.11), the cross-spectral density W_j may be expressed in the form

$$W_j(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \langle U_j^*(\boldsymbol{\rho}_1, \omega)U_j(\boldsymbol{\rho}_2, \omega)\rangle_\omega, \tag{3.1}$$

where the average is taken over an appropriate statistical ensemble of random functions U_j. Since each $U_j(\boldsymbol{\rho}, \omega)$ is a boundary value of a field that in the space between the two mirrors satisfies the Helmholtz equation (2.14), U_{j+1} and U_j will be related by a linear transformation, viz.,

$$U_{j+1}(\boldsymbol{\rho}, \omega) = \int_A L(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)U_j(\boldsymbol{\rho}', \omega)\mathrm{d}^2\rho' \tag{3.2}$$

($j = 0, 1, 2, \ldots$). Here $L(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)$ is the propagator (independent of j) for propagation of monochromatic light of frequency ω from the point $\boldsymbol{\rho}'$ on the mirror A to the point $\boldsymbol{\rho}$ on the same mirror, after a single reflection at the mirror B. Explicit approximate expressions for this propagator may be obtained with the help of the Huygens–Fresnel principle.[14] On substituting from Eq. (3.2) into Eq. (3.1), we obtain the following relation between the cross-spectral densities W_j and W_{j+1}:

$$W_{j+1}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \int_A \int_A L^*(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1', \omega) L(\boldsymbol{\rho}_2, \boldsymbol{\rho}_2', \omega) \times W_j(\boldsymbol{\rho}_1', \boldsymbol{\rho}_2', \omega) \mathrm{d}^2\rho_1' \mathrm{d}^2\rho_2'. \quad (3.3)$$

It seems reasonable to assume that, after a sufficient number of transits between the two mirrors, a steady state is reached in the sense that, for each frequency ω, $W_{j+1}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ will be equal to $W_j(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ up to a proportionality factor, $\sigma(\omega)$, say. The factor $\sigma(\omega)$ represents the loss from diffraction and reflection that occurs in one complete transit. More explicitly, we characterize steady state by the requirement that, for sufficiently large values of j,

$$W_{j+1}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \sigma(\omega) W_j(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega). \quad (3.4)$$

Since the "diagonal values" $W_{j+1}(\boldsymbol{\rho}, \boldsymbol{\rho}, \omega)$ and $W_j(\boldsymbol{\rho}, \boldsymbol{\rho}, \omega)$ represent spectral densities, they are necessarily real and positive, and hence

$$\sigma(\omega) > 0. \quad (3.5)$$

On substituting for W_{j+1} from Eq. (3.4) into Eq. (3.3) and on suppressing the suffix j, we obtain the equation

$$\int_A \int_A W(\boldsymbol{\rho}_1', \boldsymbol{\rho}_2', \omega) L^*(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1', \omega) L(\boldsymbol{\rho}_2, \boldsymbol{\rho}_2', \omega) \mathrm{d}^2\rho_1' \mathrm{d}^2\rho_2' = \sigma(\omega) W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega). \quad (3.6)$$

The integral Eq. (3.6) is the basic equation of the present theory. *We identify its solutions $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ as the boundary values of the cross-spectral densities on the mirror A of the modes that the cavity can sustain.* It should be noted that, unlike the solutions of the usual integral equation for modes of a laser cavity, the solutions of our integral Eq. (3.6) contain information about their second-order coherence properties.

4. NATURE OF THE SOLUTIONS OF THE INTEGRAL EQUATION (3.6)

As is well known, the propagator $L(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1', \omega)$ that enters our integral Eq. (3.6) is not Hermitian, a fact that already presents a number of difficulties in the conventional (monochromatic) theory of laser modes. In particular, the kernel L cannot be represented in the form of the orthogonal Mercer expansion that is so useful in connection with Fredholm integral equations with Hilbert–Schmidt kernels. There is, however, a biorthogonal generalization of the Mercer expansion, which applies to a class of non-Hermitian kernels and which, to our knowledge, has not been employed previously in the theory of laser modes. We will see that the use of this biorthogonal expansion elucidates the nature of the solutions of our integral Eq. (3.6) and of their relationship to the Fox–Li modes of the conventional theory.

