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STATIONARY MEASURES FOR SOME MARKOV CHAIN MODELS IN ECOLOGY AND ECONOMICS

by

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Abstract

Let $F \equiv \{f : f : [0,\infty) \to [0,\infty), f(0) = 0, f \text{ continuous}, \lim_{x \downarrow 0} \frac{f(x)}{x} = C$ exists in $(0,\infty), 0 < g(x) \equiv \frac{f(x)}{Cx} < 1$ for x in $(0,\infty)\}$. Let $\{f_j\}_{j\geq 1}$ be an i.i.d. sequence from F and X_0 be a nonnegative random variable independent of $\{f_j\}_{j\geq 1}$. Let $\{X_n\}_{n\geq 0}$ be the Markov chain generated by the iteration of random maps $\{f_j\}_{j\geq 1}$ by $X_{n+1} = f_{n+1}(X_n)$, $n \geq 0$. Such Markov chains arise in population ecology and growth models in economics. This paper studies the existence of nondegenerate stationary measures for $\{X_n\}$. A set of necessary conditions and two sets of sufficient conditions are provided. There are some convergence results also. The present paper is a generalization of the work on random logistics maps by Athreya and Dai [2].

Keywords: Population models, random maps, Markov chains, stationary measures.

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1 Introduction

Many models of time series arising in population studies in ecology and growth models in economics are of the form

$$X_{t+1} = f_{t+1}(X_t), \quad t = 0, 1, 2, \cdots \tag{1}$$

Here X_t, the state of the system at time t, represents the population size or density in ecology and the total output in a one sector economy in economics. The function $f_{t+1}(\cdot)$ depends on the underlying dynamics in the period $[t, t+1]$. The functions $f_{t+1}(\cdot)$ are deterministic or stochastic depending on the underlying dynamics. In the deterministic case if f_t's are the same for all t one has a discrete dynamical system

$$X_{t+1} = f(X_t), \quad t = 0, 1, 2, \cdots \tag{1$'$}$$

In this case the initial value $X_0 = x$ gives rise to an orbit $\{x, f(x), f^{(2)}(x), ..., f^{(n)}(x), ...\}$ where for $n \geq 0$, $f^{(n+1)}(x) = f(f^{(n)}(x), f^{(0)}(x) \equiv x$. The subject of discrete dynamical systems is concerned with the behavior of the orbits such as the existence of fixed points, periodic orbits, nonperiodic or chaotic behavior, existence of an equilibrium or stationary distribution π such that if X_0 is chosen to have distribution π then X_n will also have distribution π.

In the stochastic case the f_t's are random reflecting certain stochastic forces in the underlying evolutionary dynamics. In ecology these could be due to random patterns in climate, food web, predator-prey interactions, environmental changes etc. In economics these could represent stochastic shocks and or speculative behavior of the agents of the economy.

The stochastic analog of the discrete dynamical system (1)$'$ is the model (1) where $\{f_t\}$ are random but are i.i.d. or more generally a strictly stationary sequence.

When the $\{f_t\}_{t\geq 1}$ are i.i.d. and X_0 is chosen independently of $\{f_t\}_{t\geq 1}$ the sequence $\{X_t\}_{t\geq 0}$ defined by (1) becomes a Markov chain with stationary transition probabilities. The objects of interest are steady state distributions or stationary measures, convergence to them, laws of large numbers regarding the behavior of certain empirical averages etc.

In the present paper we focus on the case when the state space, i.e. the set of values of X_t is $R^+ \equiv [0, \infty)$ and the sequence $\{f_t\}_{t \geq 1}$ is a random sequence from a family F of maps from $R^+ \to R^+$ that posses two important features: (1) for small values of $x, f(x)$ is approximately linear in x reflecting the fact that ecological populations and fledgling economies grow exponentially when small and (2) for large values of $x, f(x)$ is sublinear reflecting the effect of density dependence or competition as the population grows or diminishing returns in an economy. Examples of such families include:

i) the logistic maps (Athreya and Dai(2000)

$$f_c(x) \equiv cx(1-x), \quad 0 \leq x \leq 1, \quad 0 \leq c \leq 4 \tag{2a}$$

(ii) the Ricker maps (Ricker[20])

$$f_{c,d}(x) = cxe^{-dx}, \quad 0 \leq c, \ d < \infty, \quad 0 \leq x < \infty \tag{2b}$$

(iii) the Hassel maps (Hassel[14])

$$f(x) = cx(1+x)^{-d}, \quad 0 \leq c, \ d < \infty, \quad 0 \leq x < \infty \tag{2c}$$

(iv) the Vellekoop-Hognas maps (Vellekoop-Hognas[15])

$f(x) = rx(h(x))^{-b}, \quad 0 < r, \ b < \infty$

$h(x) \geq 1$ for $x \geq 0, \quad h(0) = 1,$

h is continuously differentiable and

$$\tilde{h}(x) \equiv \frac{xh'(x)}{h(x)} \text{ is strictly increasing} \tag{2d}$$

The main thrust of this paper is to investigate the existence of nontrivial stationary measures (i.e. other than the delta measure at 0) for the case when the $\{f_t\}_{t \geq 1}$ sequence is i.i.d. with values in the set F. There are also some results on the convergence of the sequence X_n of (1) in the subcritical case. But the general questions of convergence in distribution of X_n to the stationary measure, its uniqueness or nonuniqueness, its smoothness etc. are not addressed here. Our results here are generalizations of those of Athreya and Dai [2] for the case of random logistic maps.

In the next section the general mathematical framework of iteration of random maps is described. There is also a discussion of Feller Markov chains and occupation measures. The main parts of the paper are sections 3 and 4. In section 3 a set of necessary and two set of sufficient conditions are provided for the existence of nontrivial stationary measures. In section 4 we provide some convergence results, a trichotomy into subcritical, critical and supercritical cases and a useful comparison lemma. Some open problems are mentioned in the last section.

2 Iteration of random maps

2.1 The general framework

Consider the model $\{X_t\}_{t\geq 0}$ defined by (1) where the state space is some set S and f_t's are functions that map S to itself. If f_t's are random then so are X_t's. But in order that X_t's be random variables some conditions must be imposed on the generation of f_t's. This is spelled out below.

