# Relative Growth Rates

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**If several objects grow with time the one with the highest rate of growth will dominate the rest over time. In this article this is discussed for three cases: when the number of objects is finite, countably infinite and a continuum.**

If two objects, say $A_1$ and $A_2$, grow at rates $a_1$ and $a_2$ per unit time respectively, then after $n$ units of time, one unit each of $A_1$ and $A_2$ would become $a_1^n$ and $a_2^n$ respectively. Thus, if $1 < a_1 < a_2$ then both objects would grow to be very large as $n$ gets large. On the other hand, the relative growth of $A_2$ with respect to (w.r.t.) $A_1$ will be $(a_2/a_1)^n$ and that will go to infinity while that of $A_1$ w.r.t. $A_2$ will be $(a_1/a_2)^n$ and that will go to zero.

For example, if there are two fixed deposit schemes, say, $S_1$ and $S_2$, which are compounded annually with interest rates $r_1$ and $r_2$ respectively, then one unit of money invested in each of $S_1$ and $S_2$ would grow in $n$ years to $(1 + r_1)^n$ and $(1 + r_2)^n$ respectively.

If $r_1 < r_2$ and $\frac{1+r_2}{1+r_1} = 1 + \theta$ for some $\theta > 0$ then, the relative growth of $S_2$ w.r.t. $S_1$ will be $(1 + \theta)^n$ which is atleast $1 + n\theta$. Thus, in $n > \frac{1}{\theta}$ years, one unit in scheme $S_2$ grows to more than twice to what one unit in $S_1$ grows to.

More generally, let there be $k$ objects, $A_1, A_2, \ldots, A_k$ that grow at rates $a_1, a_2, \ldots, a_k$ respectively. Suppose for some $r$, $1 \le r \le k$, $a_r > a_i$ for all $i \ne r$, i.e. $a_r$ is the unique maximum among $a_1, a_2, \ldots, a_k$. Then one unit each of $A_1, A_2, \ldots A_k$ in $n$ units of time grows into $a_1^n, a_2^n, \ldots a_k^n$.

Let

$$p_{ni} \equiv \frac{a_i^n}{\sum_{j=1}^k a_j^n}, \quad 1 \le i \le k \tag{1}$$

denote the relative proportion of the size of $A_i$ to the sum of the sizes of all $A_j$'s at time $n$. Then, it is easy to conclude that

$$p_{nr} \to 1 \quad \text{and} \quad \sum_{\substack{i \neq r \\ i=1}}^{k} p_{ni} \to 0 \tag{2}$$

as $n \to \infty$. Thus, object $A_r$ tends to dominate all the rest over time. For example, if there are $k$ countries in the world with annual population growth rates $a_1, a_2, \ldots, a_k$ and if there is one country whose growth rate is higher than others, then after some time that country will have the largest population and the proportion of people in that country to the total population of the world will go to one.

What about countably infinite number of objects? Let $A_1, A_2, A_3, \ldots,$ be a countably infinite collection of objects with growth rates $a_1, a_2, \ldots$ per unit time respectively. Assume that

(i) for some $r$, $a_r > a_i, i \neq r$

(ii)

$$\sum_{j=1}^{\infty} \left(\frac{a_j}{a_r}\right)^{n_0} < \infty \text{ for some } n_0 \ge 1 \tag{3}$$

Since one unit of object $A_i$ would become $a_i^n$ after $n$ units of time, the relative proportion of $A_i$ to the total of all objects after $n$ units of time, $p_{ni}$ say will be

$$p_{ni} \equiv \frac{a_i^n}{\sum_{j=1}^{\infty} a_j^n}$$

using condition (ii) and the fact $0 < \frac{a_j}{a_r} < 1$ for all $j \neq r$ it can be shown that

$$\lim_{n\to\infty} \sum_{j\neq r} \left(\frac{a_j}{a_r}\right)^n = 0. \tag{4}$$

Thus,

$$p_{nr} \to 1 \tag{5}$$

and

$$\sum_{i\neq r} p_{ni} \to 0 \tag{6}$$

as $n \to \infty$.

