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Quantum double suspension and spectral triples

Partha Sarathi Chakraborty [*,1], S. Sundar

The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India

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Abstract

In this paper we are concerned with the construction of a general principle that will allow us to produce regular spectral triples with finite and simple dimension spectrum. We introduce the notion of weak heat kernel asymptotic expansion (WHKAE) property of a spectral triple and show that the weak heat kernel asymptotic expansion allows one to conclude that the spectral triple is regular with finite simple dimension spectrum. The usual heat kernel expansion implies this property. The notion of quantum double suspension of a C^*-algebra was introduced by Hong and Szymanski. Here we introduce the quantum double suspension of a spectral triple and show that the WHKAE is stable under quantum double suspension. Therefore quantum double suspending compact Riemannian spin manifolds iteratively we get many examples of regular spectral triples with finite simple dimension spectrum. This covers all the odd-dimensional quantum spheres. Our methods also apply to the case of noncommutative torus.

Keywords: Local index formula; Regularity; Dimension spectrum; Heat kernel expansion; Quantum double suspension

1. Introduction

Since its inception index theorems play a central role in noncommutative geometry. Here spaces are replaced by explicit K-cycles or finitely summable Fredholm modules. Through index pairing they pair naturally with K-theory. In the foundational paper [6] Alain Connes introduced cyclic cohomology as a natural recipient of a Chern character homomorphism assigning cyclic

* Corresponding author.
E-mail addresses: parthac@imsc.res.in (P.S. Chakraborty), ssundar@imsc.res.in (S. Sundar).
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cocycles to finitely summable Fredholm modules. Then the index pairing is computed by the pairing of cyclic cohomology with K-theory. But finitely summable Fredholm modules occur as those associated with spectral triples and it was desirable to have cyclic cocycles given directly in terms of spectral data, that will compute the index pairing. This was achieved by Connes and Moscovici in [8]. Let us briefly recall their local index formula henceforth to be abbreviated as LIF. One begins with a spectral triple, i.e., a Hilbert space $\mathcal{H}$, an involutive subalgebra $\mathcal{A}$ of the algebra of bounded operators on $\mathcal{H}$ and a self adjoint operator D with compact resolvent. It is further assumed that the commutators $[D,\mathcal{A}]$ give rise to bounded operators. Such a triple is finitely summable if $|D|^{-p}$ is trace class for some positive p. The spectral triple is said to be regular if both $\mathcal{A}$ and $[D,\mathcal{A}]$ are in the domains of δ^n for all $n \geqslant 0$, where δ is the derivation $[|D|,.]$. One says that the spectral triple has dimension spectrum Σ, if for every element b in the smallest algebra $\mathcal{B}$ containing $\mathcal{A}$, $[D,\mathcal{A}]$ and closed under the derivation δ, the associated zeta function $\zeta_b(z) = Tr\, b|D|^{-z}$ a priori defined on the right half plane $\Re(z) > p$ admits a meromorphic extension to whole of complex plane with poles contained in Σ. If a spectral triple is regular and has discrete dimension spectrum then given any n-tuple of non-negative integers $k_1, k_2, \ldots, k_n$ one can consider multilinear functionals $\phi_{\mathbf{k},n}$ defined by $\phi_{\mathbf{k},n}(a_0, a_1, \ldots, a_n) = Res_{z=0}\, Tr\, a_0[D,a_1]^{(k_1)}[D,a_2]^{(k_2)} \cdots [D,a_n]^{(k_n)}|D|^{-n-2|k|-z}$, where $T^{(r)}$ stands for the r-fold commutator $[D^2,[D^2,[\cdots[D^2,T]\cdots]]]$ and $|k| = k_1 + \cdots + k_n$. In the local index formula the components of the local Chern character in the (b,B)-bicomplex is expressed as a sum $\sum_{\mathbf{k}} c_{\mathbf{k},n}\phi_{\mathbf{k},n}$, where the summation is over all n-tuples of non-negative integers and $c_{\mathbf{k},n}$'s are some universal constants independent of the particular spectral triple under consideration. Note that Remark II.1 in p. 63 of [8] says that if we consider the Dirac operator associated with a closed Riemannian spin manifold then $\phi_{\mathbf{k},n}$'s are zero for $|k| \neq 0$. Therefore most of the terms in the local Chern character are visible in truely noncommutative cases and hence should be interpreted as a signature of noncommutativity. To have a better understanding of the contribution of these terms it is desirable to have examples where these terms survive. In the foliations example the contribution of these terms becomes overwhelming and tackling them lead to new organizational principles of cyclic theory [9]. So the task of illustrating the LIF in simpler examples remained open. The first simple illustration was given by Connes in [7]. This was extended to odd-dimensional quantum spheres by Pal and Sundar in [14]. But to have a good grasp of the formula it is essential to have a systematic family of examples where one can verify the hypothesis of regularity and discreteness of the dimension spectrum. To our knowledge only Higson [10] made an attempt to this effect and gave a general scheme for verifying the meromorphic continuation. Here in this article we are also primarily concerned with the goal of developing a general procedure that will allow us to construct regular spectral triples with finite dimension spectrum from known examples. As in [10] we also draw inspiration from the classical situation. We show that a hypothesis similar to, but weaker than the heat kernel expansion which we call weak heat kernel asymptotic expansion implies regularity and discreteness of the dimension spectrum. The usual heat kernel expansion implies the weak heat kernel expansion. More importantly we show that the weak heat kernel expansion is stable under quantum double suspension, a notion introduced in [12]. Therefore by iteratively quantum double suspending compact Riemannian manifolds we get examples of noncommutative geometries which are regular with finite dimension spectrum. We show noncommutative torus satisfies the weak heat kernel expansion and there by satisfies regularity and discreteness of it's dimension spectrum.

Organization of the paper is as follows. In Section 2 we recall the basics of Mellin transform and asymptotic expansions. In the next section we introduce the weak heat kernel expansion property and show that this implies regularity and finiteness of the dimension spectrum. We also show that the usual heat kernel expansion implies the weak heat kernel expansion. In the next section we recall the notion of quantum double suspension and show that weak kernel expansion is stable under quantum double suspension. The weak heat kernel expansion of noncommutative torus is also established. In the final section we do a topological version of the theory relevant for applications in quantum homogeneous spaces. In particular we obtain the regularity and dimension spectrum of the odd-dimensional quantum spheres. This gives a conceptual explanation of the results obtained in [14].

2. Asymptotic expansions and the Mellin transform

In this section for reader's convenience we have recalled some well-known facts about Mellin transforms [16]. We begin with a few basic facts about asymptotic expansions. Let $\phi : (0,\infty) \to \mathbb{C}$ be a continuous function. We say that ϕ has an asymptotic power series expansion near 0 if there exists a sequence $(a_r)_{r=0}^{\infty}$ of complex numbers such that given N there exist $\epsilon, M > 0$ such that if $t \in (0,\epsilon)$

$$\left| \phi(t) - \sum_{r=0}^{N} a_r t^r \right| \leqslant M t^{N+1}.$$

We write $\phi(t) \sim \sum_0^\infty a_r t^r$ as $t \to 0+$. Note that the coefficients (a_r) are unique. For,

$$a_N = \lim_{t \to 0+} \frac{\phi(t) - \sum_{r=0}^{N-1} a_r t^r}{t^N}. \tag{2.1}$$

If $\phi(t) \sim \sum_{r=0}^\infty a_r t^r$ as $t \to 0+$ then ϕ can be extended continuously to $[0,\infty)$ simply by letting $\phi(0) := a_0$.

Let X be a topological space and $F : [0,\infty) \times X \to \mathbb{C}$ be a continuous function. Suppose that for every $x \in X$, the function $t \to F(t,x)$ has an asymptotic expansion near 0

$$F(t,x) \sim \sum_{r=0}^{\infty} a_r(x) t^r. \tag{2.2}$$

Let $x_0 \in X$. We say that expansion (2.2) is uniform at x_0 if given N there exist an open set $U \subset [0,\infty) \times X$ containing $(0,x_0)$ and an $M > 0$ such that for $(t,x) \in U$ one has

$$\left| F(t,x) - \sum_{r=0}^{N} a_r(x) t^r \right| \leqslant M t^{N+1}.$$

We say that expansion (2.2) is uniform if it is uniform at every point of X.

Proposition 2.1. *Let X be a topological space and $F : [0,\infty) \times X \to \mathbb{C}$ be a continuous function. Suppose that F has a uniform asymptotic power series expansion*

$$F(t,x) \sim \sum_{r=0}^{\infty} a_r(x) t^r.$$

Then for every $r \geqslant 0$, the function a_r is continuous.

Proof. It is enough to show that the function a_0 is continuous. Let $x_0 \in X$ be given. Since the expansion of F is uniform at x_0, it follows that there exist an open set U containing x_0 and $\delta, M > 0$ such that

$$\left|F(t,x) - a_0(x)\right| \leqslant Mt \quad \text{for } t < \delta \text{ and } x \in U. \tag{2.3}$$

Let $F_n(x) := F(\frac{1}{n}, x)$. Then Eq. (2.3) says that F_n converges uniformly to a_0 on U. Hence a_0 is continuous on U and hence at x_0. This completes the proof. □

The following two lemmas are easy to prove and we leave the proof to the reader.

Lemma 2.2. *Let X, Y be topological spaces. Let $F : [0,\infty) \times X \to \mathbb{C}$ and $G : [0,\infty) \times Y \to \mathbb{C}$ be continuous. Suppose that F and G has uniform asymptotic power series expansion. Then the function $H : [0,\infty) \times X \times Y \to \mathbb{C}$ defined by $H(t,x,y) := F(t,x)G(t,y)$ has uniform asymptotic power series expansion.*

Moreover if

$$F(t,x) \sim \sum_{r=0}^{\infty} a_r(x)t^r \quad \textit{and} \quad G(t,y) \sim \sum_{r=0}^{\infty} b_r(y)t^r,$$

then

$$H(t,x,y) \sim \sum_{r=0}^{\infty} c_r(x,y)t^r,$$

where

$$c_r(x,y) := \sum_{m+n=r} a_m(x)b_n(y).$$

Lemma 2.3. *Let $\phi : [1,\infty) \to \mathbb{C}$ be a continuous function. Suppose that for every N,*

$$\sup_{t \in [1,\infty)} \left|t^N \phi(t)\right| < \infty.$$

Then the function $s \mapsto \int_1^{\infty} \phi(t)t^{s-1}\,dt$ is entire.

2.1. The Mellin transform

In this section we recall the definition of the Mellin transform of a function defined on $(0,\infty)$ and analyse the relationship between the asymptotic expansion of a function and the meromorphic continuation of its Mellin transform. Let us introduce some notations. We say that a function $\phi : (0,\infty) \to \mathbb{C}$ is of rapid decay near infinity if for every $N > 0$, $\sup_{t \in [1,\infty)} |t^N \phi(t)|$ is finite. We let $\mathcal{M}_\infty$ to be the set of continuous complex valued functions on $(0,\infty)$ which has rapid decay near infinity. For $p \in \mathbb{R}$, we let

$$\mathcal{M}_p\big((0,1]\big) := \Big\{\phi : (0,1] \to \mathbb{C}\text{: } \phi \text{ is continuous and } \sup_{t\in(0,1]} t^p\big|\phi(t)\big| < \infty\Big\},$$
$$\mathcal{M}_p := \big\{\phi \in \mathcal{M}_\infty\text{: } \phi|_{(0,1]} \in \mathcal{M}_p\big((0,1]\big)\big\}.$$

Note that if $p \leqslant q$ then $\mathcal{M}_p \subset \mathcal{M}_q$ and $M_p((0,1]) \subset M_q((0,1])$.

