# Unit Ball in High Dimensions

*K B Athreya*

K B Athreya is a Visiting Professor at IMI, Mathematics Department, IISc, Bangalore. He is a professor in the Mathematics Department of the Iowa State University, USA. His research interests include mathematical analysis, probability theory and its applications and statistics. His spare time is spent listening to Indian classical music.

**In this article, we compute the volume $V_n$ of the unit ball in an $n$-dimensional space. For $n = 1, 2, 3$, the volumes are respectively $2, \pi, 4\pi/3$, which are the length of interval $[-1, 1]$, area of a unit circle and volume of the unit sphere. The numbers $V_n$ 'appear' to increase. But in fact this not so. In fact $V_n$ tends to zero as $n$ tends to infinity!**

For a positive integer $n$, the unit ball in $n$ dimension is defined as the set $S_n \equiv \{\tilde{x} : \tilde{x} = (x_1, x_2, \ldots, x_n), x_i \in R, 1 \le i \le n, \sum_1^n x_1^2 \le 1\}$ where $R$ is the set of real numbers. Thus $S_n$ consists of all points $\tilde{x} = (x_1, x_2, \ldots, x_n)$ in $n$-dimensional Euclidean space $R^n$ that are at a distance less than or equal to one from the origin. Let

$$V_n \equiv \int_{S_n} 1 \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \qquad (1)$$

be the Riemann integral of the function $f(x) \equiv 1$ over the unit ball $S_n$. Then $V_n$ can be thought of as the 'n dimensional' volume of $S_n$. In a recent issue of *Resonance*[1] the problem of showing that $V_n \to 0$ as $n \to \infty$ was posed. In this article a solution to that problem and some further results are presented.

[1] K B Athreya, The Shrinking Unit Ball, *Resonance*, Vol.11, No.11, pp.89–90, 2007.

Note that

$$\begin{aligned} V_1 &= \int_{-1}^{1} 1 \mathrm{d}x_1 = 2, \\ V_2 &= \int_{S_2} 1 \mathrm{d}x_1 \mathrm{d}x_2, \\ &= \pi \end{aligned}$$

and

$$V_3 = \int_{S_3} 1 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$

**Keywords**
Euclidean space, unit ball, Riemann integral, standard normal probability density, Bell curve, Gaussian probability density, Cesaro average.

$$= \int_{S_1}\left(\int_{S_2} 1\mathrm{d}\tilde{x}_2\mathrm{d}\tilde{x}_3)\right)(1-x_1^2)\mathrm{d}x_1,$$

where

$$\tilde{x}_2 = \frac{x_2}{(1-x_1^2)^{1/2}}, \quad \tilde{x}_3 = \frac{x_3}{(1-x_1^2)^{1/2}}.$$

Thus,

$$V_3 = \int_{S_1} V_2(\sqrt{1-x_1^2})^2\mathrm{d}x_1$$

One is led to the recurrence relation

$$\begin{aligned} V_{n+1} &= \int_{S_1} V_n\left(\sqrt{1-x_1^2}\right)^n \mathrm{d}x_1 \quad \text{for} \quad n \geq 1 \\ &= V_n\delta_n, \end{aligned} \tag{2}$$

where

$$\delta_n = \int_{-1}^{+1}(1-x^2)^{n/2}\mathrm{d}x. \tag{3}$$

It is easy to check that $\delta_1 = \frac{\pi}{2}$, $\delta_2 = \frac{4}{3}$ and hence $V_1 = 2 < V_2 = \pi < V_3 = \frac{4\pi}{3}$ suggesting that $V_n$ could be increasing with $n$. But $\delta_3 = 2\int_0^1(1-x^2)^{3/2}\mathrm{d}x = 2\int_0^{\pi/2}\cos^4\theta\mathrm{d}\theta = \frac{3}{16}\pi < 1$ implying that $V_3 < V_4$. Since $\delta_n$ decreases with $n$, it follows that $V_n$ is decreasing in $n$ for $n \geq 3$.