Let S be a nonempty set and $\mathcal{S}$ be a σ-algebra of subsets of S. Let $(\Omega, \mathcal{B}, P)$ be a probability space. Let $\{f_i\}_{i\geq 1}$ be a sequence of maps from $\Omega \times S \to S$ such that each f_i is jointly measurable, i.e.

$$\forall\ A \in \mathcal{S}, \{(\omega, x) :\ f_i(\omega, x) \in A\} \in \mathcal{B} \times \mathcal{S} \tag{3}$$

Let $X_0 : \Omega \to S$ be $(\mathcal{B}, \mathcal{S})$ measurable. Then $\{X_t\}_{t\geq 0}$ defined by

$$X_{t+1}(\omega) = f_{t+1}(\omega, X_t(\omega)) \quad t = 0, 1, 2, \cdots \tag{4}$$

is such that $\forall\ t,\ X_t : \Omega \to S$ is $(\mathcal{B}, \mathcal{S})$ measurable, i.e. it is a random variable on $(\Omega, \mathcal{B})$ and so $\{X_t\}_{t\geq 0}$ is an S-valued stochastic process defined on $(\Omega, \mathcal{B}, P)$. The proof of this (see Athreya and Stenflo [6]) is by induction and the facts:

(i) if $X : \Omega \to S$ is $(\mathcal{B}, \mathcal{S})$ measurable then the map $\pi : \omega \to (\omega, x(\omega))$ from $\Omega \to \Omega \times S$ is $(\mathcal{B}, \mathcal{B} \times \mathcal{S})$ measurable and

(ii) the composition $f \circ \pi$ is $(\mathcal{B}, \mathcal{S})$ measurable if $f : \Omega \times S \to S$ is $(\mathcal{B} \times \mathcal{S}, \mathcal{S})$ measurable.

In particular, if $F \equiv \{h_1, h_2, \ldots, h_k\},\quad k \leq \infty$ is a finite or countable collection of measurable maps from $(S, \mathcal{S})$ to itself and $\{I_j\}_{j\geq 1}$ is a sequence of random variables from some probability space $(\Omega, \mathcal{B}, P)$ to $\{1, 2, ..., k\}$ then the functions

$$f_j(\omega, x) \equiv h_{I_j(\omega)}(x) \tag{5}$$

satisfy the joint measurability condition (3). This covers the iterated function systems discussed by Barnsley [7] and also by Mitra et al [18].

More generally let F be a parametrized family of functions from S to S, say $F \equiv \{h_y(\cdot) : y \in Y\}$ such that

(i) $\forall\ y \in Y, \quad h_y : S \to S$ is $(\mathcal{S}, \mathcal{S})$ measurable

(ii) there exists a σ-algebra Σ of subsets of Y such that the map

$$h(y, x) \equiv h_y(x) \tag{6}$$

from $Y \times S \to S$ is $(\Sigma \times \mathcal{S}, \mathcal{S})$ measurable. If $Y$is finite or countable with Σ as the power set of Y, then (i) $\Longrightarrow$ (ii) as can be shown easily (see Athreya and Stenflo [6]). Now let $\{y_j\}_{j\geq 1}$ be a sequence of random variables from some probability space $(\Omega, \mathcal{B}, P)$ to (Y, Σ). Then,

$$f_j(\omega, x) \equiv h(y_j(\omega), x), \quad j \geq 1 \tag{7}$$

satisfy the joint measurability condition (3).

Examples of parametrized families include the four families mentioned in section 1. For example, for the logistic family

$$\begin{aligned} F_L \ \equiv \ \{h_c(x) \ = \ cx(1-x), \qquad & 0 \leq c \leq 4, \\ & 0 \leq x \leq 1\} \end{aligned}$$

$$\begin{aligned} S &= [0,1], \ \ \mathcal{S} \ \equiv \ \text{Borel sets of } [0,1], \\ Y &= [0,4], \ \ \Sigma \ \equiv \ \text{Borel sets of } [0,4]. \end{aligned}$$

Similarly for the Ricker family

$$\begin{aligned} F_R \ \equiv \ \{h_{c,d}(x) \ = \ cxe^{-dx}, \qquad & 0 \leq x < \infty, \\ & 0 \leq c, d < \infty\}, \end{aligned}$$

$$S = [0,\infty], \qquad \mathcal{S} = \mathcal{B}([0,\infty)),$$
$$Y = [0,\infty)\times[0,\infty), \quad \Sigma = \mathcal{B}([0,\infty)\times[0,\infty))$$

Similar identifications can be made for the Hassel family and the Vellekoop-Hognas family.

2.2 Iteration of i.i.d. random maps and Markov chains

In the paragraph above the random sequence of parameter values $\{y_j(\omega)\}_{j\geq 1}$ could be quite general. Their joint distribution was not restricted to be, for eg, stationary or Markov or i.i.d. etc. But when such restrictions are imposed then the sequence $\{X_t\}$ defined by (1) becomes more specific and amenable to further studies. Of particular importance is the case when $\{y_j(\omega)\}_{j\geq}$ are i.i.d.r.v. In this case the sequence $\{X_t\}_{t\geq 0}$ becomes a Markov chain with state space $(S, \mathcal{S})$

Theorem 1
Let F, Y, h, be as in (6). Let $\{y_j(\omega)\}_{j\geq 1}$ be a sequence of i.i.d.r.v. on some probability space $(\Omega, \mathcal{B}, P)$. Let $f_j(\omega, x)$ be as in (7). Let $X_0(\omega, x) \equiv x$

$$X_{t+1}(\omega, x) \equiv f_{t+1}(\omega, X_t(\omega, x)), \quad t \geq 0 \tag{8}$$

Then, for each $x \in S$, the sequence $\{X_t(\omega, x)\}_{t\geq 0}$ is a Markov chain with state space $(S, \mathcal{S})$, initial state x, and transition probability

$$P(x, A) \equiv P(\omega : h(y_j(\omega), x) \in A) \tag{9}$$

Remark 1 The joint measurability condition (3) ensures that for each $x, h(y_j(\omega), x)$ is an S valued random variable and so (9) is well defined.

The proof of Theorem 1 is in Athreya and Stenflo [6]. Here is a heuristic argument. Iterating (7) and suppressing ω, we get

$$X_{t+1}(x) = f_{t+1}(f_t(\ldots f_1(x))) \tag{10}$$

For each $x, X_t(x) \equiv f_t(\ldots f_1(x))$ and hence independent of f_{t+1}. So once $X_t(x)$ is given, information about previous $X_j(x)$, for $j \leq t$ gives no information about f_{t+1} making $\{X_t\}$ Markov.

Remark 2 There is a sequence dual to $\{X_t\}_{t\geq 0}$ defined by

$$\tilde{X}_t(x) = f_1(f_2, ..., f_t(x)), \quad t \geq 0 \tag{11}$$

with $\tilde{X}_0(x) \equiv x$.

It is clear that since the $\{f_i\}$ are i.i.d., for each x and t the random variables $\tilde{X}_t(x)$ and $X_t(x)$ have the same distribution. If the state space happens to be a metric space then the convergence in distribution of $\tilde{X}_t(x)$ as $t \to \infty$ implies that of $X_t(x)$. In particular, if $\tilde{X}_t(x)$ converges w.p.l. then $X_t(x)$ does converge in distribution. This idea has been used in Diaconis and Freedman [12] to prove the convergence of $X_t(x)$ in the log contractive case $Elns(f_1) < 0$ where f_1 is a Lipschitz map w.p.l. and $s(f_1)$ is its Lipschitz constant.