Definition 2.4. Let $\phi : (0,\infty) \to \mathbb{C}$ be a continuous function. Suppose that $\phi \in \mathcal{M}_p$ for some p. Then the Mellin transform of ϕ, denoted $M\phi$, is defined as follows: For $Re(s) > p$,

$$M\phi(s) := \int_0^\infty \phi(t)t^{s-1}\,dt.$$

One can show that if $\phi \in \mathcal{M}_p$ then $M\phi$ is analytic on the right half plane $Re(s) > p+2$. Also if $\phi \in \mathcal{M}_p((0,1])$ then $s \mapsto \int_0^1 \phi(t)t^{s-1}$ is analytic on $Re(s) > p+2$.

For $a < b$ and $K > 0$, let $H_{a,b,K} := \{\sigma + it\text{: } a \leqslant \sigma \leqslant b,\ |t| > K\}$.

Definition 2.5. Let F be a meromorphic function on the entire complex plane with simple poles lying inside the set of integers. We say that F has decay of order $r \in \mathbb{N}$ along the vertical strips if the function $s \mapsto s^r F(s)$ is bounded on $H_{a,b,K}$ for every $a < b$ and $K > 0$. We say that F is of rapid decay along the vertical strips if F has decay of order r for every $r \in \mathbb{N}$.

Proposition 2.6. *Let $\phi : (0,\infty) \to \mathbb{C}$ be a continuous function of rapid decay. Assume that $\phi(t) \sim \sum_0^\infty a_r t^r$ as $t \to 0+$. Then we have the following.*

1. *The function $\phi \in \mathcal{M}_0$.*
2. *The Mellin transform $M\phi$ of ϕ extends to a meromorphic function to the whole of complex plane with simple poles in the set of negative integers* $\{0,-1,-2,-3,\ldots\}$.
3. *The residue of $M\phi$ at $s = -r$ is given by $Res_{s=-r}M\phi(s) = a_r$.*
4. *The meromorphic continuation of the Mellin transform $M\phi$ has decay of order* 0 *along the vertical strips.*

Proof. By definition it follows that $\phi \in \mathcal{M}_0$. Since ϕ has rapid decay at infinity, by Lemma 2.3, it follows that the function $s \mapsto \int_1^\infty \phi(t)t^{s-1}\,dt$ is entire. Thus modulo a holomorphic function $M\phi(s) \equiv \int_0^1 \phi(t)t^{s-1}$. For $N \in \mathbb{N}$, let $R_N(t) := \phi(t) - \sum_{r=0}^N a_r t^r$. Thus modulo a holomorphic function, we have

$$M\phi(s) \equiv \sum_{r=0} \frac{a_r}{s+r} + \int_0^1 R_N(t)t^{s-1}\,dt.$$

As $R_N \in \mathcal{M}_{-(N+1)}((0,1])$ the function $s \mapsto \int_0^1 R_N(t)t^{s-1}\,dt$ is holomorphic on $Re(s) > -N+1$. Thus on $Re(s) > -N+1$, modulo a holomorphic function, one has

$$M\phi(s) \equiv \sum_{r=0}^N \frac{a_r}{s+r}. \tag{2.4}$$

This shows that $M\phi$ admits a meromorphic continuation to the whole of complex plane and has simple poles lying in the set of negative integers $\{0, -1, -2, \ldots\}$. Also (3) follows from Eq. (2.4).

Let $a < b$ and $K > 0$ be given. Choose $N \in \mathbb{N}$ such that $N + a > 0$. Then one has

$$M\phi(s) = \sum_{r=0}^{N} \frac{a_r}{s+r} + \int_0^1 R_N(t)t^{s-1}\,dt + \int_1^\infty \phi(t)t^{s-1}\,dt.$$

As the function $s \mapsto \frac{1}{s+r}$ is bounded for every $r \geqslant 0$ on $H_{a,b,K}$, it is enough to show that the functions $\psi(s) := \int_0^1 R_N(t)t^{s-1}\,dt$ and $\chi(s) := \int_1^\infty \phi(t)t^{s-1}\,dt$ are bounded on $H_{a,b,K}$.

By definition of the asymptotic expansion, it follows that there exists an $M > 0$ such that $|R_N(t)| \leqslant Mt^{N+1}$. Hence for $s := \sigma + it \in H_{a,b,K}$,

$$\left|\psi(s)\right| \leqslant \frac{M}{\sigma + N + 1} \leqslant \frac{M}{a + N + 1} \leqslant M.$$

Thus ψ is bounded on $H_{a,b,K}$.

Now for $s := \sigma + it \in H_{a,b,K}$, we have

$$\left|\chi(s)\right| \leqslant \int_1^\infty \left|\phi(t)\right| t^{\sigma-1}\,dt \leqslant \int_1^\infty \left|\phi(t)\right| t^{b-1}\,dt.$$

Since ϕ is of rapid decay, the integral $\int_1^\infty |\phi(t)|t^{a-1}\,dt$ is finite. Hence χ is bounded on $H_{a,b,K}$. This completes the proof. $\square$

Corollary 2.7. *Let $\phi : (0,\infty) \mapsto \mathbb{C}$ be a smooth function. Assume that for every n, the n-th derivative $\phi^{(n)}$ has rapid decay at infinity and admits an asymptotic power series expansion near* 0.

(1) *For every n, the Mellin transform $M\phi^{(n)}$ of $\phi^{(n)}$ extends to a meromorphic function to the whole of complex plane with simple poles in the set of negative integers* $\{0, -1, -2, -3, \ldots\}$.
(2) *The meromorphic continuation of the Mellin transform $M\phi$ is of rapid decay along the vertical strips.*

Proof. (1) follows from Proposition 2.6. To prove (2), observe that $M\phi'(s+1) = -sM\phi(s)$. For $Re(s) \gg 0$,

$$\begin{aligned} M\phi'(s+1) &:= \int_0^\infty \phi'(t)t^s\,dt \\ &= -\int_0^\infty s\phi(t)t^{s-1}\,dt \quad \text{(follows from integration by parts)} \\ &= -sM\phi(s). \end{aligned}$$

As $M\phi'$ and $M\phi$ are meromorphic, it follows that $M\phi'(s+1) = -sM\phi(s)$. Now a repeated application of this equation gives

$$M\phi(s) := (-1)^n \frac{M\phi^{(n)}(s+n)}{s(s+1)\cdots(s+n-1)}. \tag{2.5}$$

Now let $a < b$, $K > 0$ and $r \in \mathbb{N}$ be given. Now (3) of Proposition 2.6 applied to $\phi^{(r)}$, together with Eq. (2.5), implies that the function $s \mapsto s^r M\phi(s)$ is bounded on $H_{a,b,K}$. This completes the proof. □

The following proposition shows how to pass from the decay properties of the Mellin transform of a function to the asymptotic expansion property of the function.

Proposition 2.8. *Let $\phi \in \mathcal{M}_p$ for some p. Assume that the Mellin transform $M\phi$ is meromorphic on the entire complex plane with poles lying in the set of negative integers $\{0, -1, -2, \ldots\}$. Suppose that the meromorphic continuation of the Mellin transform $M\phi$ is of rapid decay along the vertical strips. Then the function ϕ has an asymptotic expansion near* 0.

Moreover if $a_r := Res_{s=-r} M\phi(s)$ then $\phi(t) \sim \sum_{r=0}^{\infty} a_r t^r$ near 0.

Proof. The proof is a simple application of the inverse Mellin transform. Let $M \gg 0$. Then one has the following inversion formula.

$$\phi(t) = \int_{M-i\infty}^{M+\infty} M\phi(s) t^{-s}\, ds.$$

Define $F_t(s) := M\phi(s)t^{-s}$. Suppose $N \in \mathbb{N}$ be given. Let $\sigma \in (-N-1, -N)$ be given. For every $A > 0$, by Cauchy's integral formula, we have

$$\int_{M-iA}^{M+iA} F_t(s)\, ds + \int_{M+iA}^{\sigma+iA} F_t(s)\, ds + \int_{\sigma+iA}^{\sigma-iA} F_t(s)\, ds + \int_{\sigma-iA}^{M-iA} F_t(s)\, ds$$
$$= \sum_{r=0}^{N} Res_{s=-r} F_t(s). \tag{2.6}$$

For a fixed t, F_t has rapid decay along the vertical strips. Thus when $A \to \infty$ the second and fourth integrals in Eq. (2.6) vanishes and we obtain the following equation

$$\phi(t) - \sum_{r=0}^{N} a_r t^r = \int_{\sigma-i\infty}^{\sigma+i\infty} M\phi(s) t^{-s}\, ds. \tag{2.7}$$

But $M\phi(\sigma + it)$ has rapid decay in t. Let $M_\sigma := \int_{-\infty}^{\infty} |M\phi(\sigma + it)|\, dt$. Then Eq. (2.7) implies that

$$\left|\phi(t) - \sum_{r=0}^{N} a_r t^r\right| \leqslant M_\sigma t^{-\sigma} \leqslant M_\sigma t^N \quad \text{for } t \leqslant 1.$$

Thus we have shown that for every N, $R_N(t) := \phi(t) - \sum_{r=0}^{N} a_r t^r = O(t^N)$ as $t \to 0$ and hence $R_{N-1}(t) = R_N(t) + a_N t^N = O(t^N)$ as $t \to 0$. This completes the proof. □

3. The weak heat kernel asymptotic expansion property and the dimension spectrum of spectral triples

In this section we consider a property of spectral triples which we call the weak heat kernel asymptotic expansion property. We show that a spectral triple having the weak heat kernel asymptotic expansion property is regular and has finite dimension spectrum lying in the set of positive integers.

Definition 3.1. Let $(\mathcal{A}, \mathcal{H}, D)$ be a $p+$ summable spectral triple for a C^*-algebra A where $\mathcal{A}$ is a dense $*$-subalgebra of A. We say that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has the weak heat kernel asymptotic expansion property if there exists a $*$-subalgebra $\mathcal{B} \subset B(\mathcal{H})$ such that:

(1) The algebra $\mathcal{B}$ contains $\mathcal{A}$.
(2) The unbounded derivation $\delta := [|D|, .]$ leaves $\mathcal{B}$ invariant. Also the unbounded derivation $d := [D, .]$ maps $\mathcal{A}$ into $\mathcal{B}$.
(3) The algebra $\mathcal{B}$ is invariant under the left multiplication by F where $F := sign(D)$.
(4) For every $b \in \mathcal{B}$, the function $\tau_{p,b} : (0, \infty) \mapsto \mathbb{C}$ defined by $\tau_{p,b}(t) = t^p\, Tr(be^{-t|D|})$ has an asymptotic power series expansion.

If the algebra $\mathcal{A}$ is unital and the representation of $\mathcal{A}$ on $\mathcal{H}$ is unital then (3) can be replaced by the condition $F \in \mathcal{B}$. The next proposition proves that an odd spectral triple that has the heat kernel asymptotic expansion property is regular and has simple dimension spectrum.

Theorem 3.2. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a $p+$ summable spectral triple which has the weak heat kernel asymptotic expansion property. Then the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is regular and has finite simple dimension spectrum. Moreover the dimension spectrum is contained in $\{1, 2, \ldots, p\}$.*

Proof. Let $\mathcal{B} \subset B(\mathcal{H})$ be a $*$-algebra for which (1)–(4) of Definition 3.1 is satisfied. The fact that $\mathcal{B}$ satisfies (1) and (2) implies that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is regular. First we assume that D is invertible. Let $b \in \mathcal{B}$ be given.