Since for $0 < |x| \leq 1$, $(1-x^2)^{n/2} \to 0$, one expects that $\delta_n \to 0$ as $n \to \infty$. This is indeed the case. Here is a proof. Fix $0 \leq \varepsilon < 1$. Then for $n \geq 1$,

$$\begin{aligned} \delta_n &= \int_{|x|\leq\varepsilon}(1-x^2)^{n/2}\mathrm{d}x + \int_{\varepsilon<|x|\leq 1}(1-x^2)^{n/2}\mathrm{d}x \\ &\leq 2\varepsilon + (1-\varepsilon^2)^{n/2}2. \end{aligned}$$

So

$$\overline{\lim}_{n\to\infty}\, \delta_n \leq 2\varepsilon, \tag{4}$$

where for any sequence $\{a_n\}_{n\geq 1}$ of real numbers, $\overline{\lim}_{n\to\infty} a_n$ (read as $\limsup_{n\to\infty} a_n$) is defined as $\lim_{n\to\infty} M_n$, where

$$M_n \equiv \ \lim_{k\to\infty} \max(a_n, a_{n+1}, \ldots, a_{n+k}).$$

Notice that for each $n, k$, $M_{n,k} \equiv \max(a_n, a_{n+1}, \ldots, a_{n+k})$ is non-decreasing in $k$ and hence $M_n \equiv \{\lim_{\mathrm{k}\to\infty} \mathrm{M}_{\mathrm{n,k}}$ exists. Again, since $M_n$ is non-increasing in $n$, $\lim_n M_n$ does exist. Thus, for any sequence $\{a_n\}_{n\geq 1}$, $\overline{\lim}_{n\to\infty} a_n$ is always well defined. Similarly $\underline{\lim}_{n\to\infty} a_n$ (read as $\liminf_{n\to\infty} a_n$) can be defined and it can be shown that a sequence $\{a_n\}_{n\geq 1}$ converges to a real number $a$, i.e., $\lim a_n = a$, if and only if $\overline{\lim}_{n\to\infty} a_n = a = \underline{\lim}_{n\to\infty} a_n$.

Going back to (4) we can conclude that since $\varepsilon > 0$ is arbitrary,

$$\lim_{n\to\infty} \delta_n \leq 0.$$

Also since $\delta_n \geq 0$ for all $n \geq 1$,

$$\lim_{n\to\infty} \delta_n \geq 0.$$

Thus,

$$\lim_{n\to\infty} \delta_n \quad \text{exists and is} \quad 0. \tag{5}$$

Now we can use the recurrence relation (2) to show that $\lim_{n\to\infty} V_n = 0$. In fact, we can show more. Namely that for any $\eta > 0$,

$$\lim_{n\to\infty} \frac{V_n}{\eta^n} = 0. \tag{6}$$

To prove (6), note that by (5) there is an $N_\eta$ such that for $n \geq N_\eta$, $0 \leq \delta_n < \eta/2$. Thus, for $n > N_\eta$,

$$V_n = V_{N_n} \prod_{\delta=N_\eta}^{n} \delta_j \leq V_{N_\eta} \left(\frac{\eta}{2}\right)^{n-N_\eta}. \tag{7}$$

This, in turn, yields, for $n > N_\eta$.

$$\frac{V_n}{\eta^n} \leq \frac{V_{N_\eta}}{\eta^{N_n}} \left(\frac{1}{2}\right)^{n-N_\eta}.$$

Since $(1/2)^{n-N_\eta} \to 0$ as $n \to \infty$, we get

$$\overline{\lim}_{n\to\infty} \frac{V_n}{\eta^n} \leq 0.$$

But $\frac{V_n}{\eta^n} \geq 0$ and so $\liminf_{n\to\infty} \frac{V_n}{\eta^n} \geq 0$. So we conclude that for any $\eta > 0$,

$$\lim_{n\to\infty} \frac{V_n}{\eta^n} = 0,$$

i.e., (6) is established.

