Think It Over

This section of Resonance presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to ‘Think It Over’, Resonance, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

The Shrinking Unit Ball

The concepts of one, two and three dimensions are familiar to us. We think of a straight line as one dimensional, a plane as two dimensional and space as three dimensional. It is not easy to visualise objects in four or higher dimensional spaces. There are many results about high dimensional spaces that are somewhat counterintuitive. Here is one such result.

For an integer \( n \geq 1 \), let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space. That is, \( \mathbb{R}^n \) is the collection of all ordered \( n \)-tuples \( \bar{x} = (x_1, x_2, \ldots, x_n) \) where each \( x_i \) is a real number. Then \( S_n = \{ \bar{x} : \bar{x} \in \mathbb{R}^n, \sum_{i=1}^{n} x_i^2 \leq 1 \} \) is called the unit ball in \( n \)-space. It is the set of all points within a distance of one unit from the origin.

Thus \( S_1 \) is the line segment \([-1, +1]\) in \( \mathbb{R}^1 \), \( S_2 \) is the unit disc \( \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\} \) in \( \mathbb{R}^2 \) and \( S_3 \) is the unit ball in \( \mathbb{R}^3 \).

Let \( V_n \) be the ‘volume’ of \( S_n \) in \( \mathbb{R}^n \). For example, \( V_1 = 2 \), the length of \( S_1 \); \( V_2 = \pi \), the area of \( S_2 \); \( V_3 = \frac{4\pi}{3} \), the

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volume of $S_3$. Noting that

$$V_1 = \int_{S_1} 1 \, dx,$$

$$V_2 = \int \int_{S_2} 1 \, dx_1 \, dx_2,$$

$$V_3 = \int \int \int_{S_3} 1 \, dx_1 \, dx_2 \, dx_3,$$

i.e., the integral of the function $f(x) \equiv 1$ over $S_1$, $S_2$ and $S_3$ respectively, one can define $V_n$ as

$$V_n \equiv \int \int \int_{S_n} 1 \, dx_1 \, dx_2 \ldots dx_n,$$

the Riemann integral of $f(x) \equiv 1$ over the set $S_n$ in $R^n$. Seeing that $V_1 = 2 < V_2 = \pi < V_3 = \frac{4\pi}{3}$ one is tempted to conclude that $V_n$ is increasing in $n$. It turns out that $V_n \to 0$ as $n \to \infty$. Can you show this?