Next, what about a continuum of objects?

One could try something like this. Let $f(x)$ be a non-negative function on some interval $[a, b]$. Associate with each point $x$ in $[a, b]$ an object $A_x$ and think of $f(x)$ as the growth rate of $A_x$. Thus one unit of $A_x$ will grow into $(f(x))^n$ in $n$ units of time. Call $f(x)$ the rate function. Let it have a unique maximum occuring at some $x_0$, $a \leq x_0 \leq b$. That is, $f(x_0) > f(x)$ for all $x \neq x_0$. Then, in analogy with (2) and (5), it is not unreasonable to ask whether

$$\frac{(f(x_0))^n}{\int_a^b (f(x))^n dx} \to 1 \tag{7}$$

and

$$\frac{(f(x'))^n}{\int_a^b (f(x))^n dx} \to 0 \quad \text{for all } x' \neq x_0. \tag{8}$$

Unfortunately, this does not hold fully.

For example, if $f(x) \equiv x$ and $[a, b] = [0, 1]$

then $x_0 = 1$ and $\int_0^1 (f(x))^n dx = \frac{1}{(n+1)}$ so that

$$\frac{(f(x_0))^n}{\int_0^1 (f(x))^n dx} = (n+1) \to \infty$$

and hence (7) does not hold.

However, for $0 \leq x' < x_0 = 1$

$$\frac{(f(x'))^n}{\int_0^1 (f(x))^n dx} = (n+1)(x')^n$$

does go to zero as $n \to \infty$, i.e. (8) holds. This suggests that an appropriate generalisation of (2) and (5) might be as follows. For every $\varepsilon > 0$,

$$p_n(x_0, \varepsilon) \equiv \frac{\int_{x_0-\varepsilon}^{x_0+\varepsilon} (f(x))^n dx}{\int_a^b (f(x))^n dx} \to 1 \qquad (9)$$

as $n \to \infty$ or equivalently

$$1 - p_n(x_0, \varepsilon) = \frac{\int_{|x-x_0| \geq \varepsilon} (f(x))^n dx}{\int_a^b (f(x))^n dx} \to 0 \qquad (10)$$

as $n \to \infty$. That is at time $n$, the proportion of objects that lie in any nondegenerate two sided neighborhood of $x_0$ to the total number of objects (rather than those that lie exactly at $x_0$) goes to one and the proportion outside any such neighborhood goes to zero.

It turns out that (9) and (10) hold under some reasonable conditions on $f(x)$. Assume that the rate function $f(.)$ is such that

(i) $f = [a, b] \to R^+ = [0, \infty)$,

(ii) for some $x_0$ in $[a, b]$, $f(x_0) > f(x)$ for all $x \neq x_0$, $x$ in $[a, b]$

(iii) $f(.)$ is continuous at $x_0$

(iv) for each $\varepsilon > 0$, there is a $\lambda_\varepsilon$ such that $0 \leq \lambda_\varepsilon < 1$ and $|x - x_0| \geq \varepsilon$ implies $0 \leq f(x) \leq \lambda_\varepsilon f(x_0)$.

(11)

Then, we claim that for each $\varepsilon > 0$, (9) and (10) hold.