One can refine this further. The proof of (6) is based on (5). Now (5) can be improved to assert that

$$\lim_{n\to\infty} \sqrt{n}\delta_n = \sqrt{2\pi}. \tag{8}$$

To see this, note that

$$\begin{aligned} \sqrt{n}\delta_n &= \sqrt{n}\int_{-1}^{1}(1-x^2)^{n/2}\mathrm{d}x \\ &= \int_{-\sqrt{n}}^{\sqrt{n}}\left(1-\frac{u^2}{n}\right)^{n/2}\mathrm{d}u \end{aligned} \tag{9}$$

by the change of variable $x \to \frac{u}{\sqrt{n}}$. Now as $n \to \infty$, $\left(1-\frac{u^2}{n}\right)^{n/2} \to \mathrm{e}^{-u^2/2}$ and the region of integration $[-\sqrt{n}, \sqrt{n}]$ goes to $(-\infty, \infty)$. This suggests that the integral in (9) converges as $n \to \infty$ to

$$I \equiv \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/2}\mathrm{d}u. \tag{10}$$

This interchange of integration and limit as $n \to \infty$ can be justified and is done so in the appendix. To evaluate $I$ note that

$$\begin{aligned} I^2 &= 4\left(\int_0^\infty \mathrm{e}^{-u^2/2}\mathrm{d}u\right)\left(\int_0^\infty \mathrm{e}^{-v^2/2}\mathrm{d}v\right) \\ &= 4\int_0^\infty\int_0^\infty \mathrm{e}^{-\frac{(u^2+v^2)}{2}}\mathrm{d}u\mathrm{d}v \\ &= 4\int_0^\infty\int_0^{\pi/2} \mathrm{e}^{-r^2/2}\ r\mathrm{d}r\mathrm{d}\theta \end{aligned} \tag{11}$$

(by changing $(u, v)$ to polar coordinates $(r, \theta)$)

$$= 4\left(\int_0^\infty \mathrm{e}^{-r^2/2} r\,\mathrm{d}r\right)\left(\int_0^{\pi/2} \mathrm{d}\theta\right) = 4\frac{\pi}{2} = 2\pi.$$

Thus $I = \sqrt{2\pi}$. So (8) is established.

Since

$$V_n = V_1\left(\prod_{J=1}^{n-1} \delta_j\right) \quad \text{for } n \geq 2,$$

it follows that

$$\frac{1}{n}\log V_n = \frac{1}{n}\log V_1 + \frac{1}{n}\sum_{J=1}^{n-1}\log(\delta_j\sqrt{j}) - \frac{1}{2n}\sum_{j=1}^{n-1}\log j.$$

This implies that

$$\frac{1}{n}\left(\log V_n + \frac{1}{2}\sum_{j=1}^{n}\log j\right) = \frac{1}{n}\log V_1 + \frac{1}{2n}\log n + \frac{1}{n}\sum_{j=1}^{n-1}\log(\delta_i\sqrt{j}).$$

It can be shown that if $\lim_{n\to\infty} a_n = a$ exists in $R$ then $\frac{1}{n}\sum_{j=1}^{n} a_j$ (called the Cesaro average) also converges to $a$.

Since (8) holds,

$$\frac{1}{n}\sum_{j=1}^{n-1}\log(\delta_j\sqrt{j}) \to \log\sqrt{2\pi}, \quad \text{as } n \to \infty.$$

Thus it has been shown that

$$\frac{1}{n}\log(V_n\sqrt{n!}) = \log\sqrt{2\pi}$$

or equivalently

$$(V_n\sqrt{n!})^{1/n} \to \sqrt{2\pi}. \qquad (12)$$

The function $\phi(u) \equiv \frac{1}{\sqrt{2\pi}}e^{-u^2/2}$ is called the standard normal (Gaussian) probability density. It is of great importance in probability theory and statistics and many areas of science. Many real world distributions such as heights of men in a given population are approximately normally distributed. In the popular science literature the graph of $\phi(\cdot)$ is called the 'bell curve'.