Since $|D|^{-q}$ is trace class for $q > p$, it follows that for every $N > p$ there exists an $M > 0$ such that $Tr(e^{-t|D|}) \leqslant Mt^{-N}\, Tr(|D|^{-N})$. Now for $1 \leqslant t < \infty$ and $N \geqslant p$ one has

$$\begin{aligned} \left|Tr\left(be^{-t|D|}\right)\right| &\leqslant \|b\|\, Tr\left(e^{-t|D|}\right) \\ &\leqslant \|b\| M t^{-N}\, Tr\left(|D|^{-N}\right). \end{aligned}$$

Thus the function $t \mapsto Tr(be^{-t|D|})$ is of rapid decay near infinity. Now observe that for $Re(s) \gg 0$

$$Tr\big(b|D|^{-s}\big) = \frac{1}{\Gamma(s)} \int_0^\infty Tr\big(be^{-t|D|}\big) t^{s-1}\, dt. \tag{3.8}$$

By assumption the function $\phi(t) := t^p\, Tr(be^{-t|D|})$ has an asymptotic power series expansion near 0. By Eq. (3.8), it follows that $M\phi(s) = \Gamma(s+p)\, Tr(b|D|^{-s-p})$. Now Proposition 2.6 implies that the function $s \mapsto \Gamma(s)\, Tr(b|D|^{-s})$ is meromorphic with simple poles lying inside $\{n \in \mathbb{Z}\colon n \leqslant p\}$. As $\frac{1}{\Gamma(s)}$ is entire and has simple zeros at $\{k\colon k \leqslant 0\}$, it follows that the function $s \to Tr(b|D|^{-s})$ is meromorphic and has simple poles with poles lying in $\{1, 2, \ldots, p\}$.

Suppose D is not invertible. Let P denote the projection onto the kernel of D which is finite dimensional. Let $D' := D + P$ and b be an element in $\mathcal{B}^\infty$. Now note that

$$Tr\big(be^{-t|D'|}\big) = Tr(PbP)e^{-t} + Tr\big(be^{-t|D|}\big).$$

Hence the function $t \to t^p\, Tr(be^{-t|D'|})$ has an asymptotic power series expansion. Thus the function $s \to Tr(b|D'|^{-s})$ is meromorphic with simple poles lying in $\{1, 2, \ldots, p\}$. Observe that for $Re(s) \gg 0$, $Tr(b|D'|^{-s}) = Tr(b|D|^{-s})$. Hence the function $s \to Tr(b|D|^{-s})$ is meromorphic with simple poles lying in $\{1, 2, \ldots, p\}$. This completes the proof. $\square$

Remark 3.3. If $Tr(be^{-t|D|}) \sim \sum_{r=-p}^{\infty} a_r(b)t^r$ then (3) of Proposition 2.6 implies that

$$Res_{z=k}\, Tr\big(b|D|^{-z}\big) = \frac{1}{k!} a_{-k}(b) \quad \text{for } 1 \leqslant k \leqslant p,$$
$$Tr\big(b|D|^{-z}\big)_{z=0} = a_0(b).$$

Remark 3.4. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple which has the weak heat kernel asymptotic expansion property. Then the dimension spectrum Σ is finite and lies in the set of positive integers. We call the greatest element in the dimension spectrum as the dimension of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. If Σ is empty we set the dimension to be 0.

Now in the next proposition we show that the usual heat kernel asymptotic expansion implies the weak heat kernel asymptotic expansion.

Theorem 3.5. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a $p+$ summable spectral triple for a C^*-algebra A. Suppose that $\mathcal{B}$ is a $*$-subalgebra of $B(\mathcal{H})$ satisfying* (1)–(4) *of Definition* 3.1. *Assume that for every $b \in \mathcal{B}$, the function $\sigma_{p,b} : (0,\infty) \to \mathbb{C}$ defined by $\sigma_{p,b}(t) := t^p\, Tr(be^{-t^2D^2})$ has an asymptotic power series expansion.*

Then for every $b \in \mathcal{B}$, the function $\tau_{p,b} : t \mapsto t^p\, Tr(be^{-t|D|})$ has an asymptotic power series expansion.

Proof. It is enough to consider the case where D is invertible. Let $b \in \mathcal{B}$ be given. Let ψ denote the Mellin transform of the function $t \mapsto Tr(be^{-t^2D^2})$ and χ denote the Mellin transform of the function $t \mapsto Tr(be^{-t|D|})$. Then a simple change of variables shows that

$\psi(s) = \frac{\Gamma(\frac{s}{2})}{2} Tr(b|D|^{-s})$. But then $\chi(s) = \Gamma(s)\, Tr(b|D|^{-s})$. Thus we obtain the equation

$$\chi(s) = \frac{2\Gamma(s)}{\Gamma(\frac{s}{2})}\psi(s).$$

But we have following duplication formula for the gamma function

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\,\Gamma(2s).$$

Hence one has

$$\chi(s) = \frac{1}{\sqrt{\pi}}2^s\Gamma\left(\frac{s+1}{2}\right)\psi(s).$$

Now Proposition 2.6 implies that ψ has decay of order 0 along the vertical strips and has simple poles lying inside $\{n \in \mathbb{Z}\colon n \leqslant p\}$. Since the gamma function has rapid decay along the vertical strips, it follows that χ has rapid decay along the vertical strips and has poles lying in $\{n \in \mathbb{Z}\colon n \leqslant p\}$. If $\tilde{\chi}$ denotes the Mellin transform of $\tau_p(.,b)$ then $\tilde{\chi}(s) = \chi(s+p)$. Hence $\tilde{\chi}$ has rapid decay along the vertical strips and has poles lying in the set of negative integers. Now Proposition 2.8 implies that the map $t \to t^p\, Tr(be^{-t|D|})$ has an asymptotic power series expansion near 0. This completes the proof. □

4. Stability of the weak heat kernel expansion property and the quantum double suspension

Let us recall the definition of the quantum double suspension of a unital C^*-algebra. The quantum double suspension is first defined in [12] and our equivalent definition is as in [13]. Let us fix some notations. We denote the left shift on $\ell^2(\mathbb{N})$ by S which is defined on the standard orthonormal basis (e_n) as $Se_n = e_{n-1}$ and p denote the projection $|e_0\rangle\langle e_0|$. The number operator on $\ell^2(\mathbb{N})$ is denoted by N and defined as $Ne_n := ne_n$. We denote the C^*-algebra generated by S in $B(\ell^2(\mathbb{N}))$ by $\mathscr{T}$ which is the Toeplitz algebra. Note that $SS^* = 1$ and $p = 1 - S^*S$. Let $\sigma : \mathscr{T} \to C(\mathbb{T})$ be the symbol map which sends S to the generating unitary z. Then one has the following exact sequence

$$0 \to \mathcal{K} \to \mathscr{T} \xrightarrow{\sigma} C(\mathbb{T}) \to 0.$$

Definition 4.1. Let A be a unital C^*-algebra. Then the quantum double suspension of A denoted $\Sigma^2(A)$ is the C^*-algebra generated by $A \otimes p$ and $1 \otimes S$ in $A \otimes \mathscr{T}$.

Let A be a unital C^*-algebra. One has the following exact sequence.

$$0 \to A \otimes \mathcal{K}\big(\ell^2(\mathbb{N})\big) \to \Sigma^2(A) \xrightarrow{\rho} C(\mathbb{T}) \to 0$$

where ρ is just the restriction of $1 \otimes \sigma$ to $\Sigma^2(A)$.

Remark 4.2. It can be easily shown that $\Sigma^2(C(\mathbb{T})) = C(SU_q(2))$ and more generally one can show that $\Sigma^2(C(S_q^{2n-1})) = C(S_q^{2n+1})$. We refer to [12] or Lemma 3.2 of [14] for the proof. Thus the odd-dimensional quantum spheres can be obtained from the circle $\mathbb{T}$ by applying the quantum double suspension recursively.

Let $\mathcal{A}$ be a dense $*$-subalgebra of a C^*-algebra A. Define

$$\Sigma^2_{alg}(\mathcal{A}) := span\big\{a \otimes k, 1 \otimes S^n, 1 \otimes S^{*m}\colon a \in \mathcal{A},\ k \in \mathcal{S}\big(\ell^2(\mathbb{N})\big),\ n, m \geqslant 0\big\}$$

where $\mathcal{S}(\ell^2(\mathbb{N})) := \{(a_{mn})\colon \sum_{m,n}(1+m+n)^p|a_{mn}| < \infty \text{ for every } p\}$.

Then $\Sigma^2_{alg}(\mathcal{A})$ is just the $*$-algebra generated by $\mathcal{A} \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$ and $1 \otimes S$. Clearly $\Sigma^2_{alg}(\mathcal{A})$ is a dense subalgebra of $\Sigma^2(A)$.

Definition 4.3. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple and denote the sign of the operator D by F. Then the spectral triple $(\Sigma^2_{alg}(\mathcal{A}), \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2(D) := (F \otimes 1)(|D| \otimes 1 + 1 \otimes N))$ is called the quantum double suspension of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

4.1. Stability of the weak heat kernel expansion

We consider the stability of the weak heat kernel expansion under quantum double suspension. First observe that the following are easily verifiable.

(1) The spectral triple $(\mathcal{S}(\ell^2(\mathbb{N})), \ell^2(\mathbb{N}), N)$ has the weak heat kernel asymptotic expansion with dimension 0.
(2) Let $(\mathcal{A}_i, \mathcal{H}_i, D_i)$ be a spectral triple with the weak heat kernel asymptotic expansion property with dimension p_i for $1 \leqslant i \leqslant n$. Then the spectral triple $(\bigoplus_{i=1}^n \mathcal{A}_i, \bigoplus_{i=1}^n \mathcal{H}_i, \bigoplus_{i=1}^n D_i)$ has the weak heat kernel expansion property with dimension $p := max\{p_i\colon 1 \leqslant i \leqslant n\}$.
(3) If $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple with the weak heat kernel asymptotic expansion property and has dimension p then $(\mathcal{A}, \mathcal{H}, |D|)$ also has the weak heat kernel asymptotic expansion with the same dimension p.
(4) Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with the weak heat kernel asymptotic expansion property with dimension p. Then the amplification $(\mathcal{A} \otimes 1, \mathcal{H} \otimes \ell^2(\mathbb{N}), |D| \otimes 1 + 1 \otimes N)$ also has the asymptotic expansion property with dimension $p + 1$.

We start by proving the stability of the weak heat kernel expansion under tensoring by compacts.

Proposition 4.4. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with the weak heat kernel asymptotic expansion property of dimension p. Then $(\mathcal{A} \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N})), \mathcal{H} \otimes \ell^2(\mathbb{N}), D_0 := (F \otimes 1)(|D| \otimes 1 + 1 \otimes N))$ also has the weak heat kernel asymptotic expansion property with dimension p.*

Proof. Let $\mathcal{B} \subset B(\mathcal{H})$ be a $*$-subalgebra for which (1)–(4) of Definition 3.1 are satisfied. We denote $\mathcal{B} \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$ by $\mathcal{B}_0$. We show that $\mathcal{B}_0$ satisfies (1)–(4) of Definition 3.1. Clearly (1) holds.

We denote the unbounded derivation $[|D_0|, .]$, $[|D|, .]$ and $[N, .]$ by δ_{D_0}, δ_D and δ_N respectively. By assumption δ_D leaves $\mathcal{B}$ invariant. Clearly $\mathcal{B} \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$ is contained in the domain of δ_{D_0} and $\delta_{D_0} = \delta_D \otimes 1 + 1 \otimes \delta_N$ on $\mathcal{B} \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$. Similarly one can show that the unbounded derivation $[D_0, .]$ maps $\mathcal{A} \otimes_{alg} S(\ell^2(\mathbb{N}))$ into $\mathcal{B}_0$ invariant.