Here is a proof of this claim. Fix $\varepsilon > 0$. Let $\lambda_\varepsilon$ be as in condition (iv) above. By condition (iii) above, i.e. the continuity of $f(.)$, and the fact $f(x_0) > 0$, for any $0 \leq \lambda < 1$, we can choose an $\eta > 0$ such that $|x - x_0| < \eta$ implies $f(x) > \lambda f(x_0)$. Now choose the $\lambda$ to be less than $\lambda_\varepsilon$. Then the denominator of (10) is

$$\geq \int_{|x-x_0|<\eta} (f(x))^n dx \geq \lambda^n (f(x_))^n \eta$$

and by condition (iv) the numerator of (10) is

$$\leq (\lambda_\varepsilon f(x_0))^n (b-a)$$

Thus

$$0 \leq 1 - p_n(x_0, \varepsilon) \leq \left(\frac{\lambda}{\lambda_\varepsilon}\right)^n \frac{(b-a)}{\eta}$$

Since $0 \leq \frac{\lambda}{\lambda_\varepsilon} < 1$, the rightside above goes to zero as $n \to \infty$, establishing (10) which implies (9). Note further that for each $\varepsilon > 0$

$$\begin{aligned} 0 \leq \frac{\int_a^b (f(x))^n dx}{(f(x_0))^n} &\leq \int_{|x-x_0|<\varepsilon} \left(\frac{f(x)}{f(x_0)}\right)^n dx \\ &+ \int_{\varepsilon \leq |x-x_0|} \left(\frac{f(x)}{f(x_0)}\right)^n dx \\ &\leq 2\varepsilon + \lambda_\varepsilon^n (b-a) \end{aligned}$$

and hence that

$$0 \leq \overline{\lim_{n\to\infty}} \frac{\int_a^b (f(x))^n dx}{(f(x_0))^n} \leq 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary this says that

$$\lim_{n\to\infty} \frac{\int_a^b (f(x))^n dx}{(f(x_0))^n} = 0$$

Thus in (9) both the numerator and denominator when divided by $(f(x_0))^n$ go to zero as $n \to \infty$ and yet their ratio goes to one.

One can refine (9) if we know more about the behaviour of $f(x)$ near the optimum value $x_0$. The assertion in (9) says that for each $\varepsilon > 0$ the proportion of objects in the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ goes to one.

One could ask whether we could find smaller intervals $O_n$ around $x_0$ that get small and shrink to $x_0$ as $n \to \infty$ such that the proportion in $n$ still goes to one. For example, would $O_n = (x_0 - \varepsilon_n, x_0 + \varepsilon_n)$ where $n\varepsilon_n \to 0$ do? What if $\sqrt{n}\varepsilon_n \to 0$.

The following special case was considered by S Shirali in an earlier issue of *Resonance* [1]. Let $f(x) = (\cos x)^n$ be the rate function on the interval $[a, b] = [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $f$ satisfies (11) and has a unique maximum of 1 at $x_0 = 0$. Thus

$$\int_{-\pi/2}^{\pi/2} (\cos x)^n dx \to 0 \text{ as } n \to \infty$$

Shirali [1] asserts that for any $-\infty < \alpha < \beta < \infty$

$$\frac{\int_{\frac{\alpha}{\sqrt{n}}}^{\frac{\beta}{\sqrt{n}}} (\cos x)^n dx}{\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (\cos x)^n d_2} \to \Phi(\beta) - \Phi(\alpha) \qquad (12)$$

as $n \to \infty$, where

$$\Phi(x) \equiv \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \qquad (13)$$

is the standard normal (gaussian) cumulative distribution function, well known in probability theory and statistics and in many areas of science and engineering.

Thus, the proportion of objects in the neighborhood $\left(\frac{\alpha}{\sqrt{n}}, \frac{\beta}{\sqrt{n}}\right)$ after $n$ units of time converges as $n \to \infty$ to a quantity that is strictly postive but also strictly less than one. This implies that most of the objects are centered around the origin at a distance of the order $\sqrt{n}$. This also implies that the proportion of objects in

$O_n \equiv (-\varepsilon_n, \varepsilon_n)$ goes to zero if $\sqrt{n}\varepsilon_n \to 0$ (i.e. $\varepsilon_n \to 0$ faster than $\frac{1}{\sqrt{n}}$) and to one if $\sqrt{n}\varepsilon_n \to \infty$.