The function $\phi(u) \equiv 1/\sqrt{2\pi}\, e^{-u^2/2}$ is called the standard normal (Gaussian) probability density. It is of great importance in probability theory and statistics and many areas of science.

Thus, we have established the following:

**Theorem 1.** Let for $n \geq 1$ $S_n \equiv \{\tilde{x} : \tilde{x} = (x_1, x_2, \ldots, x_n),$ $x_i \in R,\ 1 \leq i \leq n,\ \sum_1^n x_1^2 \leq 1\}$ be the unit ball in the $n$-dimensional Euclidean space $R^n$, where $R$ is the set of real numbers. Let $V_n \equiv \int_{S_n} 1\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n$, the Riemann integral of the function $f(x) \equiv 1$ over $S_n$ be the '$n$-dimensional volume' of $S_n$. Then

(i) $V_1 = 2$, $V_2 = \pi$, $V_3 = \frac{4\pi}{3}$ and for $n \geq 2$ $V_n = V_1 \prod_{J=1}^{n-1} \delta_j$, where $\delta_k = \int_{-1}^{1}(1-x^2)^{k/2}\mathrm{d}x$;

(ii) $\sqrt{n}\delta_n \to \sqrt{2\pi}$ as $n \to \infty$;

(iii) For any $\eta > 0$, $\frac{V_n}{\eta^n} \to 0$ as $n \to \infty$;

(iv) $(V_n\sqrt{n!})^{1/n} \to \sqrt{2\pi}$ as $n \to \infty$.

Along the same lines one can establish the following generalization.

**Theorem 2.** Fix $0 < p < \infty$. Let for $n \geq 1$.

$$S_{n,p} \equiv \{\tilde{x} : \tilde{x} = (x_1, x_2, \ldots, x_n), x_i \in R, 1 \leq i \leq n, \sum_1^n |x_i|^p \leq 1\}$$

In the '∞-norm' the volume of the unit ball in $2^n$ in the $n$-dimensional space. This is actually an $n$-cube.

be the unit ball in the $n$-dimensional Euclidean space $R^n$ with the '$p^{th}$ norm'. Let $V_{n,p} \equiv \int_{S_{n,p}} 1\mathrm{d}x, \mathrm{d}x_2 \dots \mathrm{d}x_n$ be 'the $n$-dimensional volume' of $S_{n,p}$.

Then

(i) $V_{1,p} = 2$, $V_{n,p} = V_{1,p} \prod_{j=1}^{n-1} \delta_{j,p}$, where $\delta_{k,p} = \int_{-1}^{1} (1 - |x_1|^p)^{n/p} \mathrm{d}x_1$;

(ii) $n^{1/p}\delta_{n,p} \to 2p^{1/p-1} \int_0^\infty \mathrm{e}^{-u} u^{1/p-1} \mathrm{d}u \equiv c(p)$, say;

(iii) For any $\eta > 0$, $\frac{V_{n,p}}{\eta^n} \to 0$;

(iv) $(V_n(n!)^{1/p})^{1/n} \to c(p)$ as $n \to \infty$.

**Remarks**

1. Note that if one defines the unit ball in '∞-norm' as $S_{n,\infty} = \{\tilde{x} : \tilde{x} = (x_1, x_2, \dots, x_n),\ |x_i| \le 1$ for all $1 \le i \le n\}$ then $S_{n,\infty}$ is the cube with sides $[-1, +1]$ in all $n$ directions and its volume $V_{n,\infty}$ is $2^n$ and hence does not go to zero.