As $F_0 := sign(D_0) = F \otimes 1$, (3) is clear. Now (4) follows from Lemma 2.2 and the equality $t^p\, Tr(b \otimes k)e^{-t|D_0|} = t^p\, Tr(be^{-t|D|})\, Tr(ke^{-tN})$. This completes the proof. □

Now we consider the stability of the heat kernel asymptotic expansion under the double suspension.

Theorem 4.5. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with the weak heat kernel asymptotic expansion property of dimension p. Assume that the algebra $\mathcal{A}$ is unital and the representation on $\mathcal{H}$ is unital. Then the spectral triple $(\Sigma^2(\mathcal{A}), \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2(D))$ also has the weak heat kernel asymptotic expansion property with dimension $p + 1$.*

Proof. We denote $\Sigma^2(D)$ by D_0. Let $\mathcal{B}$ be a $*$-subalgebra of $B(\mathcal{H})$ for which (1)–(4) of Definition 3.1 are satisfied. For $f = \sum_n \lambda_n z^n \in C^\infty(\mathbb{T})$, we let $\sigma(f) := \sum_{n\geqslant 0} \lambda_n S^n + \sum_{n>0} \lambda_{-n} S^{*n}$. We denote the projection $\frac{1+F}{2}$ by P. We let $\mathcal{B}_0$ to denote the algebra $\mathcal{B} \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$ as in Proposition 4.4. As in Proposition 4.4, we let $\delta_{D_0}, \delta_D, \delta_N$ to denote the unbounded derivations $[|D_0|, .]$, $[|D|, .]$ and $[N, .]$ respectively. Define

$$\tilde{\mathcal{B}} := \bigl\{b + P \otimes \sigma(f) + (1 - P) \otimes \sigma(g)\colon\ b \in \mathcal{B}_0,\ f, g \in C^\infty(\mathbb{T})\bigr\}.$$

Now it is clear that $\tilde{\mathcal{B}}$ satisfies (1) of Definition 3.1.

We have already shown in Proposition 4.4 that $\mathcal{B}_0$ is closed under δ_{D_0} and $d_0 := [D_0, .]$ maps $\mathcal{A} \otimes S(\ell^2(\mathbb{N}))$ into $\mathcal{B}_0$. Now note that

$$\begin{aligned}
\delta_{D_0}\bigl(P \otimes \sigma(f)\bigr) &= P \otimes \sigma\bigl(if'\bigr),\\
\delta_{D_0}\bigl((1 - P) \otimes \sigma(g)\bigr) &= (1 - P) \otimes \sigma\bigl(ig'\bigr),\\
\bigl[D_0, P \otimes \sigma(f)\bigr] &= P \otimes \sigma\bigl(if'\bigr),\\
\bigl[D_0, (1 - P) \otimes \sigma(g)\bigr] &= -(1 - P) \otimes \sigma\bigl(ig'\bigr).
\end{aligned}$$

Thus it follows that δ_{D_0} leaves $\tilde{\mathcal{B}}$ invariant and $d_0 := [D_0, .]$ maps $\Sigma_2(\mathcal{A})$ into $\tilde{\mathcal{B}}$.

Since $F_0 := sign(D_0) = F \otimes 1$, it follows from definition that $F_0 \in \tilde{\mathcal{B}}$. Now we show that $\tilde{\mathcal{B}}$ satisfies (4).

We have already shown in Proposition 4.4 that given $b \in \mathcal{B}_0$, the function $\tau_{p,b}(t) = t^p\, Tr(be^{-t|D_0|})$ has an asymptotic expansion. Hence the function $\tau_{p+1,b}$ has an asymptotic expansion for every $b \in \mathcal{B}_0$. Now note that

$$\tau_{p+1,P\otimes\sigma(f)}(t) = \biggl(\int f(\theta)\, d\theta\biggr) t^p\, Tr\bigl(Pe^{-t|D|}\bigr) t\, Tr\bigl(e^{-tN}\bigr), \tag{4.9}$$

$$\tau_{p+1,(1-P)\otimes\sigma(g)}(t) = \biggl(\int g(\theta)\, d\theta\biggr) t^p\, Tr\bigl((1 - P)e^{-t|D|}\bigr) t\, Tr\bigl(e^{-tN}\bigr). \tag{4.10}$$

Now recall that we have assumed that $\mathcal{A}$ is unital and hence $P \in \mathcal{B}$. Hence $t^p\, Tr(xe^{-t|D|})$ has an asymptotic power series expansion for $x \in \{P, 1 - P\}$. Thus $t\, Tr(e^{-tN})$ has an asymptotic power series expansion. From Eqs. (4.9), (4.10) and from the earlier observation that $\tau_{p+1,b}$ has an asymptotic power series expansion for $b \in \mathcal{B}_0$, it follows that for every $b \in \tilde{\mathcal{B}}$, the function $\tau_{p+1,b}$ has an asymptotic power series expansion. This completes the proof. □

4.2. Higson's differential pair and the heat kernel expansion

Now we discuss some examples of spectral triples which satisfy the weak heat kernel asymptotic expansion property. In particular we discuss the spectral triple associated to noncommutative torus and the classical spectral triple associated to a spin manifold. Let us recall Higson's notion of a differential pair as defined in [11].

Consider a Hilbert space $\mathcal{H}$ and a positive, selfadjoint and an unbounded Δ on $\mathcal{H}$. We assume that Δ has compact resolvent. For $k \in \mathbb{N}$, we let $\mathcal{H}_k$ be the domain of the operator $\Delta^{\frac{k}{2}}$. The vector space $\mathcal{H}_k$ is given a Hilbert space structure by identifying $\mathcal{H}_k$ with the graph of the operator $\Delta^{\frac{k}{2}}$. Denote the intersection $\bigcap_k \mathcal{H}_k$ by $\mathcal{H}_\infty$. An operator $T : \mathcal{H}_\infty \to \mathcal{H}_\infty$ is said to be of analytic order $\leqslant m$ where $m \in \mathbb{Z}$ if T extends to a bounded operator from $\mathcal{H}_{k+m} \to \mathcal{H}_k$ for every k. We say an operator T on $\mathcal{H}_\infty$ has analytic order $-\infty$ if T has analytic order less than $-m$ for every $m > 0$. The following definition is due to Higson [11].

Definition 4.6. Let Δ be a positive, unbounded, selfadjoint operator on a Hilbert space $\mathcal{H}$ with compact resolvent. Suppose that $\mathcal{D} := \bigcup_{p \geqslant 0} \mathcal{D}_q$ is a filtered algebra of operators on $\mathcal{H}_\infty$. The pair $(\mathcal{D}, \Delta)$ is called a differential pair if the following conditions hold.

1. The algebra $\mathcal{D}$ is invariant under the derivation $T \to [\Delta, T]$.
2. If $X \in \mathcal{D}_q$, then $[\Delta, X] \in \mathcal{D}_{q+1}$.
3. If $X \in \mathcal{D}_q$, then the analytic order of $X \leqslant q$.

Now let us recall Higson's definition of pseudodifferential operators.

Definition 4.7. Let $(\mathcal{D}, \Delta)$ be a differential pair. We denote the orthogonal projection onto the kernel of Δ by P. Then P is of finite rank as Δ has compact resolvent. Let $\Delta_1 := \Delta + P$. Then Δ_1 is invertible.

A linear operator T on $\mathcal{H}_\infty$ is called a basic pseudodifferential operator of order $\leqslant k$ is for every ℓ there exist m and $X \in \mathcal{D}_{m+k}$ such that

$$T = X \Delta_1^{-\frac{m}{2}} + R$$

where R has analytic order less than or equal to ℓ.

A finite linear combinations of basic pseudodifferential operator of order $\leqslant k$ is called a pseudodifferential operator of order $\leqslant k$.

We denote the set of pseudodifferential operators of order $\leqslant 0$ by $\Psi_0(\mathcal{D}, \Delta)$. It is proved in [11] that the pseudodifferential operators of order $\leqslant 0$ form an algebra. We need the following proposition due to Higson. Denote the derivation $T \mapsto [\Delta^{\frac{1}{2}}, T]$ by δ.

Proposition 4.8. *Let $(\mathcal{D}, \Delta)$ be a differential pair. The derivation δ leaves the algebra $\Psi_0(\mathcal{D}, \Delta)$ invariant.*

Let $(\mathcal{D}, \Delta)$ be a differential pair. Assume that $\Delta^{-\frac{r}{2}}$ is trace class for some $r > 0$. We say that the analytic dimension of $(\mathcal{D}, \Delta)$ is p if

$$p := \inf\left\{q > 0:\ \Delta^{\frac{-r}{2}} \text{ is trace class for every } r > q\right\}.$$

Let us make the following definition of the heat kernel expansion for a differential pair.

Definition 4.9. Let $(\mathcal{D}, \Delta)$ be a differential pair of analytic dimension p. We say that $(\mathcal{D}, \Delta)$ has a heat kernel expansion if for $X \in \mathcal{D}_m$, the function $t \mapsto t^{p+m}\, Tr(Xe^{-t^2\Delta})$ has an asymptotic expansion near 0.

Now we show that if $(\mathcal{D}, \Delta)$ has the heat kernel expansion then the algebra $\Psi_0(\mathcal{D}, \Delta)$ has the weak heat kernel expansion.

Proposition 4.10. *Let $(\mathcal{D}, \Delta)$ be a differential pair of analytic dimension p having the heat kernel expansion. Denote the operator $\Delta^{\frac{1}{2}}$ by $|D|$. Then for every $b \in \Psi_0(\mathcal{D}, \Delta)$, the function $t \mapsto t^p\, Tr(be^{-t|D|})$ has an asymptotic power series expansion.*

Proof. First observe that if $R : \mathcal{H}_\infty \to \mathcal{H}_\infty$ is an operator of analytic order $< -p - n - 1$ then $R|D|^{n+1}$ is trace class and hence by Taylor's series

$$Tr\left(Re^{-t|D|}\right) = \sum_{k=0}^{n} \frac{(-1)^k\, Tr(R|D|^k)}{k!} t^k + O\left(t^{n+1}\right)$$

for t near 0. Thus it is enough to show the result when $b = X\Delta_1^{-\frac{m}{2}}$. For an operator T on $\mathcal{H}_\infty$, let $\zeta_T(s) := Tr(T|D|^{-s})$. Then $\zeta_b(s) := \zeta_X(s+m)$. As in Proposition 3.5 one can show that $\Gamma(s)\zeta_X(s)$ has rapid decay along the vertical strips. Now

$$\Gamma(s)\zeta_b(s) = \frac{\Gamma(s)}{\Gamma(s+m)}\Gamma(s+m)\zeta_X(s+m).$$

Hence $\Gamma(s)\zeta_b(s)$ has rapid decay along the vertical strips. But $\Gamma(s)\zeta_b(s)$ is the Mellin transform of $Tr(be^{-t|D|})$. Hence by Proposition 2.8, it follows that $t^p\, Tr(be^{-t|D|})$ has an asymptotic power series expansion. This completes the proof. $\square$

We make use of the following proposition to prove that spectral triple associated to the NC torus and that of a spin manifold posses the weak heat kernel expansion property.