Remark 3 The above Theorem 1 suggests a natural converse question. Given a transition probability function $P(\cdot,\cdot)$ on some state space $(S, \mathcal{S})$ is there an iteration of i.i.d. maps that generates a Markov chain with transition function P? Kifer [16] showed that the answer is yes if the state space S is a complete separable metric space (Polish). For a proof see Athreya and Stenflo [6] who establish the following.

Theorem 2 Let S be a metric space that is Borel isomorphic to a Borel subset of R. Let $P : S \times \mathcal{S} \to [0, 1]$ be a transition probability function. That is

(i) $\forall\ x$ in $S, P(x, \cdot)$ is a probability measure on $(S, \mathcal{S})$ and

(ii) $\forall\ A \in \mathcal{S}, P(\cdot, A)$ is an $\mathcal{S}$ measurable map on S.

Then there exists a function $h : [0, 1] \times S \to S$ that is jointly measurable such that $\mu\{\omega : \omega \in [0, 1], h(\omega, x) \in A\} = P(x, A)$ where μ is the Lebesgue measure on the Borel σ-algebra of [0,1]$\times S$.

Thus Y of (6) can be taken to be $[0, 1]$. It is clear that if $\{y_i\}_{i\geq 1}$ are i.i.d. uniform [0,1] random variables then the family $\{f_j(\omega, x)\}_{j\geq 1}$ defined by $f_j(\omega, x) \equiv h(y_j(\omega), x))$ generates a Markov chain with transition function $P(\cdot, \cdot)$.

2.3 Feller (Markov) chains, occupation and stationary measures

Let $\{X_n\}_{n\geq o}$ be a Markov chain with a metric state space (S, d) and a transition function $P(\cdot, \cdot)$ (as defined in Theorem 2).

Definition 1 $\{X_n\}_{n\geq 0}$ is called a Feller (Markov) chain (or P is called a Feller transition function) if $x_n \to x$ implies $P(x_n, \cdot) \to P(x, \cdot)$ in distribution or equivalently

$$E(k(X_1)|X_0 = x) \equiv \int_S k(y)P(x, dy) \equiv (Pk)(x) \tag{12}$$

is continuous in x for all functions $k : S \to R$ that are bounded and continuous.

If $\{X_n\}_{n\geq 0}$ is generated by an iteration scheme as in (8) with $\{f_t\}_{t\geq 1}$ i.i.d. with $f_1(\omega, \cdot)$ being continuous w.p.l. then it is Feller. Indeed, since $(Pk)(x) = Ek(f_1(\omega, x))$ and $f_1(\omega, \cdot)$ is continuous and k is bounded and continuous the assertion follows by the bounded convergence theorem. Note that all the four families listed in (2) consist of continuous functions.

Definition 2 Let $\{X_n\}_{n\geq 0}$ be a Markov chain with transition function P. Let for all $A \in \mathcal{S}$,

$$L_n(A) \equiv \frac{1}{n}\sum_0^{n-1} I_A(X_j) \quad \text{and} \tag{13}$$

$$\begin{aligned} \mu_{n,x}(A) &= \tfrac{1}{n}\sum_0^{n-1} P(X_j \in A | x_0 = x) \\ &\equiv \tfrac{1}{n}\sum_0^{n-1} P^j(x, A). \end{aligned} \tag{14}$$

Then $L_n(\cdot)$ is called the empirical measure and $\mu_{n,x}(\cdot)$ the occupation measure for the chain $\{X_n\}$.

Definition 3 A sequence $\{\mu_n\}$ of probability measures on (S, d) is said to converge weakly or in distribution to a probability distribution μ if

$$\int k(x)\mu_n(dx) \to \int k(x)\mu(dx) \tag{15}$$

for all $k : S \to R$, bounded and continuous.

Definition 4 A sequence $\{\nu_n\}$ of subprobability measures on $(S, d)(i.e.\nu_n(S) \leq 1)$ is said to converge vaguely to a subprobability distribution ν if (15) holds for all $k : S \to R$, bounded, continuous and vanishing outside a compact set.

Definition 5 A measure μ on (S, d) is stationary for the transition function P if

$$\mu(A) = (\mu P)(A) \equiv \int P(x, A)\mu(dx) \text{for all} A \in \mathcal{S} \tag{16}$$

One way of finding stationary measures for P is to consider all weak or vague limits of the occupation measures $\{\mu_{n,x}(\cdot)\}$.

Theorem 3 Let $\{X_n\}$ be a Feller Markov chain with transition function P. Suppose for some intial distribution of X_0, there is a subsequence $\{n_k\}$ such that $\mu_{n_k,X_0}(\cdot)$ converges weakly ie in distribution to a probability measure μ. Then μ is stationary for P.

Proof Let g be a bounded continuous function from $S \to R$. Then

$$|\int g(y)\mu_{n_k,X_0}(dy) - \int g(y)\mu_{n_k+1,X_0}(dy)|$$

$$\leq 2\frac{||g||}{n_k}$$

where $||g|| = \sup\{|g(x)| : x \in S\}$.

Also $\int g(y)\mu_{n_k+1,X_0}(dy) = \frac{1}{n_k+1}\sum_0^{n_k} E_{X_0}g(X_j)$

$$= \frac{1}{n_k+1}\sum_1^{n_k} E_{X_0}(Pg)(X_{j-1}) \quad + \quad \frac{1}{n_k+1}Eg(X_0)$$

$$= \frac{n_k}{n_k+1}\frac{1}{n_k}\sum_0^{n_k-1} E_{X_0}(Pg)(X_j) \quad + \quad \frac{1}{n_k+1}Eg(X_0).$$

By Feller property Pg is bounded and continuous.

So $\frac{1}{n_k}\sum_0^{n_k-1} E_{X_0}(Pg)(X_j) \to \int (Pg)(y)\mu(dy)$

Also by hypothesis

$$\int g(y)\mu_{n_k,X_0}(dy) \to \int g(y)\mu(dy)$$

$$\begin{aligned} \text{Thus} \quad \int g(y)\mu(dy) &= \int Pg)(y)\mu(dy) \\ &= \int g(z)(\mu P)(dz) \quad \text{(by Fubini)} \qquad (17) \\ \text{where} \quad (\mu P)(A) &\equiv \int P(x,A)\mu(dx). \end{aligned}$$

This being true for all bounded continuous g it follows that $\mu = \mu P$, ie (16) holds, ie μ is stationary.

□

Theorem 4 Under the set up of Theorem 3 suppose that there is a subsequence $\{n_k\}$ such that $\mu_{n_k,X_0}(\cdot)$ converges vaguely to a subprobability measure μ (ie $\mu(S) \leq 1$) and that there exists an "approximate identity", ie, a sequence $\{g\}$ of continuous functions such that for each $r, g_r(x) \in [0,1] \forall\, x$ in $S, g_r(\cdot)$ has compact support and for each x in $S, g_r(x)$ increases to one as $r \to \infty$.