Another way of interpreting Shirali's assertion (12) is that if at time $n$ one chooses an object at random with uniform distribution among all the objects and its location is $X_n$ then the probability that $X_n$ lies between $\left(\frac{\alpha}{\sqrt{n}}, \frac{\beta}{\sqrt{n}}\right)$ converges to $\Phi(\beta) - \Phi(\alpha)$ or that the probability that $\sqrt{n}X_n$ lies in $(\alpha, \beta)$ converges to $\Phi(\beta) - \Phi(\alpha)$. In the language of probability theory it says that $\sqrt{n}X_n$ *converges in distribution* to a standard normal distribution.

Now let us prove (12). Note that

$$\sqrt{n}\int_{\frac{\alpha}{\sqrt{n}}}^{\frac{\beta}{\sqrt{n}}} (\cos x)^n dx = \int_{\alpha}^{\beta} \left(\cos\frac{u}{\sqrt{n}}\right)^n du$$

Now write

$$\begin{aligned} \cos\frac{u}{\sqrt{n}} &= \left(1 + \frac{\left(\cos\frac{u}{\sqrt{n}} - 1\right)n}{n}\right) \\ &= \left(1 + \frac{\lambda_n(u)}{n}\right) \text{ say} \qquad (14) \end{aligned}$$

Now for $u \neq 0$

$$\lambda_n(u) = \frac{\left(\cos\frac{u}{\sqrt{n}} - 1\right)}{\frac{u^2}{2n}}\frac{u^2}{2}.$$

Since $\frac{\cos\theta - 1}{\theta^2} \to -1/2$ as $\theta \to 0$, $\lambda_n(u) \to -\frac{u^2}{2}$ as $n \to \infty$.

Next,

$$\left(1 + \frac{x_n}{n}\right)^n \to e^x \quad \text{if } x_n \to x.$$

So

$$\varphi_n(u) \equiv \left(\cos\frac{u}{\sqrt{n}}\right)^n \to e^{-u^2/2} \equiv \varphi(u),$$

say as $n \to \infty$.

It can be shown that $\varphi_n(.)$ converges uniformly to $\varphi(u)$ on any finite interval $[\alpha, \beta]$, i.e. $\max\{|\varphi_n(u) - \varphi(u)| \,:\, \alpha \leq u \leq \beta\} \to 0$ as $n \to \infty$.

This implies that

$$\int_\alpha^\beta \left(\cos \frac{u}{\sqrt{n}}\right)^n \to \int_\alpha^\beta e^{-u^2/2} du$$

i.e.

$$\sqrt{n} \int_{\frac{\alpha}{\sqrt{n}}}^{\frac{\beta}{\sqrt{n}}} (\cos x)^n dx \to \int_\alpha^\beta e^{-u^2/2} du. \tag{15}$$

A little more effort can be used to show that

$$\sqrt{n} \int_{-n}^{n} (\cos x)^n dx \to \int_{-\infty}^{\infty} e^{-u^2/2} du. \tag{16}$$

Now

$$\int_{-\infty}^{\infty} e^{-u^2/2} du = 2 \int_0^\infty e^{-u^2/2} du = 2I, \text{ say,}$$

then

$$\begin{aligned} I^2 &= \left(\int_0^\infty e^{-u^2/2} du\right) \left(\int_0^\infty e^{-v^2/2} dv\right) \\ &= \int_0^\infty \int_0^\infty e^{-\frac{u^2+v^2}{2}} du dv \\ &= \int_0^\infty \int_0^{\pi/2} e^{-r^2/2} r dr d\theta \\ &= \left(\int_0^\infty e^{-r^2/2} r dr\right) \left(\int_0^{\pi/2} d\theta\right) \\ &= \left(\int_0^\infty e^{-t} dt\right) \frac{\pi}{2} = \frac{\pi}{2} \end{aligned}$$

Thus

$$2I = 2\sqrt{\frac{\pi}{2}} = \sqrt{2\pi}. \tag{17}$$

Following the above arguments it is not difficult to establish the following:

Let $f(x)$ be a Riemann integrable function on $[-1,1]$ such that (i) $f(0)=1$, (ii) $0 \leq f(x) \leq 1$ for all $|x| \leq 1$, (iii) $f$ is continuous at 0 and (iv) for any $\varepsilon > 0$, there is a $0 < \lambda_\varepsilon < 1$ such that for all $|x| \geq \varepsilon$, $|f(x)| \leq \lambda_\varepsilon$. Then

(a) $A_n = \int_{-1}^{1} (f(x))^n dx \to 0$ as $n \to \infty$.

(b) For each $\varepsilon > 0$,

$$p_n(\varepsilon) \equiv \frac{\int_{-\varepsilon}^{\varepsilon} (f(x))^n dx}{A_n} \to 1 \quad \text{as} \quad n \to \infty$$

(c) If in addition to condition (i) and (iv) above if $f$ satisfies

(v) For some $\alpha > 0$

$$\lim_{n \to 0} \frac{1 - f(x)}{|x|^\alpha} \equiv c_\alpha, \quad 0 < c_\alpha < \infty,$$

then

$$\begin{aligned} B_n(a,b) &= \frac{\int_{a/n^{1/\alpha}}^{b/n^{1/\alpha}} (f(x))^n dx}{An} \\ &\to \frac{\int_a^b e^{-c_\alpha |u|^\alpha} du}{\int_{-\infty}^{\infty} e^{-c_\alpha |u|^\alpha} du} \end{aligned} \tag{18}$$

Note that

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-c_\alpha |u|^\alpha} du &= 2 \int_0^\infty e^{-c_\alpha v} \frac{1}{\alpha} v^{\frac{1}{\alpha}-1} dv \\ &= \frac{2}{\alpha c_\alpha^{1/\alpha}} \int_0^\infty e^{-u} u^{\frac{1}{\alpha}-1} du \\ &= \frac{2}{\alpha} \cdot \frac{1}{c_\alpha^{1/\alpha}} \cdot \Gamma(1/\alpha) \end{aligned}$$

where $\Gamma(p)$, the *gamma function*, is defined for $p > 0$ as

$$\Gamma(p) = \int_0^\infty e^{-u} u^{p-1} du.$$

It can be shown that $\Gamma(1) = 1$,

$$\Gamma(p+1) = p\Gamma(p) \quad \text{for all } p > 0, \quad \Gamma(1/2) = \sqrt{\pi}.$$

Thus, if the rate function satisfies (i) to (v) above and one chooses an organism at random from the big collection then its location, $X_n$ will be such that it is of order $n^{-1/\alpha}$ and $n^{1/\alpha}X_n$ will converge in distribution to a probability distribution on $(-\infty, \infty)$ with probability density function proportional to $e^{-c_\alpha |u|^\alpha}$.

Another formulation of (5) is the following:

Let $h(.)$ be a Riemann integrable function on some bounded interval $[a, b]$ such that it achieves a unique maximum at $x_0$. Let for $\varepsilon > 0$, $t > 0$

$$p(x_0, \varepsilon, t) \equiv \frac{\int_{x_0-\varepsilon}^{x_0+\varepsilon} e^{th(x)} dx}{\int_a^b e^{th(x)} dx}$$

Then for each $\varepsilon > 0$,

$$p(x_0, \varepsilon, t) \to 1 \text{ as } t \to \infty.$$

Thus a probability distribution on $[a, b]$ with probability density proportional to $e^{th(x)}$ converges in distribution to the unit mass at $x_0$ where $h(\cdot)$ achieves its unique maximum. This is known in physics as the Laplace's method of locating the maximum of $h$ and is related to what is called simulated annealing in optimization problems in operations research.

## Suggested Reading

[1] S A Shirali, Solution to On a Use of Normal Distribution, *Resonance*, Vol.11, pp.88–90, 2006.

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