The biorthogonal expansion of a non-Hermitian kernel $L(\boldsymbol{\rho}, \boldsymbol{\rho}', \omega)$ may be introduced in the following manner.[15] Let $\{\alpha_n(\omega)\}$ and $\{\phi_n(\boldsymbol{\rho}, \omega)\}$ be the eigenvalues and the eigenfunctions, respectively, of the Fredholm integral equation

$$\int_A L(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) \phi_n(\boldsymbol{\rho}_2, \omega) \mathrm{d}^2\rho_2 = \alpha_n(\omega) \phi_n(\boldsymbol{\rho}_1, \omega), \quad (4.1)$$

and let $\{\beta_n(\omega)\}$ and $\{\chi_n(\boldsymbol{\rho}, \omega)\}$ be the eigenvalues and the eigenfunctions, respectively, of the corresponding integral equation with the adjoint kernel $L^*(\boldsymbol{\rho}_2, \boldsymbol{\rho}_1, \omega)$:

$$\int_A L^*(\boldsymbol{\rho}_2, \boldsymbol{\rho}_1, \omega) \chi_n(\boldsymbol{\rho}_2, \omega) \mathrm{d}^2\rho_2 = \beta_n(\omega) \chi_n(\boldsymbol{\rho}_1, \omega). \quad (4.2)$$

The kernel $L(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ for laser resonators is defined on a finite domain A and is, for each ω, a continuous function in both the spatial variables $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$. Such kernels belong to a class of square-integrable kernels for which the following theorems are known to hold[16]:

(1) To each eigenvalue α_n of Eq. (4.1), there corresponds an eigenvalue β_n of Eq. (4.2) and

$$\beta_n = \alpha_n^*. \quad (4.3)$$

Moreover, the ranks (degrees of degeneracy) of α_n and of β_n are the same.

(2) The corresponding eigenfunctions of the two equations are (with suitable normalization) orthonormal over the domain A, i.e.,

$$\int_A \phi_n^*(\boldsymbol{\rho}, \omega) \chi_m(\boldsymbol{\rho}, \omega) \mathrm{d}^2\rho = \delta_{nm}. \quad (4.4)$$

The biorthogonal expansion of the kernel $L(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ that we employ to study the nature of the solution of the integral Eq. (3.6) is the expansion[15]

$$L(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \sum_n \alpha_n(\omega) \phi_n(\boldsymbol{\rho}_1, \omega) \chi_n^*(\boldsymbol{\rho}_2, \omega). \quad (4.5)$$

This expansion should not be confused with the Schmidt expansion (which is biorthogonal in a different sense) of non-Hermitian kernels that was used previously in the theory of laser resonator modes.[17–19]

If we substitute from Eq. (4.5) into Eq. (3.6) and interchange the orders of integrations and summations, we obtain the relation

$$\sum_n \sum_m \alpha_n^*(\omega) \alpha_m(\omega) w_{nm}(\omega) \phi_n^*(\boldsymbol{\rho}_1, \omega) \phi_m(\boldsymbol{\rho}_2, \omega) = \sigma(\omega) W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega), \quad (4.6)$$

where

$$w_{nm} = \int_A \int_A W(\boldsymbol{\rho}_1', \boldsymbol{\rho}_2', \omega) \chi_n(\boldsymbol{\rho}_1', \omega) \times \chi_m^*(\boldsymbol{\rho}_2', \omega) \mathrm{d}^2\rho_1' \mathrm{d}^2\rho_2'. \quad (4.7)$$

Next let us multiply both sides of Eq. (4.6) by $\chi_N(\boldsymbol{\rho}_1, \omega) \chi_M^*(\boldsymbol{\rho}_2, \omega)$, integrate with respect to $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$, and make use of biorthogonality relations (4.4) and of expression (4.7) that defines the w_{nm}. We then obtain the relation

$$\sum_n \sum_m \alpha_n^*(\omega) \alpha_m(\omega) w_{nm}(\omega) \delta_{nN} \delta_{mM} = \sigma(\omega) w_{NM}(\omega), \quad (4.8)$$

which implies that

$$[\sigma(\omega) - \alpha_N^*(\omega) \alpha_M(\omega)] w_{NM} = 0 \quad \text{(no summation).} \quad (4.9)$$

We see from Eq. (4.9) that either $w_{NM}(\omega) = 0$ or $\sigma(\omega) = \alpha_N^*(\omega) \alpha_M(\omega)$. The first case ($w_{NM} = 0$) is of no interest since the corresponding term (with $n = N$, $m = M$) does not contribute to the double sum on the left-hand side of Eq. (4.6).