Then $\mu = \mu P$, ie, (16) holds.

Proof For each $g : S \to R^+$ continuous and with compact support

$$\int g(y)\mu_{n_k,X_0}(dy) \to \int g(y)\mu(dy) \tag{18}$$

As in the proof of Theorem 3

$$\begin{aligned} |\textstyle\int g(y)\mu_{n_k,X_0}(dy) &- \textstyle\int g(y)\mu_{n_k+1,X_0}(dy)| \leq \frac{2\|g\|}{n_k} \to 0. \\ \text{and} \quad |\textstyle\int g(y)\mu_{n_k,X_0}(dy) &- \textstyle\int (Pg)(y)\mu_{n_k,X_0}(dy)| \to 0. \end{aligned}$$

Also since $0 \leq g_r(\cdot) \leq 1$, and $g(\cdot) \geq 0$

$$\int (Pg)(y)\mu_{n_k,X_0}(dy) \geq \int (Pg)(y)g_r(y)\mu_{n_k,X_0}(dy).$$

Now $(Pg)(\cdot)$ is continuous since P is Feller.

Also $(Pg(y))g_r(y)$ is continuous with compact support. So

$$\int (Pg)(y)g_r(y)\mu_{n_k,x_0}(dy) \to \int (Pg)(y)g_r(y)\mu(dy). \tag{19}$$

Thus from (18) and (19) we get

$$\int g(y)\mu(dy) \geq \int (Pg)(y)g_r(y)\mu(dy)$$

Since $0 \leq g_r(\cdot) \uparrow 1$, by the monotone convergence theorem

$$\int (Pg)(y)g_r(y)\mu(dy) \uparrow \int (Pg)(y)\mu(dy).$$

Thus, for all $g : S \to R^+$ and continuous with compact support

$$\begin{aligned} \textstyle\int g(y)\mu(dy) &\geq \textstyle\int (Pg)(y)\mu(dy) \\ &= \textstyle\int g(z)(\mu P)(dz). \end{aligned}$$

This implies $\mu(A) \geq (\mu P)(A)$ for all $A \in \mathcal{S}$..

But $\quad \mu P(S) \;=\; \int_S P(x,S)\mu(dx) = \mu(S).$

Thus $\qquad \mu \;=\; \mu P.$

□

A natural problem is to find a sufficient condition for $\{\mu_{n,X_0}\}$ to have at least one vague limit point μ that is not the trivial measure 0. This is provided by the so called Foster-Lyaponov condition. See Meyn and Tweedie [17].

Theorem 5 Suppose there exist a function $V : S \to R^+$, a set $K \subset S$ and constants $0 < \alpha, M < \infty$
such that

$$\left.\begin{array}{llcl} i) & \forall\ x \notin K, \ \ E(V(X_1)|X_0 = x) & - & V(x) \ \leq -\alpha \\ ii) & \forall\ x \in S, \ \ E(V(X_1)|X_0 = x) & - & V(x) \ \leq M \end{array}\right\} \tag{20}$$

$$\text{Then,}\ \ lim\mu_{n,X_0}(K) \geq \frac{\alpha}{\alpha + M} \tag{21}$$

Proof Let E_x stand for expectation when $X_0 = x$.

For $j \geq 1$,

$$\begin{aligned} E_x V(X_j) \ - \ E_x V(X_{j-1}) &= E_x(PV)(X_{j-1}) - V(X_{j-1}) \\ &\leq -\alpha P_x(X_{j-1} \notin K) + M P_x(X_{j-1} \in K) \\ &= -\alpha + (\alpha + M) P_x(X_{j-1} \in K). \end{aligned}$$

Adding over $j = 1, 2, ..., n$ and dividing by n yields

$$\frac{1}{n}(E_x V(X_n) - V(x)) \leq -\alpha + (\alpha + M)\mu_{n,x}(K).$$

Since $V(\cdot) \geq 0$, letting $n \to \infty$ yields (21).

□

Remark 4 In many applications K would be a compact subset of S. From (21) it follows that for any vague limit point μ of μ_{n,X_0}, $\mu(K) > 0$ ensuring its nontriviality.

3 Stationary measures

Let the collection F of functions $f : [0, L) \to [0, L), L \leq \infty$ be such that

i) f is continuous

ii) $f(0) = 0$

iii) $\lim\limits_{x \downarrow 0} \frac{f(x)}{x} \equiv f'_+(0)$ exists and is positive and finite

iv) $g(x) \equiv \frac{1}{f'_+(0)} \frac{f(x)}{x}$ satisfies $0 < g(x) < 1$ for $0 < x < \infty$.

Let $(\Omega, \mathcal{B}, P)$ be a probability space.

Let $\{f_j(\omega, x\}_{j \geq 1}$ be a collection of random maps from $\Omega \times [0, \infty) \to [0, \infty)$ that are jointly measurable, ie that are $(\mathcal{B} \times \mathcal{B}[0, \infty), \mathcal{B}[0, \infty))$ measurable and for each $j, f_j(\omega, \cdot) \in F$ with probability one. Consider the random dynamical system generated by the iteration scheme:

$$\begin{aligned} X_{t+1}(\omega, x) &\equiv f_{t+1}(\omega, X_t(\omega, x)), t \geq 0 \\ X_0(\omega, x) &\equiv x. \end{aligned} \tag{22}$$

Since $f_j(\omega, \cdot) \in F$ w.p.l. the model (22) reflects the two features mentioned in the introduction, ie for small values of X_t, X_{t+1} is proportional to X_t with proportionality constant $f'_{t+1}(0) \equiv C_{t+1}$, say, and for large values of X_t, this is reduced by the factor $g(X_t)$.

The class F includes the logistic, Ricker, Hassel, Vellekoop-Hognas families mentioned in the introduction.

For the logistic family $f_c(x) = cx(1-x), L = 1, f'_+(0) = c$, and $g(x) = 1 - x$ for $0 \leq x \leq 1$.

For the Ricker family, $L = \infty, f_{c,d}(x) = cxe^{-dx}, \;\; f'_+(0) = c, g(x) = e^{-dx}, 0 \leq x < \infty$.

For the Hassel family, $L = \infty, f_{c,d}(x) = cx(1+x)^{-d}, f'_+(0) = c$ and $g(x) = (1+x)^{-d}$.

For the Vellekoop-Hognas family, $L = \infty$, $f(x) = rx(h(x))^{-b}, f'_+(0) = r, g(x) = (h(x))^{-b}$.

Our first result gives a necessary condition for the existence of a nondegenerate stationary distribution π (ie $\pi(0,\infty) > 0$) for the Markov chain $\{X_t\}$ in (22) generated by the case when $\{f_j\}_{j\geq 1}$ are i.i.d..