Proposition 4.11. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a finitely summable spectral triple and $\Delta := D^2$. Suppose that there exists an algebra of operators $D := \bigcup_{p\geqslant 0} \mathcal{D}_p$ such that (D, Δ) is a differential pair of analytic dimension p. Assume that $(\mathcal{D}, \Delta)$ satisfies the following*:

1. *the algebra $\mathcal{D}_0$ contains $\mathcal{A}$ and $[D, \mathcal{A}]$,*
2. *the differential pair $(\mathcal{D}, \Delta)$ has the heat kernel expansion property,*
3. *the operator $D \in \mathcal{D}_1$.*

Then the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has the weak heat kernel asymptotic expansion property.

Proof. Without loss of generality, we can assume that D is invertible. We let $\mathcal{B}$ be the algebra of pseudodifferential operators of order 0 associated to $(\mathcal{D}, \Delta)$. Now Proposition 4.8 together with the fact that $\mathcal{D}_0 \subset \mathcal{B}$ shows that $\mathcal{B}$ contains $\mathcal{A}$ and $[D, \mathcal{A}]$ and is invariant under $\delta := [|D|, .]$. Since $D \in \mathcal{D}_1$, it follows that $F := D\Delta^{\frac{-1}{2}} \in \mathcal{B}$. Now (4) of Definition 3.1 follows from Proposition 4.10. This completes the proof. □

4.3. Examples

Now we discuss some examples of spectral triples which satisfy the weak heat kernel asymptotic expansion. We start with the classical example.

Let M be a Riemannian spin manifold and $S \to M$ be a spinor bundle. We denote the Hilbert space of square integrable sections on $L^2(M, S)$ by $\mathcal{H}$. We represent $C^\infty(M)$ on $\mathcal{H}$ by multiplication operators. Let D be the Dirac operator associated with Levi–Civita connection. Then the triple $(C^\infty(M), \mathcal{H}, D)$ is a spectral triple. Then the operator D^2 is then a generalised Laplacian [1]. Let $\mathcal{D}$ denote the usual algebra of differential operators on S. Then $(\mathcal{D}, \Delta)$ is a differential pair. Moreover Proposition 2.4.6 in [1] implies that $(\mathcal{D}, \Delta)$ has the heat kernel expansion. Also $D \in \mathcal{D}_1$. Now Proposition 4.11 implies that the spectral triple $(C^\infty(M), \mathcal{H}, D)$ has the weak heat kernel asymptotic expansion.

4.3.1. The spectral triple associated to the NC torus

Let us recall the definition of the noncommutative torus which we abbreviate as NC torus. Throughout we assume that $\theta \in [0, 2\pi)$.

Definition 4.12. The C^*-algebra A_θ is defined as the universal C^*-algebra generated by two unitaries u and v such that $uv = e^{i\theta}vu$.

Define the operators U and V on $\ell^2(\mathbb{Z}^2)$ as follows

$$\begin{aligned} Ue_{m,n} &:= e_{m+1,n}, \\ Ve_{m,n} &:= e^{-in\theta}e_{m,n+1} \end{aligned}$$

where $\{e_{m,n}\}$ denotes the standard orthonormal basis on $\ell^2(\mathbb{Z}^2)$. Then it is well known that $u \to U$ and $v \to V$ give a faithful representation of the C^*-algebra A_θ.

Consider the positive selfadjoint operator Δ on $\mathcal{H} := \ell^2(\mathbb{Z}^2)$ defined on the orthonormal basis $\{e_{m,n}\}$ by $\Delta(e_{m,n}) = (m^2 + n^2)e_{m,n}$. For a polynomial $P = p(m, n)$, define the operator T_P on $\mathcal{H}_\infty$ by $T_P(e_{m,n}) := p(m, n)e_{m,n}$. The group $\mathbb{Z}^2$ acts on the algebra of polynomials as follows. For $x := (a, b) \in \mathbb{Z}^2$ and $P := p(m, n)$, define $x.P := p(m - a, n - b)$. We denote $(1, 0)$ by e_1 and $(0, 1)$ by e_2.

Note that if P is a polynomial of degree $\leqslant k$, then $T_P\Delta^{-\frac{k}{2}}$ is bounded on $Ker(\Delta)^\perp$. Thus it follows that if P is a polynomial of degree $\leqslant k$ then T_P has analytic order $\leqslant k$.

Also note that

$$\Delta_1^{\frac{k}{2}} U \Delta_1^{\frac{-k}{2}} e_{m,n} := \frac{((m+1)^2 + n^2)^{\frac{k}{2}}}{(m^2 + n^2)^{\frac{k}{2}}} e_{m+1,n} \quad \text{if } (m, n) \neq 0.$$

Thus it follows that U is of analytic order $\leqslant 0$. Similarly one can show that V is of analytic order $\leqslant 0$. Now note the following commutation relationship

$$UT_P := T_{e_1.P}U, \tag{4.11}$$

$$VT_P := T_{e_1.P}V. \tag{4.12}$$

Thus it follows that $[\Delta, U^\alpha V^\beta] = T_Q U^\alpha V^\beta$ for some degree 1 polynomial Q.

Let us define $\mathcal{D}_p := span\{T_{P_{\alpha,\beta}} U^\alpha V^\beta\colon\ deg(P_{\alpha,\beta}) \leqslant k\}$ and let $\mathcal{D} := \bigcup_p \mathcal{D}_p$. The above observations can be rephrased into the following proposition.

Proposition 4.13. *The pair* $(\mathcal{D}, \Delta)$ *is a differential pair of analytic dimension* 2.

Now we show that the differential pair $(\mathcal{D}, \Delta)$ has the heat kernel expansion.

Proposition 4.14. *The differential pair* $(\mathcal{D}, \Delta)$ *has the heat kernel expansion property.*

Proof. Let $X \in \mathcal{D}_q$ be given. It is enough to consider the case when $X := T_P U^\alpha V^\beta$. First note that $Tr(Xe^{-t\Delta}) = 0$ unless $(\alpha, \beta) = 0$. Now let $X := T_P$. Again it is enough to consider the case when P is a monomial. Let $P = p(m,n) = m^{k_1} n^{k_2}$. Now

$$Tr\big(T_P e^{-t\Delta}\big) = \bigg(\sum_{m\in\mathbb{Z}} m^{k_1} e^{-tm^2}\bigg)\bigg(\sum_{n\in\mathbb{Z}} n^{k_2} e^{-tn^2}\bigg).$$

Now the asymptotic expansion follows from applying Proposition 2.4.6 in [1] to the standard Laplacian on the circle. This completes the proof. □

Let $\mathcal{A}_\theta$ be the $*$-algebra generated by U and V. We consider the direct sum representation of A_θ on $\mathcal{H} \oplus \mathcal{H}$. Define $D := \left[\begin{smallmatrix} 0 & T_{m-in} \\ T_{m+in} & 0 \end{smallmatrix}\right]$. Then D is selfadjoint on $\mathcal{H} \oplus \mathcal{H}$ and $D^2 = \left[\begin{smallmatrix} \Delta & 0 \\ 0 & \Delta \end{smallmatrix}\right]$. It is well known that $(\mathcal{A}_\theta, \mathcal{H} \oplus \mathcal{H}, D)$ is a $2+$ summable spectral triple.

Proposition 4.15. *The spectral triple* $(\mathcal{A}_\theta, \mathcal{H} \oplus \mathcal{H}, D)$ *has the weak heat kernel asymptotic expansion property.*

Proof. Let $(\mathcal{D}, \Delta)$ be the differential pair considered in Proposition 4.13. Then the amplification $(\mathcal{D}' := M_2(\mathcal{D}), D^2)$ is a differential pair. Note that $D \in \mathcal{D}'_1$. Clearly $\mathcal{A}_\theta \in \mathcal{D}'$. Note the commutation relations

$$[T_{m\pm in}, U] = U,$$

$$[T_{m\pm in}, V] = \pm iV.$$

This implies that $[D, \mathcal{A}_\theta] \subset \mathcal{D}'_0$. Since $(\mathcal{D}, \Delta)$ has the heat kernel expansion, it follows that the differential pair $(M_2(\mathcal{D}), D^2)$ also has the heat kernel expansion. Now Proposition 4.11 implies that the spectral triple $(\mathcal{A}_\theta, \mathcal{H} \oplus \mathcal{H}, D)$ has the weak heat kernel expansion. This completes the proof. □

4.3.2. The torus equivariant spectral triple on the odd-dimensional quantum spheres

In this section we recall the spectral triple for the odd-dimensional quantum spheres given in [5]. We begin with some known facts about odd-dimensional quantum spheres. Let $q \in (0,1]$. The C^*-algebra $C(S_q^{2\ell+1})$ of the quantum sphere $S_q^{2\ell+1}$ is the universal C^*-algebra generated by elements $z_1, z_2, \dots, z_{\ell+1}$ satisfying the following relations (see [12]):

$$\begin{aligned} z_i z_j &= q z_j z_i, \quad 1 \leqslant j < i \leqslant \ell+1, \\ z_i^* z_j &= q z_j z_i^*, \quad 1 \leqslant i \neq j \leqslant \ell+1, \\ z_i z_i^* - z_i^* z_i + \left(1-q^2\right) \sum_{k>i} z_k z_k^* &= 0, \quad 1 \leqslant i \leqslant \ell+1, \\ \sum_{i=1}^{\ell+1} z_i z_i^* &= 1. \end{aligned}$$

We will denote by $\mathcal{A}(S_q^{2\ell+1})$ the $*$-subalgebra of A_ℓ generated by the z_j's. Note that for $\ell = 0$, the C^*-algebra $C(S_q^{2\ell+1})$ is the algebra of continuous functions $C(\mathbb{T})$ on the torus and for $\ell = 1$, it is $C(SU_q(2))$.

There is a natural torus group $\mathbb{T}^{\ell+1}$ action τ on $C(S_q^{2\ell+1})$ as follows. For $w = (w_1, \dots, w_{\ell+1})$, define an automorphism τ_w by $\tau_w(z_i) = w_i z_i$.

Recall that N is the number operator on $\ell^2(\mathbb{N})$ and S is the left shift on $\ell^2(\mathbb{N})$. We also use the same notation S for the left shift on $\ell^2(\mathbb{Z})$. We let $\mathcal{H}_\ell$ denote the Hilbert space $\ell^2(\mathbb{N}^\ell \times Z)$. Let $Y_{k,q}$ be the following operators on $\mathcal{H}_\ell$:

$$Y_{k,q} = \begin{cases} \underbrace{q^N \otimes \cdots \otimes q^N}_{k-1 \text{ copies}} \otimes \sqrt{1-q^{2N}} S^* \otimes \underbrace{I \otimes \cdots \otimes I}_{\ell+1-k \text{ copies}} & \text{if } 1 \leqslant k \leqslant \ell, \\ \underbrace{q^N \otimes \cdots \otimes q^N}_{\ell \text{ copies}} \otimes S^* & \text{if } k = \ell+1. \end{cases} \tag{4.13}$$

Then $\pi_\ell : z_k \mapsto Y_{k,q}$ gives a faithful representation of $C(S_q^{2\ell+1})$ on $\mathcal{H}_\ell$ for $q \in (0,1)$ (see Lemma 4.1 and Remark 4.5 [12]). We will denote the image $\pi_\ell(C(S_q^{2\ell+1}))$ by $A_\ell(q)$ or by just A_ℓ.

Let $\{e_\gamma \colon \gamma \in \Gamma_{\Sigma_\ell}\}$ be the standard orthonormal basis for $\mathcal{H}_\ell$. We recall the following theorem from [5].