The other case implies that the eigenvalues of the integral Eq. (3.6) are

$$\sigma_{NM}(\omega) = \alpha_N^*(\omega)\alpha_M(\omega). \tag{4.10}$$

Let us assume, for the moment, that σ_{NM} is nondegenerate in the sense that there are no other pairs $\alpha_{N'}(\omega)$, $\alpha_{M'}(\omega)$ of eigenvalues of integral Eq. (4.1) for which

$$\alpha_N^*(\omega)\alpha_M(\omega) = \alpha_{N'}^*(\omega)\alpha_{M'}(\omega). \tag{4.11}$$

Formula (4.9) implies that with a particular choice

$$\sigma_{kl}(\omega) = \alpha_k^*(\omega)\alpha_l(\omega) \tag{4.12}$$

of a nondegenerate eigenvalue of our integral Eq. (3.6), $w_{NM} = 0$ unless $k = N$, $l = M$, and the expansion (4.6) then reduces the single term

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = w_{kl}(\omega)\phi_k^*(\boldsymbol{\rho}_1)\phi_l(\boldsymbol{\rho}_2). \tag{4.13}$$

Now the cross-spectral density function $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ is necessarily Hermitian, i.e.,

$$W(\boldsymbol{\rho}_2, \boldsymbol{\rho}_1, \omega) = W^*(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega), \tag{4.14}$$

as is readily deduced from Eqs. (2.1) and (2.2). Applied to expression (4.13), this condition implies that

$$w_{kl}\phi_k^*(\boldsymbol{\rho}_2, \omega)\phi_l(\boldsymbol{\rho}_1, \omega) = w_{kl}^*\phi_k(\boldsymbol{\rho}_1, \omega)\phi_l^*(\boldsymbol{\rho}_2, \omega), \tag{4.15}$$

i.e.,

$$\frac{\phi_l(\boldsymbol{\rho}_1,\omega)}{\phi_k(\boldsymbol{\rho}_1, \omega)} = \frac{w_{kl}^*}{w_{kl}}\frac{\phi_l^*(\boldsymbol{\rho}_2,\omega)}{\phi_k^*(\boldsymbol{\rho}_2, \omega)}\,. \tag{4.16}$$

With respect to the dependence on the two spatial variables, the left-hand side of Eq. (4.16) is a function of $\boldsymbol{\rho}_1$ only, whereas the right-hand side is a function of $\boldsymbol{\rho}_2$ only. This is possible only if each side is independent of the spatial variables. If we denote each side by $\gamma_{kl}(\omega)$, it then follows that

$$\phi_l(\boldsymbol{\rho}, \omega) = \gamma_{kl}(\omega)\phi_k(\boldsymbol{\rho}, \omega). \tag{4.17}$$

On substituting from Eq. (4.17) into Eq. (4.13), we obtain at once the following expression for W:

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = w_{kl}(\omega)\gamma_{kl}(\omega)\phi_k^*(\boldsymbol{\rho}_1, \omega)\phi_k(\boldsymbol{\rho}_2, \omega). \tag{4.18}$$

If we substitute from Eq. (4.18) into Eq. (4.7), we obtain the formula

$$w_{nm} = w_{kl}\gamma_{kl}\int_A \phi_k^*(\boldsymbol{\rho}_1', \omega)\chi_n(\boldsymbol{\rho}_1', \omega)\mathrm{d}^2\rho_1' \times \int_A \phi_k(\boldsymbol{\rho}_2', \omega)\chi_m^*(\boldsymbol{\rho}_2'\omega)\mathrm{d}^2\rho_2'. \tag{4.19}$$