Theorem 6 Let $\{f_j\}_{j\geq 1}$ be i.i.d. Let

$$C_j(\omega) \equiv \lim_{x\downarrow 0}\frac{f_j(\omega,x)}{x} \in (0,\infty) \tag{23}$$

$$g_j(\omega,x) = \begin{cases} \frac{f_j(\omega,x)}{C_j(\omega)x} & x>0 \\ 1 & x=0 \end{cases} \tag{24}$$

$$\text{Assume} \quad E(lnC_1)^+ < \infty \tag{25}$$

Suppose there exists a stationary probability measure π for the Markov chain $\{X_t\}$ defined by (22) such that $\pi(0,\infty) > 0$.

Then

$$i) \quad E(lnC_1)^- < \infty, \int E|ln\ g_1(\omega,x)|\ \pi(dx) < \infty \tag{26}$$

$$\text{and} \quad ii) \quad E(lnC_1) = -\int (E\,ln\ g_1(\omega,x))\ \pi(dx) \tag{27}$$

and hence is strictly positive.

Proof Let X_0 have distribution π. Then, since π is stationary measure for $\{X_n\}, X_1 = f_1(\omega, X_0)$ also has distribution π.

Since $X_1 = f_1(\omega, X_0)$ can be written as

$$X_1 = C_1(\omega)X_0 g_1(\omega, X_0) \tag{28}$$

taking logarithms yields (suppressing ω)

$$lnX_1 = lnC_1 + lnX_0 + lng_1(X_0). \tag{29}$$

Let

$$Z \equiv (lnC_1)^- + (-lng_1(X_0)). \tag{30}$$

Since $0 \leq g_1(\cdot) \leq 1, Z$ is a nonnegative random variable. From (29)

$$lnX_0 - lnX_1 + (lnC_1)^+ = Z. \tag{31}$$

If it was known that $E|lnX_0| < \infty$, then taking expectations in (31) and using (25) one could conclude that (26) and (27) hold. Since it is not known that $E|lnX_0| < \infty$ an alternate approach is required. A truncation argument works. Let, for $k = 1, 2......$,

$$\phi_k(x) = \begin{cases} x & \text{if} \quad |x| \leq k \\ k & \text{if} \quad x > k \\ -k & \text{if} \quad x < -k \end{cases}$$

It is clear that each $\phi_k(\cdot)$ is bounded and $|\phi_k(x) - \phi_k(y)| \leq |x - y|$ for all k, x, y.

It is easy to verify that if $\eta \geq 0$ and $x - y + \eta \geq 0$ then $\phi_k(x) - \phi_k(y) + \eta \geq 0$ (just by considering the nine possibilities arising out of x and y each being $< -k$, in $[-k, k]$ or $> k$).

Let

$$Z_k = \phi_k(lnX_0) - \phi_k(lnX_1) + (lnC_1)^+ \tag{32}$$

Since Z is ≥ 0 and $(lnC_1)^+ \geq 0$ it follows that $Z_k \geq 0$.

Also $Z_k \to Z w.p.l.$ as $k \to \infty$. By stationarity of π and boundedness of ϕ_k and the hypothesis $E(lnC_1)^+ < \infty$ we get $EZ_k = E(lnC_1)^+$. Letting $k \to \infty$ and using Fatou's lemma yields

$$EZ \leq \lim EZ_k = E(lnC_1)^+ < \infty. \tag{33}$$

Since $Z = (lnC_1)^- + (-lng_1(X_0)$ and both terms are nonnegative, (33) yields $E(lnC_1)^- < \infty$ and $E(-lng_1(X_0)) < \infty$. Thus (26) is established. Since $EZ < \infty$ and by hypothesis $E(lnC_1)^+ < \infty$ we get from (31) that

$$E|lnX_0 - lnX_1| < \infty. \tag{34}$$

Also $|\phi_k(lnX_0) - \phi_k(lnX_1)| \leq |lnX_0 - lnX_1|$

and $0 \leq Z_k \leq |lnX_0 - lnX_1| + (lnC_1)^+ \equiv \tilde{Z} say..$

From (34), $E\tilde{Z} < \infty$ and so by the dominated convergence theorem we get

$$EZ_k \to EZ$$

$$\text{ie} \quad E(lnC_1)^+ = E(lnC_1)^- + E(-lng_1(X_0)).$$

All the terms involved being finite, this yields

$$\begin{aligned} E(lnC_1) &= -E(lng_1(X_0)) \\ &= -\textstyle\int Elng_1(x)\pi(dx) \end{aligned}$$

establishing (27). Since $\pi(0,\infty) > 0$ and w.p.1, $0 < g_1(x) < 1$ for $0 < x < \infty$, it follows that $ElnC_1 > 0$.

□

Corollary 1 In the set up of Theorem 6 if $ElnC_1 \leq 0$ then

i) the only stationary probability measure on $[0,\infty)$ is the delta measure at 0.

ii) For any initial distribution X_0, the occupation measure $\mu_{n,X_0}(A) \equiv \frac{1}{n}\sum\limits_0^{n-1} P(X_j \in A)$ converges to zero for all A such that its closure is a $\subset (0,\infty)$ and hence for such A the empirical measure $L_n(A) = \frac{1}{n}\sum\limits_0^{n-1} I(X_j \in A) \to 0$ in probability.

Proof

i) Suppose there is a stationary measure μ with $\mu(0,\infty) > 0$.
Let $\tilde{\mu}(A) \equiv \frac{\mu(A\cap(0,\infty))}{\mu(0,\infty)}$ for $A \in \mathcal{B}(0,\infty)$. Then $\tilde{\mu}$ is a probability measure on $(0,\infty)$. Also $\mu = \mu\{0\}\delta_0 + (1-\mu\{0\})\tilde{\mu}$. Since δ_0 and μ are both stationary for P so is $\tilde{\mu}$. By Theorem 6 this implies $ElnC_1 > 0$.

ii) Since $f_j(\omega, \cdot)$ are continuous w.p.l. the Markov chain is Feller. Also $S = [0, \infty)$ admits an approximate identify in the sense of Theorem 4. So, if μ is any vague limit point of the occupation measures $\{\mu_{n,X_0}(\cdot)\}$ then μ is stationary for P. By (i) $\mu(0, \infty)$ must be zero.

Finally, since $EL_n(A) = \mu_{n,X_0}(A)$, and $\mu_{n,X_0}(A) \to 0$ for all $A \subset \overline{A} \subset (0, \infty)$, $L_n(A) \to 0$ in probability.

□

Next we present two sets of sufficient conditions for the existence of a stationary measure π with $\pi(0, \infty) > 0$ for the Markov chain $\{X_t\}$ in (22).