Theorem 4.16. *(See [5].) Let D_ℓ be the operator $e_\gamma \to d(\gamma) e_\gamma$ on $\mathcal{H}_\ell$ where the d_γ's are given by*

$$d(\gamma) = \begin{cases} \gamma_1 + \gamma_2 + \cdots + \gamma_\ell + |\gamma_{\ell+1}| & \text{if } \gamma_{\ell+1} \geqslant 0, \\ -(\gamma_1 + \gamma_2 + \cdots + \gamma_\ell + |\gamma_{\ell+1}|) & \text{if } \gamma_{\ell+1} < 0. \end{cases}$$

Then $(\mathcal{A}(S_q^{2\ell+1}), \mathcal{H}_\ell, D_\ell)$ is a non-trivial $(\ell+1)$ summable spectral triple.

But to deduce that the spectral triple $(\mathcal{A}(S_q^{2\ell+1}), \mathcal{H}_\ell, D_\ell)$ satisfies the weak heat kernel asymptotic expansion, we need a topological version of Definition 3.1 and Proposition 4.5. We do this in the next section.

5. Smooth subalgebras and the weak heat kernel asymptotic expansion

First we recall the definition of smooth subalgebras of C^*-algebras. For an algebra A (possibly non-unital), we denote the algebra obtained by adjoining a unit to A by A^+.

Definition 5.1. Let A be a unital C^*-algebra. A dense unital $*$-subalgebra $\mathcal{A}^\infty$ is called smooth in A if:

1. The algebra $\mathcal{A}^\infty$ is a Fréchet $*$-algebra.
2. The unital inclusion $\mathcal{A}^\infty \subset A$ is continuous.
3. The algebra $\mathcal{A}^\infty$ is spectrally invariant in A i.e. if an element $a \in \mathcal{A}^\infty$ is invertible in A then $a^{-1} \in \mathcal{A}^\infty$.

Suppose A is a non-unital C^*-algebra. A dense Fréchet $*$-subalgebra $\mathcal{A}^\infty$ is said to be smooth in A if $(\mathcal{A}^\infty)^+$ is smooth in A^+.

We also assume that our smooth subalgebras satisfy the condition that if $\mathcal{A}^\infty \subset A$ is smooth then $\mathcal{A}^\infty \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N}^k)) \subset A \otimes \mathcal{K}(\ell^2(\mathbb{N}^k))$ is smooth.

Let A be a unital C^*-algebra and $\mathcal{A}^\infty$ be a smooth unital $*$-subalgebra of A. Assume that the topology on $\mathcal{A}^\infty$ is given by the countable family of seminorms $(\|\cdot\|_p)$. Let us denote the operator $1 \otimes S$ by α. Define the smooth quantum double suspension of $\mathcal{A}^\infty$ as follows

$$\Sigma^2(\mathcal{A}^\infty) := \Big\{ \sum_{j,k\in\mathbb{N}} \alpha^{*j}(a_{jk} \otimes p)\alpha^k + \sum_{k\geqslant 0} \lambda_k \alpha^k + \sum_{k>0} \lambda_{-k}\alpha^{*k} \colon a_{jk} \in \mathcal{A}^\infty,$$
$$\sum_{j,k} (1+j+k)^n \|a_{jk}\|_p < \infty,\ (\lambda_k) \text{ is rapidly decreasing} \Big\}. \tag{5.14}$$

Now let us topologize $\Sigma^2(\mathcal{A}^\infty)$ by defining a seminorm $\| \ \|_{n,p}$ for every $n, p \geqslant 0$. For an element

$$a := \sum_{j,k\in\mathbb{N}} \alpha^{*j}(a_{jk} \otimes p)\alpha^k + \sum_{k\geqslant 0} \lambda_k \alpha^k + \sum_{k>0} \lambda_{-k}\alpha^{*k}$$

in $\Sigma^2(\mathcal{A}^\infty)$ we define $\|a\|_{n,p}$ by

$$\|a\|_{n,p} := \sum_{j,k\in\mathbb{N}} \big(1+|j|+|k|\big)^n \|a_{jk}\|_p + \sum_{k\in\mathbb{Z}} \big(1+|k|\big)^n |\lambda_k|.$$

It is easily verifiable that:

1. The subspace $\Sigma^2(\mathcal{A}^\infty)$ is a dense $*$-subalgebra of $\Sigma^2(A)$.

2. The topology on $\Sigma^2(\mathcal{A}^\infty)$ induced by the seminorms $(\| \|_{n,p})$ makes $\Sigma^2(\mathcal{A}^\infty)$ a Fréchet $*$-algebra.
3. The unital inclusion $\Sigma^2(\mathcal{A}^\infty) \subset \Sigma^2(A)$ is continuous.

The next proposition proves that the Fréchet algebra $\Sigma^2(\mathcal{A}^\infty)$ is infact smooth in $\Sigma^2(A)$.

Proposition 5.2. *Let A be a unital C^*-algebra and let $\mathcal{A}^\infty \subset A$ be a unital smooth subalgebra such that $\mathcal{A}^\infty \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N}^k)) \subset A \otimes \mathcal{K}(\ell^2(\mathbb{N}^k))$ is smooth for every $k \in \mathbb{N}$. Then the algebra $\Sigma^2(\mathcal{A}^\infty) \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N}^k))$ is smooth in $\Sigma^2(A) \otimes \mathcal{K}(\ell^2(\mathbb{N}^k))$ for every $k \geqslant 0$.*

Proof. Let us denote the restriction of $1 \otimes \sigma$ to $\Sigma^2(A)$ by ρ. Recall the $\sigma : \mathscr{T} \to C(\mathbb{T})$ is the symbol map sending S to the generating unitary. Then one has the following exact sequence at the C^*-algebra level

$$0 \to A \otimes \mathcal{K}\big(\ell^2(\mathbb{N})\big) \to \Sigma^2(A) \xrightarrow{\rho} C(\mathbb{T}) \to 0.$$

At the subalgebra level one has the following "sub" exact sequence

$$0 \to \mathcal{A}^\infty \hat{\otimes}_\pi \mathcal{S}\big(\ell^2(\mathbb{N})\big) \to \Sigma^2\big(\mathcal{A}^\infty\big) \xrightarrow{\rho} C^\infty(\mathbb{T}) \to 0.$$

Since $\mathcal{A}^\infty \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N})) \subset A \otimes \mathcal{K}(\ell^2(\mathbb{N}))$ and $C^\infty(\mathbb{T}) \subset C(\mathbb{T})$ are smooth, it follows from Theorem 3.2, part 2 [15] that $\Sigma^2(\mathcal{A}^\infty)$ is smooth in $\Sigma^2(A)$. One can prove that $\Sigma^2(\mathcal{A}^\infty) \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N}^k))$ is smooth in $\Sigma^2(A) \otimes \mathcal{K}(\ell^2(\mathbb{N}^k))$ for every $k > 0$ along the same lines first by tensoring the C^*-algebra exact sequence by $\mathcal{K}(\ell^2(\mathbb{N}^k))$ and then by tensoring the Fréchet algebra exact sequence by $\mathcal{S}(\ell^2(\mathbb{N}^k))$ and appealing to Theorem 3.2, part 2 of [15]. This completes the proof. □

5.1. The topological weak heat kernel expansion

We need the following version of the weak heat kernel expansion.

Definition 5.3. Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a $p+$ summable spectral triple for a C^*-algebra A where $\mathcal{A}^\infty$ is smooth in A. We say that the spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ has the topological weak heat kernel asymptotic expansion property if there exists a $*$-subalgebra $\mathcal{B}^\infty \subset B(\mathcal{H})$ such that:

(1) The algebra $\mathcal{B}^\infty$ has a Fréchet space structure and endowed with it, it is a Fréchet $*$-algebra.
(2) The algebra $\mathcal{B}^\infty$ contains $\mathcal{A}^\infty$.
(3) The inclusion $\mathcal{B}^\infty \subset B(\mathcal{H})$ is continuous.
(4) The unbounded derivations $\delta := [|D|, .]$ leaves $\mathcal{B}^\infty$ invariant and is continuous. Also the unbounded derivation $d := [D, .]$ maps $\mathcal{A}^\infty$ into $\mathcal{B}^\infty$ in a continuous fashion.
(5) The left multiplication by the operator $F := sign(D)$ denoted L_F leaves $\mathcal{B}^\infty$ invariant and is continuous.
(6) The function $\tau_p : (0, \infty) \times \mathcal{B}^\infty \to \mathbb{C}$ defined by $\tau_p(t, b) = t^p\, Tr(be^{-t|D|})$ has a uniform asymptotic power series expansion.

We need the following analog of Proposition 4.4 and Proposition 4.5. First we need the following two lemmas.

Lemma 5.4. *Let E be a Fréchet space and $F \subset E$ be a dense subspace. Let $\phi : (0,\infty) \times E \to \mathbb{C}$ be a continuous function which is linear in the second variable. Suppose that $\phi : (0,\infty) \times F \to \mathbb{C}$ has a uniform asymptotic power series expansion then $\phi : (0,\infty) \times E \to \mathbb{C}$ has a uniform asymptotic power series expansion.*

Proof. Suppose that $\phi(t, f) \sim \sum_{r=0}^{\infty} a_r(f)t^r$. Then $a_r : F \to \mathbb{C}$ is linear and is continuous for every $r \in \mathbb{N}$. Since F is dense in E, for every $r \in \mathbb{N}$, the function a_r admits a continuous extension to the whole of E which we still denote it by a_r. Now fix $N \in \mathbb{N}$. Then there exist a neighbourhood U of E containing 0 and $\epsilon, M > 0$ such that

$$\left|\phi(t, f) - \sum_{r=0}^{N} a_r(f)t^r\right| \leqslant Mt^{N+1} \quad \text{for } 0 < t < \epsilon,\ f \in U \cap F. \tag{5.15}$$

Since $\phi(t, .)$ and $a_r(.)$ are continuous and as F is dense in E, Eq. (5.15) continues to hold for every $f \in U$. This completes the proof. □

Lemma 5.5. *Let E_1, E_2 be Fréchet spaces and let $F_i : (0,\infty) \times E_i \to \mathbb{C}$ be continuous and linear in the second variable for $i = 1, 2$. Consider the function $F : (0,\infty) \times E_1 \hat{\otimes}_\pi E_2 \to \mathbb{C}$ be defined by $F(t, e_1 \otimes e_2) = F_1(t, e_1)F(t, e_2)$. Assume that F is continuous. If F_1 and F_2 have uniform asymptotic expansions then F has a uniform asymptotic power series expansion.*

Proof. By Lemma 5.4, it is enough to show that $F : (0,\infty) \times E_1 \otimes_{alg} E_2 \to \mathbb{C}$ has a uniform asymptotic power series expansion. Let $\theta : E_1 \times E_2 \to E_1 \otimes_{alg} E_2$ be defined by $\theta(e_1, e_2) = e_1 \otimes e_2$. Consider the map $G : (0,\infty) \times E_1 \times E_2 \to \mathbb{C}$ defined by $G(t, e_1, e_2) := F(t, \theta((e_1, e_2)))$. By Lemma 2.2, it follows that G has a uniform asymptotic power series expansion say

$$G(t, e) \sim \sum_{r=0}^{\infty} a_r(e)t^r.$$

The maps $a_r : E_1 \times E_2 \to \mathbb{C}$ are continuous bilinear. We let $\tilde{a}_r : E_1 \hat{\otimes}_\pi E_2 \to \mathbb{C}$ be the linear maps such that $\tilde{a}_r \circ \theta := a_r$. Let $N \in \mathbb{N}$ be given. Then there exist $\epsilon, M > 0$ and open sets U_1, U_2 containing 0 in E_1, E_2 such that

$$\left|G(t, e) - \sum_{r=0}^{N} a_r(e)t^r\right| \leqslant Mt^{N+1} \quad \text{for } 0 < t < \epsilon,\ e \in U_1 \times U_2. \tag{5.16}$$

Without loss of generality, we can assume that $U_i := \{x \in E_i\colon\ p_i(x) < 1\}$ for a seminorm p_i of E_i. Now Eq. (5.16) implies that

$$\left|F\bigl(t, \theta(e)\bigr) - \sum_{r=0}^{N} \tilde{a}_r\bigl(\theta(e)\bigr)t^r\right| \leqslant Mt^{N+1} \quad \text{for } 0 < t < \epsilon,\ e \in U_1 \times U_2. \tag{5.17}$$

Hence for $t \in (0, \epsilon)$ and $x \in \theta(U_1 \times U_2)$,

$$\left| F(t, x) - \sum_{r=0}^{N} \tilde{a}_r(x) t^r \right| \leqslant M t^{N+1}. \tag{5.18}$$

Since $\tilde{a}_r$ is linear and F is linear in the second variable, it follows that Eq. (5.18) continues to hold for x in the convex hull of $\theta(U_1 \times U_2)$ which is nothing but the unit ball determined by the seminorm $p_1 \otimes p_2$ in $E_1 \otimes_{alg} E_2$. This completes the proof. □

In the next proposition, we consider the stability of the weak heat kernel asymptotic expansion property for tensoring by smooth compacts.