2. The fact that $V_{n,2} \to 0$ as $n \to \infty$ was cited by Professor John Hoperoft of Cornell University, a well-known computer scientist, as one of many counter intuitive results about high-dimensional Euclidean spaces and the need to include the study of these in the curriculum for computer science students in this information age.

## Appendix

Here we show that

$$\lim_{n\to\infty}\int_{-\sqrt{n}}^{\sqrt{n}}\left(1-\frac{u^2}{n}\right)^{n/2}\mathrm{d}u=\int_{-\infty}^{\infty}\mathrm{e}^{-u^2/2}\mathrm{d}u. \qquad \text{(A1)}$$

The function $\phi(x)=1-x-\mathrm{e}^{-x}$ on $R$ satisfies $\phi'(x)=-1+\mathrm{e}^{-x}$ and hence $<0$ for all $x>0$ and $>0$ for all $x<0$ and $=0$ for $x=1$. Thus, $\phi(.)$ is increasing in $(-\infty,0)$, decreasing in $(0,\infty)$ and 0 at $x=0$. This yields

$$\left(1-\frac{u^2}{n}\right)\leq \mathrm{e}^{-u^2/n}$$

for all $u$ real. For any $k>1$ and $n>k^2$

$$\begin{aligned}\int_{-\sqrt{n}}^{\sqrt{n}}\left(1-\frac{u^2}{n}\right)^{n/2}\mathrm{d}n &= \int_{|u|\leq k}\left(1-\frac{u^2}{2}\right)^{n/2}\mathrm{d}u\\ &\quad+\int_{k<|u|\leq\sqrt{n}}\left(1-\frac{u^2}{2}\right)^{n/2}\mathrm{d}u.\end{aligned}$$

Then function $\psi_n(u)\equiv\left(1-\frac{u^2}{n}\right)^{n/2}$ converges uniformly to $\psi(u)=\mathrm{e}^{-u^2/2}$ in $[-k_1,k]$ and both $\psi_n$ and $\psi$ are continuous functions. So, for each $1<k<\infty$,

$$\int_{|u|\leq k}\left(1-\frac{u^2}{n}\right)^{n/2}\mathrm{d}u\to\int_{|u|\leq k}\mathrm{e}^{-u^2/2}\mathrm{d}u.$$

Also

$$\begin{aligned}\int_{k<|u|\leq\sqrt{n}}\left(1-\frac{u^2}{n}\right)^{n/2}\mathrm{d}u &\leq \int_{k<|u|}\mathrm{e}^{-u^2/2}\mathrm{d}u\\ &\leq \int_{k<|u|}\mathrm{e}^{-u/2}\mathrm{d}u\\ &\qquad\text{(since } k>1\text{)}\\ &= 4\mathrm{e}^{-k/2}.\end{aligned}$$

Thus

$$\left|\int_{-\sqrt{n}}^{\sqrt{n}}\left(1-\frac{u^2}{n}\right)^{2n}\mathrm{d}u-\int_{-\infty}^{\infty}\mathrm{e}^{-u^2/2}\mathrm{d}u\right|$$
$$\leq\left|\int_{|u|\leq k}\left(1-\frac{u^2}{n}\right)^{2}\mathrm{d}u-\int_{|u|\leq k}\mathrm{e}^{-u^2/2}\mathrm{d}u\right|$$
$$+2\int_{|u|>k}\mathrm{e}^{-u^2/2}\mathrm{d}u$$

This yields,

$$\overline{\lim_{n\to\infty}}\left|\int_{-\sqrt{n}}^{\sqrt{n}}\left(1-\frac{u^2}{n}\right)^{2n}\mathrm{d}u-\int_{-\infty}^{\infty}\mathrm{e}^{-u^2/2}\mathrm{d}u\right|\leq 8e^{-k/2}.$$

Now letting $k\uparrow\infty$ yields (A1).

*Address for Correspondence*
K B Athreya
IMI
Mathematics Department
Indian Institute of Science
Bangalore 560 012, India.
Email:
kba@math.iisc.ernet.in