Theorem 7 Let $\{f_j\}\{C_j\}, \{g_j\}$ be as in Theorem 6. Let $D_j(\omega) \equiv \sup_{x \geq 0} f_j(\omega, x)$. Assume

i) $k(x) = -Eln g_1(x) < \infty$ for all $0 < x < \infty$

ii) $\lim_{x \downarrow 0} k(x) = 0$

iii) $k(\cdot)$ be nondecreasing

iv) $E|lnC_1| < \infty, ElnC_1 > 0$

v) $E(lnD_1)^+ < \infty$

vi) $E|k(D_1)| < \infty$

Then, there exists a stationary distribution π for the Markov chain $\{X_t\}$ defined by (22) such that $\pi(0, L) = 1$.

Proof Suppressing ω, (22) becomes

$$X_{j+1} = C_{j+1} X_j g_{j+1}(X_j) \tag{35}$$

and so $lnX_{j+1} - lnX_j = lnC_{j+1} + lng_{j+1}(X_j)$

Adding this over $j = 0, 1, ..., n-1$

$$lnX_n - lnX_0 = \sum_1^n lnC_j + \sum_1^n lng_j(X_{j-1}) \tag{36}$$

Since $\quad X_j = f_j(X_{j-1}) \leq D_j$

$lnX_n \leq lnD_n$

Also $\quad E|lng_j(X_{j-1})| = -Elng_j(X_{j-1}) = Ek(X_{j-1})$

$\leq Ek(D_{j-1}) < \infty$ by (iii) and (vi).

Also $\quad E|lnC_1| < \infty$ by (iv)

So the rightside of (36) has a finite expectation. Now choose X_0 such that $E|lnX_0| < \infty$, for eg deterministic $X_0 \neq 0$.

Dividing (36) by n and taking expectations yields

$$\frac{1}{n}ElnX_n - \frac{1}{n}ElnX_0 = ElnC_1 + \frac{1}{n}\sum_1^n Ek(X_{j-1}) \tag{37}$$

But $\quad \frac{1}{n}ElnX_n \leq \frac{1}{n}E(lnD_n)^+ \to 0$ by (v)

and $\quad \frac{1}{n}ElnX_0 \to 0.$

By hypothesis (iv) $ElnC_1 > 0$

Let (H) be the condition that $\{\mu_{n,X_0}(\cdot)\}$ has no vague limit point μ with $\mu(0,\infty) > 0$. We shall show that if (H) holds then

$$\frac{1}{n}\sum_1^n Ek(X_{j-1}) \to 0 \tag{38}$$

Thus in (37) the leftside goes to zero but the rightside goes to a positive quantity. This contradiction shows that there is a vague limit point μ of $\{\mu_{n,X_0}(\cdot)\}$ with $\mu(0,\infty) > 0$. Then $\tilde{\mu}(A) \equiv (\mu(0,\infty))^{-1}\mu(A \cap (0,\infty))$ will be a stationary probability measure for P with $\tilde{\mu}(0,\infty) = 1$.

Now fix $\epsilon > 0, \eta > 0$. Then

$$\begin{aligned} Ek(X_j) &\le E(k(X_j) : X_j \le \in) \\ &+ E(k(D_j) : X_j > \in, |k(D_j)| \le M_\eta) \\ &+ E(k(D_j) : |k(D_j)| > M_\eta) \end{aligned}$$

where M_η is chosen so that

$$E(|k(D_j)| : |k(D_j)| > M_\eta) < \eta$$

(using hypothesis (vi)).

Thus,

$$\frac{1}{n}\sum_1^n Ek(X_{j-1}) \le \sup_{x \le \in} k(x) + M_\eta \mu_{n,X_0}(\in, \infty) + \eta$$

implying

$$\overline{\lim_n} \ \frac{1}{n}\sum_1^n Ek(X_{j-1})$$

$$\le \sup_{x \le \in} k(x) + \eta$$

since, if (H) holds, $\overline{\lim_n} \ \mu_{n,X_0}(\in, \infty) = 0 \ \forall \ \ \in > 0,$. By (ii) $\sup_{x \le \in} k(x) \to 0$ as $\in \to 0$. Thus (38) is established and hence the theorem is proved.

□

Remark 5 (Special cases)

1. If $f(\omega, x) = C(\omega)x(1-x), 0 \le x \le 1$ is a logistic map then $D(\omega) \equiv \sup_x f(\omega, x) = \frac{C(\omega)}{4}$ and $g(\omega, x) = (1-x)I_{[0,1]}(x)$. Thus, $k(D) = -ln(1-D) = -ln(1-\frac{C}{4})$. So if f_j is logistic w.p.l. then the hypothesis i) - vi) of Theorem 7 reduce to $-Eln(1-\frac{C}{4}) < \infty$ (see Athreya and Dai [2]).

2. If $f(\omega, x) = C(\omega)xe^{-d(\omega)x}, 0 \le x < \infty$ is a Ricker map then $D(\omega) = C(\omega)/d(\omega)$, $g(\omega, x) = e^{-d(\omega)x}$, $k(x) = E(d(\omega))x$. So if f_j is Ricker w.p.l. the hypothesis i) - vi) of Theorem 7 reduce to

$$\begin{aligned} Ed(\omega) < \infty, \quad E\tfrac{C(\omega)}{d(\omega)} &< \infty \\ E|lnC(\omega)| < \infty, \quad ElnC(\omega) &> 0. \end{aligned}$$

Similar reductions can be made in the other two cases, ie Hassel maps and Vellekoop-Hognas maps.

Now we give a second set of sufficient conditions.

Theorem 8 Let $\{f_j\}, \{C_j\}, \{g_j\}$ be as in Theorem 6. Suppose

i) $\lim_{x \to 0} E ln C_1 g_1(x) \equiv \beta_1$ exists and is > 0

ii) $\lim_{x \to 0} E(ln C_1 x g_1(x))^+ = 0$

iii) $\lim_{x \to L} E ln C_1 g_1(x) \equiv \beta_2$ exists and is < 0

iv) $\lim_{x \to L} E(ln C_1 x g_1(x))^- = 0$

v) $\tilde{k}(x) \equiv E|ln C_1 g_1(x)|$ is bounded on $[a, b]$ for all $0 < a < b < L$.

Then there exists a stationary measure π for P satisfying $\pi(0, L) = 1$.

Proof Since P is Feller we can apply Theorem 5. Let $V(x) \equiv |ln x|$.

We shall now show that there exists $\alpha, M, a, b \in (0, \infty)$ such that

$$E(V(X_1)|X_0 = x) - V(x) \leq -\alpha \text{ for all } x \notin [a, b] \tag{39}$$

$$\leq M \text{ for all } x \in [0, L] \tag{40}$$

Again suppressing ω and noting that

$$X_1 = C_1 X_0 g(X_0)$$

we see that

a) for $x < 1$

$E_x|ln X_1| - |ln x|$

$= -E ln C_1 g_1(x) + 2E(ln C_1 x g_1(x))^+$

and for $x > 1$

$$E_x|lnX_1| - lnx|$$

$$= ElnC_1g_1(x) + 2E(lnC_1xg_1(x))^-$$

By hypothesis (i) - (iv)

$$\lim_{x \to 0} E_x|lnX_1| - |lnx| = -\beta_1 < 0,$$

$$\lim_{x \to L} E_x|lnX_1| - |lnx| = \beta_2 < 0$$

Choose $0 < a < b < \infty$ such that

$$\text{for } x \leq a, \quad E_x|lnX_1| - |lnx| \leq \tfrac{-\beta_1}{2},$$

$$\text{for } x \geq b, \quad E_x|lnX_1| - |lnx| \leq \tfrac{\beta_2}{2}.$$

Next for $a < x < b$

$$|E_x|lnX_1| - |lnx|| \leq E_x|lnX_1 - lnx|$$

$$= E_x|lnC_1g_1(x)| = \tilde{k}(x)$$

which is bounded in $[a, b]$ by hypothesis (v).