Proposition 5.6. *Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple where the algebra $\mathcal{A}^\infty$ is a smooth subalgebra of C^*-algebra. Assume that $(\mathcal{A}^\infty, \mathcal{H}, D)$ has the topological weak heat kernel expansion property with dimension p. Then the spectral triple $(\mathcal{A}^\infty \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N})), \mathcal{H} \otimes \ell^2(\mathbb{N}),$ $D_0 := (F \otimes 1)(|D| \otimes 1 + 1 \otimes N))$ also has the weak heat kernel asymptotic expansion property with dimension p where $F := sign(D)$.*

Proof. Let $\mathcal{B}^\infty \subset B(\mathcal{H})$ be a $*$-subalgebra for which (1)–(6) of Definition 5.3 are satisfied. We denote $\mathcal{B}^\infty \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N}))$ by $\mathcal{B}_0^\infty$. We show that $\mathcal{B}_0^\infty$ satisfies (1)–(6) of Definition 5.3. First note that the natural representation of $\mathcal{B}_0^\infty$ in $\mathcal{H} \otimes \ell^2(\mathbb{N})$ is injective. Thus (3) is clear. Also (1) and (2) are obvious. Now let us now prove (4).

We denote the unbounded derivation $[|D_0|, .]$, $[|D|, .]$ and $[N, .]$ by δ_{D_0}, δ_D and δ_N respectively. By assumption δ_D leaves $\mathcal{B}$ invariant and is continuous. It is also easy to see that δ_N leaves $\mathcal{S}(\ell^2(\mathbb{N}))$ invariant and is continuous. Let $\delta' := \delta_D \otimes 1 + 1 \otimes \delta_N$. Then $\delta' : \mathcal{B}_0^\infty \to \mathcal{B}_0^\infty$ is continuous. Clearly $\mathcal{B}^\infty \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$ is contained in the domain of δ and $\delta = \delta'$ on $\mathcal{B}^\infty \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$. Now let $a \in \mathcal{B}_0^\infty$ be given. Then there exists a sequence (a_n) in $\mathcal{B}^\infty \otimes_\pi \mathcal{S}(\ell^2(\mathbb{N}))$ such that (a_n) converges to a in $\mathcal{B}_0^\infty$. Since δ' is continuous on $\mathcal{B}_0^\infty$ and the inclusion $\mathcal{B}_0^\infty \subset B(\mathcal{H})$ is continuous, it follows that $\delta_{D_0}(a_n) = \delta'(a_n)$ converges to $\delta'(a)$. As δ_{D_0} is a closed derivation, it follows that $a \in Dom(\delta_{D_0})$ and $\delta_{D_0}(a) = \delta'(a)$. Hence we have shown that δ_{D_0} leaves $\mathcal{B}_0^\infty$ invariant and is continuous. Similarly one can show that the unbounded derivation $d_0 := [D_0, .]$ maps $\mathcal{A} \hat{\otimes}_\pi S(\ell^2(\mathbb{N}))$ into $\mathcal{B}_0^\infty$ invariant in a continuous manner.

As $F_0 := sign(D_0) = F \otimes 1$, (5) is clear. Consider the function $\tau_p : (0, \infty) \times \mathcal{B}_0^\infty \to \mathbb{C}$ defined by $\tau_p(t, b) := t^p\, Tr(b e^{-t|D_0|})$. Then $\tau_p(t, b \otimes k) = \tau_p(t, b)\tau_0(t, k)$. Hence by Lemma 5.5, it follows that τ_p has a uniform asymptotic power series expansion. This completes the proof. □

Now we consider the stability of the weak heat kernel asymptotic expansion under the double suspension.

Theorem 5.7. *Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple with the topological weak heat kernel asymptotic expansion property of dimension p. Assume that the algebra $\mathcal{A}^\infty$ is unital and the representation on $\mathcal{H}$ is unital. Then the spectral triple $(\Sigma^2(\mathcal{A}^\infty), \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2(D))$ also has the topological weak heat kernel asymptotic expansion property with dimension $p+1$.*

Proof. We denote the operator $\Sigma^2(D)$ by D_0. Let $\mathcal{B}^\infty$ be $*$-subalgebra of $B(\mathcal{H})$ for which (1)–(6) of Definition 5.3 are satisfied. For $f = \sum_{n \in \mathbb{Z}} \lambda_n z^n \in C(\mathbb{T})$, we let $\sigma(f) :=$

$\sum_{n\geqslant 0}\lambda_n S^n+\sum_{n>0}\lambda_{-n}S^{*n}$. We denote the projection $\frac{1+F}{2}$ by P. We assume here that $P\neq\pm 1$ as the case $P=\pm 1$ is similar. We let $\mathcal{B}_0^\infty$ to denote the algebra $\mathcal{B}^\infty\mathbin{\hat{\otimes}_\pi}\mathcal{S}(\ell^2(\mathbb{N}))$ as in Proposition 5.6. As in Proposition 5.6, we let $\delta_{D_0},\delta_D,\delta_N$ to denote the unbounded derivations $[|D_0|,.]$, $[|D|,.]$ and $[N,.]$ respectively. Define

$$\tilde{\mathcal{B}}^\infty:=\bigl\{b+P\otimes\sigma(f)+(1-P)\otimes\sigma(g)\colon\ b\in\mathcal{B}_0^\infty,\ f,g\in C^\infty(\mathbb{T})\bigr\}.$$

Then $\tilde{\mathcal{B}}^\infty$ is isomorphic to the direct sum $\mathcal{B}_0^\infty\oplus C^\infty(\mathbb{T})\oplus C^\infty(\mathbb{T})$. We give $\tilde{\mathcal{B}}^\infty$ the Fréchet space structure coming from this decomposition. It is easy to see that $\tilde{\mathcal{B}}^\infty$ is a Fréchet $*$-subalgebra of $B(\mathcal{H}\otimes\ell^2(\mathbb{N}))$. Clearly $(\pi\otimes 1)(\Sigma^2(\mathcal{A}^\infty))\subset\tilde{\mathcal{B}}^\infty$. Thus we have shown that (1) and (2) of Definition 5.3 are satisfied. Since $\mathcal{B}_0^\infty$ is represented injectively on $\mathcal{H}\otimes\ell^2(\mathbb{N})$, it follows that $\tilde{\mathcal{B}}$ satisfies (3).

We have already shown in Proposition 5.6 that $\mathcal{B}_0^\infty$ is closed under δ_{D_0} and is continuous. Also we have shown that $d_0:=[D_0,.]$ maps $\mathcal{A}\mathbin{\hat{\otimes}_\pi}S(\ell^2(\mathbb{N}))$ into $\mathcal{B}_0^\infty$ continuously. Now note that

$$\begin{aligned}
\delta_{D_0}\bigl(P\otimes\sigma(f)\bigr)&=P\otimes\sigma\bigl(if'\bigr),\\
\delta_{D_0}\bigl((1-P)\otimes\sigma(g)\bigr)&=(1-P)\otimes\sigma\bigl(ig'\bigr),\\
\bigl[D_0,P\otimes\sigma(f)\bigr]&=P\otimes\sigma\bigl(if'\bigr),\\
\bigl[D_0,(1-P)\otimes\sigma(g)\bigr]&=-(1-P)\otimes\sigma\bigl(ig'\bigr).
\end{aligned}$$

Thus it follows that δ_{D_0} leaves $\tilde{\mathcal{B}}^\infty$ invariant and is continuous. Also, it follows that $d_0:=[D_0,.]$ maps $\Sigma^2(\mathcal{A}^\infty)$ into $\tilde{\mathcal{B}}$ in a continuous manner.

Since $F_0:=sign(D_0)=F\otimes 1$, it follows from definition that $F_0\in\tilde{\mathcal{B}}^\infty$. Now we show that $\tilde{\mathcal{B}}^\infty$ satisfies (6).

We have already shown in Proposition 5.6 that the function $\tau_p:(0,\infty)\otimes\mathcal{B}_0^\infty\to\mathbb{C}$ defined by $\tau_p(t,b):=t^p\,Tr(be^{-t|D_0|})$ has a uniform asymptotic power series expansion. Hence τ_{p+1} restricted to $\mathcal{B}_0^\infty$ has a uniform asymptotic power series expansion. Now note that

$$\tau_{p+1}\bigl(P\otimes\sigma(f)\bigr)=\biggl(\int f(\theta)\,d\theta\biggr)t^p\,Tr\bigl(Pe^{-t|D|}\bigr)t\,Tr\bigl(e^{-tN}\bigr),\tag{5.19}$$

$$\tau_{p+1}\bigl((1-P)\otimes\sigma(g)\bigr)=\biggl(\int g(\theta)\,d\theta\biggr)t^p\,Tr\bigl((1-P)e^{-t|D|}\bigr)t\,Tr\bigl(e^{-tN}\bigr).\tag{5.20}$$

Now recall that we have assumed that $\mathcal{A}^\infty$ is unital and hence $P\in\mathcal{B}^\infty$. Hence $t^p\,Tr(xe^{-t|D|})$ has an asymptotic power series expansion for $x\in\{P,1-P\}$. Also $t\,Tr(e^{-tN})$ has an asymptotic power series expansion. Now Eqs. (5.19) and (5.20), together with the earlier observation that τ_{p+1} restricted to $\mathcal{B}_0^\infty$ has a uniform asymptotic power series expansion, imply that the function $\tau_{p+1}:(0,\infty)\times\tilde{\mathcal{B}}^\infty\to\mathbb{C}$ has a uniform asymptotic power series expansion. This completes the proof. □

Let $(C^\infty(\mathbb{T}),L^2(\mathbb{T}),\frac{1}{i}\frac{d}{d\theta})$ be the canonical spectral triple on the circle. Via Fourier transform if we identify $L^2(\mathbb{T})$ with $L^2(\mathbb{Z})$ then $\frac{1}{i}\frac{d}{d\theta}$ becomes the number operator. Let F be the sign of the number operator. Then with $\mathcal{B}_\infty = \{f_0 + Ff_1 + R\colon\ f_0, f_1 \in C^\infty(\mathbb{T}),\ R \text{ infinitely smoothing}\}$ this spectral triple satisfies topological WHKAE. Hong and Szymanski proved [12] that by iteratedly quantum double suspending $C(\mathbb{T})$ we get the odd-dimensional quantum spheres. It follows that the iterated quantum double suspension $C^\infty(S_q^{2\ell+1}) := \Sigma^{2\ell}(C^\infty(\mathbb{T}))$ is dense in $C(S_q^{2\ell+1})$. Now if we quantum double suspend the spectral triple $(C^\infty(\mathbb{T}), L^2(\mathbb{T}), \frac{1}{i}\frac{d}{d\theta})$ we get the torus equivariant spectral triple on $C(S_q^{2\ell+1})$ [5]. Now Theorem 5.7 implies that the torus equivariant spectral triple for the odd-dimensional quantum sphere $C(S_q^{2\ell+1})$ satisfies topological weak heat kernel asymptotic expansion property with dimension $\ell+1$. Hence by Theorem 3.2 this is regular with finite dimension spectrum. This gives a conceptual proof of Proposition 3.9 in [14].