If we make again use of the biorthomality relations (4.4), Eq. (4.19) reduces to the formula

$$w_{nm}(\omega) = w_{kl}(\omega)\gamma_{kl}\delta_{kn}\delta_{km}, \tag{4.20}$$

which implies that

$$w_{nm}(\omega) = 0 \quad \text{unless} \quad n = m = k \tag{4.21}$$

and

$$w_{kk}(\omega) = w_{kl}(\omega)\gamma_{kl}(\omega). \tag{4.22}$$

On substituting from Eq. (4.22) into Eq. (4.18), we see that the admissable solutions [which we now denote by[20] $W_k(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$] of our integral Eq. (3.6) are given by

$$W_k(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = w_{kk}(\omega)\phi_k^*(\boldsymbol{\rho}_1, \omega)\phi_k(\boldsymbol{\rho}_2, \omega). \tag{4.23}$$

It is clear from Eq. (4.12) that the corresponding eigenvalues [which we denote by $\sigma_k(\omega)$] are

$$\sigma_k(\omega) = \alpha_k^*(\omega)\alpha_k(\omega). \tag{4.24}$$

The factor $w_{kk}(\omega)$ in Eq. (4.23) depends on normalization. Let us normalize the ϕ_k so that

$$\int_A |\phi_k(\boldsymbol{\rho}, \omega)|^2\mathrm{d}^2\rho = 1. \tag{4.25}$$

Then the right-hand side of Eq. (4.23) can be identified with the two-dimensional version of the Mercer expansion (2.3) of W_k, which now consists of a single term; and one evidently has

$$\phi_k(\boldsymbol{\rho}, \omega) \equiv \psi_k(\boldsymbol{\rho}, \omega), \tag{4.26}$$

$$w_{kk}(\omega) = \lambda_k(\omega). \tag{4.27}$$

Thus each solution of our integral equation (3.6) is also a mode in the sense of the general theory of coherent-mode representation of fields of any state of coherence, discussed in Section 2. We note that Eqs. (4.23), (4.25) and (4.27) imply that

$$\int_A W_k(\boldsymbol{\rho}, \boldsymbol{\rho}, \omega)\mathrm{d}^2\rho = \lambda_k(\omega). \tag{4.28}$$

Since $W_k(\boldsymbol{\rho}, \boldsymbol{\rho}, \omega)$ represents the spectral density at frequency ω at the point $\boldsymbol{\rho}$, Eq. (4.28) implies, roughly speaking, that $\lambda_k(\omega)$ is a measure of the rate at which energy at frequency ω is propagated, in steady state, from the mirror A into the cavity.

The integral equation (4.1) for the functions $\phi_n(\boldsymbol{\rho}, \omega)$ is precisely the integral equation of the elementary (monochromatic) theory of laser resonators modes formulated by Fox and Li.[1,14] We therefore refer to the functions of $\phi_n(\boldsymbol{\rho}, \omega)$ as the Fox–Li modes. The preceding analysis shows that the Fox–Li modes have a broader significance than would appear from the manner in which they were originally introduced.

Since each solution

$$W_k(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \lambda_k(\omega)\phi_k^*(\boldsymbol{\rho}_1, \omega)\phi_k(\boldsymbol{\rho}_2, \omega) \tag{4.29}$$

of our integral Eq. (3.6) factorizes with respect to the spatial variables $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$, its degree of spectral coherence at frequency ω is unimodular [cf. Eqs. (2.6)–(2.8)]. Hence the solution (4.29) is the cross-spectral density of a field distribution that is spatially completely coherent at frequency ω over the surface of the mirror A. Stated somewhat differently, *if there is no degeneracy, our integral Eq. (3.6) only admits of solutions that represent light that, at each frequency, is spatially completely coherent within the framework of second-order correlation theory.*

The factorization of the cross-spectral density into a product of a function of $\boldsymbol{\rho}_1$ and a function of $\boldsymbol{\rho}_2$ is known to be both a necessary and a sufficient condition for complete (second-order) spatial coherence.[21] Hence, if the laser operates on more than one transverse mode, the output cannot then be spatially fully coherent across the mirror surface. These conclusions are in agreement with results of experiments.[6]