Thus (39) and (40) are verified and so by Theorem 5 there exists a stationary measure $\tilde{\pi}$ for P such that $\tilde{\pi}(0, L) > 0$. Normalizing $\tilde{\pi}$ by $\tilde{\pi}(0, L)$ yields the desired measure π.

□

Remark 6 In all the four special cases (logistic, Ricker, etc) the function

$$g_j(x) \equiv \frac{f_j(x)}{C_j x} \to 0 \text{ as } x \to \infty$$

This says that for large x the growth is sublinear. But in some ecological context such as arising in resource management procedures it is more realistic to keep $g_j(x)$ bounded away from zero as $x \to \infty$. Similarly in some growth models in economics the possibility of $f_j(x) \to \infty$ as $x \to \infty$ is not unrealistic.

The next corollary is easy to verify.

Corollary 2 In the set up of Theorem 8 assume:

i) $E|lnC_1| < \infty, ElnC_1 > 0.$

ii) With probability one $\lim_{x \downarrow 0} g_1(x) = 1, \lim_{x \uparrow L} g_1(x) = \eta > 0$ and there exists $0 < a$ such that and $a \leq \inf_x g_1(x) \leq \sup_x g_1(x) \leq 1$

iii) $ElnC_1 + Eln\eta < 0$

Then there exists a stationary π for P satisfying $\pi(0, L) = 1$.

4 Convergence results

The last section dealt with the existence of stationary measures for the Markov chain $\{X_t\}$ generated by (22) or equivalently by the iteration scheme

$$X_{t+1} = C_{t+1} X_t g_{t+1}(X_t) \quad t = 0, 1, 2, \cdots \tag{41}$$

where the pair $(C_t, g_t(\cdot)_{t \geq 1})$ are i.i.d. with $0 < C_t < \infty$, $g_t(\cdot)$ being w.p.l. a continuous function with $g_t(0) = 1$ and independent of X_0.

The convergence questions that we consider here are

i) the almost sure convergence of the sequence $\{X_t\}$ as $t \to \infty$, ie convergence of the trajectories

ii) the convergence of $\{X_t\}$ in probability and

iii) the convergence of the distribution of X_t

Since the state space of the Markov chain $\{X_t\}$ is uncountable one has to look for results from general state space Markov chains theory. There is a body of results available for the case when the chain is Harris irreducible (see Numnelin[19]). Unfortunately, many of the iterated random maps cases turn out to be not irreducible especially among those where the collection of functions F sampled from is finite or countable. In these cases if the maps are interval maps that are monotone then the Dubins-Freedman theory [13] can be appealed to. The papers by Bhattacharya and Rao [9], Bhattacharya and Majumdar[8] and Bhattacharya and Waymire [10] have nice accounts of this in the random logistic maps case.

On the other hand if the distribution of C_t is smooth, for eg, absolutely continuous then $\{X_t\}$, turns out to be (under some more hypothesis) Harris irreducible. For the random logistics case Bhattacharya and Rao [9], Bhattacharya and Waymire [10] have some nice results under such assumptions.

Motivated by Theorem 6 we give the following definition.

Definition The Markov chain $\{X_t\}$ of (22) or (41) is subcritical, critical or supercritical according as $ElnC_1 < 0, = 0$ or > 0.

In the subcritical case, $\{X_n\}$ converges to zero w.p.l. In fact, a slightly more general result holds.

For the rest of this section $\{X_t\}_{t\geq 0}$ will be as in (41).

Theorem 9 Suppose

$$\overline{\lim}\ \frac{1}{n}\sum_1^n lnC_j(\omega) \equiv d(\omega) < 0 \quad w.p.1. \tag{42}$$

Then

$$X_n(\omega) = O(\rho^n) \quad w.p.1. \tag{43}$$

for any $\rho > e^{d(\omega)}$ and hence $X_n(\omega) \to 0$ w.p.l.

Proof Since $f_j \in F$

$$\begin{aligned} X_{n+1} &= C_{n+1}X_n g_{n+1}(X_n) \leq C_{n+1}X_n \\ &\leq C_{n+1}C_n ... C_1 X_0 \end{aligned}$$

Thus

$$\frac{1}{n}lnX_n \leq \frac{1}{n}lnX_0 + \frac{1}{n}\sum_1^n lnC_j.$$

Now (42) $\Longrightarrow$ (43).

Corollary If $ElnC_1 < 0$ then (42) and hence (43) holds.

The geometric decay of $\{X_n\}$ can be exploited to establish the log normality of X_n.

Theorem 10 Assume

i) $g_j(\cdot)$ is nonincreasing in $[0, \delta]$ w.p.l. for some $\delta > 0$

ii) $ElnC_1 < 0, E(lnC_1)^2 < \infty$

iii) $0 \leq k(x) = -Elng_1(x) < \infty$ for all x and nondecreasing

iv) $\int_0^1 \frac{k(x)}{x} dx < \infty$

Then,

$$\frac{lnX_n - nElnC_1}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1) \tag{44}$$

where $\sigma^2 = V(lnC_1)$.

Proof From (41)

$$lnX_n - lnX_0 = \sum_1^n lnC_j + \sum_1^n lng_j(X_{j-1}) \tag{45}$$

Since g_j is nonincreasing in $[0, \delta]$ w.p.l. and (43) holds, $1 \geq g_j(X_{j-1}) \geq g_j(\alpha\lambda^j)$ for j large, some constant α and $0 < \lambda < 1$.

But $\quad -Elng_j(\alpha\lambda^j) \leq k(\alpha\lambda^j)$

and so $\quad E(-\sum_{j=1}^{\infty} lng_j(\alpha\lambda^j)) \leq \sum_1^{\infty} k(\alpha\lambda^j)$

which is finite by by (iii).

Thus, $\quad -\sum_{j=1}^{\infty} lng_j(\alpha\lambda^j) < \infty$ w.p.l.

$$\Longrightarrow \qquad \frac{1}{\sqrt{n}} \sum_{j=1}^{n} lng_j(X_{j-1}) \to 0 \text{ w.p.l.} \tag{46}$$

By the central limit theorem

$$\frac{\sum_{1}^{n} lnC_j - nElnC_1}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1).$$

Now (45) and (46) yield (44).