5.2. The equivariant spectral triple on odd-dimensional quantum spheres

In this section, we show that the equivariant spectral triple on $S_q^{2\ell+1}$ constructed in [4] has the topological weak heat kernel asymptotic expansion. First let us recall that the odd-dimensional quantum spheres can be realised as the quantum homogeneous space. Throughout we assume $q \in (0,1)$. The C^*-algebra of the quantum group $SU_q(n)$ denoted by $C(SU_q(n))$ is defined as the universal C^*-algebra generated by $\{u_{ij}\colon\ 1 \leqslant i, j \leqslant \ell\}$ satisfying the following conditions

$$\sum_{k=1}^{n} u_{ik}u_{jk}^* = \delta_{ij}, \qquad \sum_{k=1}^{n} u_{ki}^* u_{kj} = \delta_{ij}, \tag{5.21}$$

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n} E_{i_1 i_2\cdots i_n} u_{j_1 i_1}\cdots u_{j_n i_n} = E_{j_1 j_2\cdots j_n} \tag{5.22}$$

where

$$E_{i_1 i_2\cdots i_n} := \begin{cases} 0, & \text{if } i_1, i_2, \ldots, i_n \text{ are not distinct} \\ (-q)^{\ell(i_1, i_2, \ldots, i_n)} & \end{cases}$$

where for a permutation σ on $\{1,2,\ldots,n\}$, $\ell(\sigma)$ denotes its length. The C^*-algebra has the quantum group structure with the comultiplication being defined by

$$\Delta(u_{ij}) := \sum_k u_{ik} \otimes u_{kj}.$$

Call the generators of $SU_q(n-1)$ as v_{ij}. The map $\phi : C(SU_q(n)) \to C(SU_q(n-1))$ defined by

$$\phi(u_{ij}) := \begin{cases} v_{i-1,j-1} & \text{if } 2 \leqslant i, j \leqslant n, \\ \delta_{ij} & \text{otherwise} \end{cases} \tag{5.23}$$

is a surjective unital C^*-algebra homomorphism such that $\Delta \circ \phi = (\phi \otimes \phi)\Delta$. In this way the quantum group $SU_q(n-1)$ is a subgroup of the quantum group $SU_q(n)$. The C^*-algebra of the quotient $SU_q(n)/SU_q(n-1)$ is defined as

$$C\big(SU_q(n)/SU_q(n-1)\big) := \big\{a \in C\big(SU_q(n)\big)\colon\ (\phi \otimes 1)\Delta(a) = 1 \otimes a\big\}.$$

Also the C^*-algebra $C(SU_q(n)/SU_q(n-1))$ is generated by $\{u_{1j}\colon 1 \leqslant j \leqslant n\}$. Moreover the map $\psi : C(S_q^{2n-1}) \to C(SU_q(n)/SU_q(n-1))$ defined by $\psi(z_i) := q^{-i+1}u_{1i}$ is an isomorphism.

Let h be the Haar state on the quantum group $C(SU_q(\ell+1))$ and let $L^2(SU_q(\ell+1))$ be the corresponding *GNS* space. We denote the closure of $C(S_q^{2\ell+1})$ in $L^2(SU_q(\ell+1))$ by $L^2(S_q^{2\ell+1})$. Then $L^2(S_q^{2\ell+1})$ is invariant under the regular representation of $SU_q(\ell+1)$. Thus we get a covariant representation for the dynamical system $(C(S_q^{2\ell+1}), SU_q(\ell+1), \Delta)$. We denote the representation of $C(S_q^{2\ell+1})$ on $L^2(S_q^{2\ell+1})$ by π_{eq}. In [4] $SU_q(\ell+1)$ equivariant spectral triples for this covariant representation were studied and a non-trivial one was constructed. It is proved in [14] that the Hilbert space $L^2(S_q^{2\ell+1})$ is unitarily equivalent to $\ell^2(\mathbb{N}^\ell \times \mathbb{Z} \times \mathbb{N}^\ell)$. Then the self-adjoint operator D_{eq} constructed in [4] is given on the orthonormal basis $\{e_\gamma\colon \gamma \in \mathbb{N}^\ell \times \mathbb{Z} \times \mathbb{N}^\ell\}$ by the formula $D_{eq}(e_\gamma) := d_\gamma e_\gamma$ where d_γ is given by

$$d_\gamma := \begin{cases} \sum_{i=1}^{2\ell+1} |\gamma_i| & \text{if } (\gamma_{\ell+1}, \gamma_{\ell+2}, \dots, \gamma_{2\ell+1}) = 0 \text{ and } \gamma_{\ell+1} \geqslant 0, \\ -\sum_{i=1}^{2\ell+1} |\gamma_i| & \text{else.} \end{cases}$$

In [14], a smooth subalgebra $C^\infty(S_q^{2\ell+1}) \subset C(S_q^{2\ell+1})$ is defined and it is shown that the spectral triple $(C^\infty(S_q^{2\ell+1}), L^2(S_q^{2\ell+1}), D_{eq})$ is a regular spectral triple with simple dimension spectrum $\{1, 2, \dots, 2\ell+1\}$. Now we show that the spectral triple $(C^\infty(S_q^{2\ell+1}), L^2(S_q^{2\ell+1}), D_{eq})$ has the topological weak heat kernel expansion.

We use the same notations as in [14]. Let $\mathcal{A}_\ell^\infty := \Sigma^{2\ell}(C^\infty(\mathbb{T}))$. It follows from Corollary 4.2.3 that $C^\infty(S_q^{2\ell+1}) \subset \mathcal{A}_\ell^\infty$. Let $(C^\infty(S_q^{2\ell+1}), \pi_\ell, \mathcal{H}_\ell, D_\ell)$ be the torus equivariant spectral triple. Let N be the number operator on $\ell^2(\mathbb{N}^\ell)$ defined by

$$N e_\gamma := \left(\sum_{i=1}^{\ell} \gamma_i \right) e_\gamma.$$

Let us denote the Hilbert space $\ell^2(S_q^{2\ell+1})$ by $\mathcal{H}$. We identify $\mathcal{H}_\ell := \ell^2(\mathbb{N}^\ell \times \mathbb{Z})$ with the subspace $\ell^2(\mathbb{N}^\ell \times \mathbb{Z} \times \{0\})$ and we denote the orthogonal complement in $\mathcal{H}$ by $\mathcal{H}'_\ell$. Then $\ell^2(S_q^{2\ell+1}) = \mathcal{H}_\ell \oplus \mathcal{H}'_\ell$. Define the unbounded operator D_{torus} on $\mathcal{H}$ by the equation

$$D_{torus} := \begin{bmatrix} D_\ell & 0 \\ 0 & -|D_\ell| \otimes 1 - 1 \otimes N \end{bmatrix}.$$

Then in [14], it is shown that $D_{eq} = D_{torus}$. We denote representation $\pi_\ell \oplus (\pi_{e\ll} \otimes 1)$ of $C(S_q^{2\ell+1})$ on $\mathcal{H}$ by π_{torus}.

Let $\mathscr{T}^\infty := \Sigma^2_{smooth}(\mathbb{C})$ and let $\mathscr{T}_\ell^\infty := \mathscr{T}^\infty \hat{\otimes} \mathscr{T}^\infty \hat{\otimes} \cdots \hat{\otimes} \mathscr{T}^\infty$ denote the Fréchet tensor product of ℓ copies. The main theorem in [14] is the following.

Theorem 5.8. *For every $a \in C^\infty(S_q^{2\ell+1})$, the difference $\pi_{eq}(a) - \pi_{torus}(a) \in OP_{D_\ell}^{-\infty} \hat{\otimes} \mathscr{T}_\ell$.*

Let $P_\ell := \frac{1+F_\ell}{2}$ where $F_\ell := Sign(D_\ell)$. We denote the rank one projection $|e_0\rangle\langle e_0|$ on $\ell^2(\mathbb{N}^\ell)$ by P where $e_0 := e_{(0,0,\dots,0)}$.

Proposition 5.9. *The equivariant spectral triple* $(C^\infty(S_q^{2\ell+1}), \mathcal{H}, D_{eq})$ *has the topological weak heat kernel expansion.*

Proof. Let $\mathcal{J} := OP_{D_\ell}^{-\infty} \hat{\otimes} \mathscr{T}_\ell^\infty$. In [14], the following algebra is considered.

$$\mathcal{B} := \big\{a_1 P_\ell \otimes P + a_2 P_\ell \otimes (1-P) + a_3(1-P_\ell) \otimes P + a_4(1-P_\ell) \otimes (1-P) + R\colon \\ a_1, a_2, a_3, a_4 \in \mathcal{A}_\ell^\infty,\ R \in \mathcal{J}\big\}.$$

The algebra $\mathcal{B}$ is isomorphic to $\mathcal{A}_\ell^\infty \oplus \mathcal{A}_\ell^\infty \oplus \mathcal{A}_\ell^\infty \oplus \mathcal{A}_\ell^\infty \oplus \mathcal{J}$. We give $\mathcal{B}$ the Fréchet space structure coming from this decomposition. In [14], it is shown that $\mathcal{B}$ contains $C^\infty(S_q^{2\ell+1})$ and is closed under $\delta := [|D_{eq}|, .]$ and $d := [D, .]$. Moreover it is shown that δ and d are continuous on $\mathcal{B}$. Note that $F_{eq} := F_\ell \otimes P - 1 \otimes (1-P)$. Hence by definition $F_{eq} \in \mathcal{B}$. Now note that the torus equivariant spectral triple $(\mathcal{A}_\ell^\infty, \mathcal{H}_\ell, D_\ell)$ has the topological weak heat kernel asymptotic expansion. Thus it is enough to show that the map $\tau_{2\ell+1} : (0,\infty) \times \mathcal{J} \to \mathbb{C}$ defined by $\tau_{2\ell+1}(t,b) := t^{2\ell+1}\, Tr(be^{-t|D_{eq}|})$ has uniform asymptotic expansion.

But this follows from the fact that $(OP_{D_\ell}^{-\infty}, \mathcal{H}_\ell, D_\ell)$ and $(\mathscr{T}^\infty, \ell^2(\mathbb{N}), N)$ have the topological weak heat kernel expansion and by using Lemma 5.5. This completes the proof. □

Remark 5.10. The method in [14] can be applied to show that the equivariant spectral triple on the quantum $SU(2)$ constructed in [2] has the heat kernel asymptotic expansion property with dimension 3 and hence deducing the dimension spectrum computed in [7]. It has been shown in [3] that the isospectral triple studied in [17] differs from the equivariant one (with multiplicity 2) constructed in [2] only be a smooth perturbation. As a result it will follow that (since the extension $\mathcal{B}^\infty$ for the equivariant spectral triple satisfying Definition 5.3 contains the algebra of smoothing operators) the isospectral spectral triple also has the weak heat kernel expansion with dimension 3.

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