We have assumed so far that there is no degeneracy in the sense that there are no four eigenvalues of Eq. (4.1) that obey the relation (4.11). If there is a degeneracy, we would find in place of Eq. (4.13) an expression of the form

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \sum_k \sum_l c_{kl}(\omega)\phi_k^*(\boldsymbol{\rho}_1, \omega)\phi_l(\boldsymbol{\rho}_2, \omega), \quad (4.30)$$

where the c_{kl} are arbitrary coefficients. These coefficients must, however, satisfy the constraint $c_{lk}(\omega) = c_{kl}^*(\omega)$, because the cross-spectral density is necessarily Hermitian [Eq. (4.14)]. If we diagonalize the matrix $[c_{kl}(\omega)]$, $c = U^\dagger \Lambda u$, and normalize the $\phi_k(\boldsymbol{\rho}, \omega)$ in accordance with Eq. (4.25), the degenerate solution (4.30) takes the form of its Mercer expansion

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \sum_k \Lambda_k(\omega) f_k^*(\boldsymbol{\rho}_1, \omega) f_k(\boldsymbol{\rho}_2, \omega), \quad (4.31)$$

where

$$f_k(\boldsymbol{\rho}_1, \omega) = \sum_l u_{kl}(\omega)\phi_l(\boldsymbol{\rho}, \omega). \quad (4.32)$$

In such a case, the field across the mirror A is no longer spatially completely coherent.

In our steady-state condition (3.6), the frequency ω is arbitrary. It seems reasonable to assume that, when a laser operates in a steady state, the condition (3.6) will hold for every frequency component that is present in the spectrum of the laser light. It then follows on substituting from Eq. (4.29) into the Fourier inverse of Eq. (2.2) (where $\mathbf{r}_1$ and $\mathbf{r}_2$ are, of course, now replaced by $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$, respectively) that the mutual-coherence function of a laser resonator mode is given by

$$\Gamma_k(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \tau) = \int_0^\infty \lambda_k(\omega)\phi_k^*(\boldsymbol{\rho}_1, \omega)\phi_k(\boldsymbol{\rho}_2, \omega)e^{-i\omega\tau}d\omega, \quad (4.33)$$

if there is no degeneracy. If there is a degeneracy, one must use expression (4.31) in place of Eqs. (4.23) and (4.27), and one then obtains for the mutual coherence function of a degenerate mode the formula

$$\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \tau) = \sum_k \int_0^\infty \Lambda_k(\omega) f^*_k(\boldsymbol{\rho}_1, \omega) f_k(\boldsymbol{\rho}_2, \omega)e^{-i\omega\tau}d\omega. \quad (4.34)$$

The spectrum of a laser mode at a point $\boldsymbol{\rho}$ of the mirror A is equal to

$$S_k(\boldsymbol{\rho}, \omega) \equiv W_k(\boldsymbol{\rho}, \boldsymbol{\rho}, \omega) = \lambda_k(\omega)|\phi_k(\boldsymbol{\rho}, \omega)|^2 \quad (4.35)$$

if there is no degeneracy and by

$$S(\boldsymbol{\rho}, \omega) = \sum_k \lambda_k(\omega)|f_k(\boldsymbol{\rho}, \omega)|^2 \quad (4.36)$$

if there is a degeneracy.

5. SUMMARY

In this paper we have introduced transverse modes of an empty open laser cavity as solutions of an integral equation [Eq. (3.6)] that expresses a steady-state condition for the cross-spectral density of the light over a mirror of the cavity. Unlike the usual integral equation for laser resonator modes, the present theory takes into account the frequency spectrum of laser light. We have found that the mutual coherence function of each mode is expressible as a Fourier transform of an expression that involves, in a simple way, the Fox–Li modes of the usual theory [Eqs. (4.33) and (4.34)]. We have also found that if there is no degeneracy, each mode is, within the framework of second-order correlation theory, completely spatially coherent over the mirror surface, at each frequency, and that if several modes are excited, the light is then necessarily partially coherent. We showed further that laser resonator modes are also modes in the sense of a general mode representation of fields of any state of coherence.[8–10]

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