□

Next we turn to the critical case.

In the critical case the occupation measures $\mu_{n,x}(\cdot)$ defined by (14) all converge in distribution to δ_0. This implies that for every $\in > 0, \frac{1}{n}\sum_{0}^{n-1} P_x(X_j \geq \in) \to 0$, ie $a_n \equiv P_x(X_n \geq \in) \to 0$ in the Cesaro sense. A natural question is whether it can be improved to full convergence or equivalently does $X_n \to 0$ in probability for all $0 < x < \infty$. For the logistic case, ie when f_1 is a logistic map w.p.l. Athreya and Dai [2] have shown this by a comparison argument. This is extended below to the present content assuming that w.p.l. f_1 is unimodal with a common mode α such that f_1 is nondecreasing in $[0, \alpha]$ and nonincreasing in $[\alpha, \infty)$.

Theorem 11 Let $E(lnC_1)^+ < \infty$ and $ElnC_1 = 0$. Assume further that there exists an $0 < \alpha < \infty$ such that w.p.l. f_1 is nondecreasing in $[0, \alpha]$ and nonincreasing in $[\alpha, \infty)$.

Then

$$X_n \xrightarrow{p} 0 \text{ for any initial value } X_0 = x \tag{47}$$

The proof makes use of the following.

Theorem 12 (Comparison Lemma) Let $\{f_i\}_{i\geq 1}$ be i.i.d. and unimodal as in the above theorem. Let X_0 be independent of $\{f_i\}_{i\geq 1}$.

Let $\{X_n\}, \{Y_n\}\{\tilde{Y}_n\}$ and $\{Z_n\}$ $n \geq 0$ be defined by

$$\begin{aligned}
X_{n+1} &= f_{n+1}(X_n) \\
Y_{n+1} &= \min\{f_{n+1}(Y_n), \alpha\}, \quad Y_0 = \min\{X_0, \alpha\} \\
\tilde{Y}_{n+1} &= \min\{f_{n+1}(\tilde{Y}_n), \alpha\}, \quad \tilde{Y}_0 = \alpha \\
Z_n &= \min\{X_n, \alpha\}
\end{aligned}$$

Then for all $n \geq 0, \tilde{Y}_n \geq Y_n \geq Z_n$ w.p.l.

Proof Since $Y_0 \leq \tilde{Y}_0 = \alpha$, and f_1 is nondecreasing in $[0, \alpha], f_1(Y_0) \leq f_1(\tilde{Y}_0)$ implying $Y_1 \equiv \min(f_1(Y_0), \alpha) \leq \min(f_1(\tilde{Y}_0), \alpha) \equiv \tilde{Y}_1$. Now induction yields $\tilde{Y}_n \geq Y_n$ for all n.

If $X_0 \leq \alpha$, then $Y_0 = X_0$ and so

$$f_1(Y_0) = f_1(X_0) = X_1$$

implying $Y_1 = \min\{f_1(Y_0), \alpha\} = \min\{X_1, \alpha\} = Z_1$.

If $X_0 > \alpha$, then $Y_0 = \alpha$ and so

$$f_1(Y_0) = f_1(\alpha) \geq f_1(X_0) = X_1$$

implying $Y_1 = \min\{f_1(Y_0), \alpha\} \geq \min\{X_1, \alpha\} = Z_1$.

Thus $Y_1 \geq Z_1$. Induction yields $Y_n \geq Z_n$ for all n.

□

Remark 7 This comparison lemma does not require any conditions as $ElnC_1$.

Corollary 3 For any $0 < \in < \alpha$, and $n \geq 1$

$$\begin{aligned} i) \quad P_x(X_n \geq \in) &\leq P_x(Z_n \geq \in) \\ &\leq P_x(Y_n \geq \in) \leq P(\tilde{Y}_n \geq \in) \end{aligned}$$

$$ii) \quad P(\tilde{Y}_{n+1} > \in) \leq P(\tilde{Y}_n \geq \in)$$

Proof Clearly i) follows from the comparison lemma. Next, by the Markov property of $\{\tilde{Y}_n\}$

$$P(\tilde{Y}_{n+1} \geq \in) = EP(\tilde{Y}_n \geq \in |\tilde{Y}_1) \leq P(\tilde{Y}_n \geq \in).$$

Proof of Theorem 11 By Corollary 3 i) it suffices to show that $P(\tilde{Y}_n \geq \in) \to 0$. But since this is nondecreasing in n this is equivalent to showing

$$\frac{1}{n} \sum_{0}^{n-1} P(\tilde{Y}_j \geq \in) \to 0 \tag{48}$$

But the occupation measure sequence $\mu_n^{\tilde{Y}}(\cdot)$ defined by $\mu_n^{\tilde{Y}}(\cdot) \equiv \frac{1}{n} \sum_{0}^{n-1} P(\tilde{Y}_j \geq \in)$ can be shown to have a nontrivial limit point only if $ElnC_1 > 0$ (as in the proof of Theorem 2). Thus $\mu_n^{Y}([\in, \infty)) \to 0$ implying (48).

Turning now to the supercritical case we know from Theorems 6 and 7 that under some mild additional hypothesis there is a stationary measure π such that $\pi(0, \infty) = 1$. If the chain $\{X_n\}$ of (41) happens to be Harris recurrent then the stationary measure π if one exists is necessarily unique and the distribution of X_n converges to π in variation norm.

5 Some open problems

The case of iteration of random logistic maps has been well studied by a number of authors (see Athreya and Bhattacharya [1] for a review). Many of those results have been extended to the general class F of section 3 but many more remain. A few of them are outlined below.

i) Harris irreducibility: Find appropriate conditions on the distribution of $f_1(\omega, \cdot)$ and in particular $(C_1(\omega), g_1(\omega, \cdot))$ to ensure that $\{X_n\}$ is Harris recurrent.

ii) Persistence in the critical case: In the critical logistic case Athreya and Schuh [5] showed that even though $X_n \to 0$ in probability, X_n does not go to zero w.p.l. In fact, there exists a level $0 < \beta < 1$ such that for all x outside of a countable set in $(0, 1) P_x(X_n \geq \beta$ for infinitely many $n \geq 1) = 1$. An interesting problem is to extend this to the present setting.

iii) Nonuniqueness: There are examples (see Athreya and Dai [3]) in the random logistic case when C_1 takes only two values there are two nondegenerate stationary measures. It should be possible to extend that construction to the present more general setting.

iv) Statistical Inference: Suppose the sequence $\{X_n\}$ has been observed for $0 \leq n \leq N$. Using this data one should be able to do statistical inference on the distribution of $(C_1, g_1(\cdot))$.

Also if it is known that it is supercritical and admits a unique stationary measure π then estimating π from the data $\{X_n = 0 \leq n \leq N\}$ would be very useful. See Athreya and Majumdar [4]

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