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FROM C^*-ALGEBRA EXTENSIONS TO COMPACT QUANTUM METRIC SPACES, QUANTUM SU(2), PODLEŚ SPHERES AND OTHER EXAMPLES

PARTHA SARATHI CHAKRABORTY

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Abstract

We construct compact quantum metric spaces starting from a C^*-algebra extension with a positive splitting. As special cases, we discuss Toeplitz algebras, quantum SU(2) and Podleś spheres.

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1. Introduction

In noncommutative geometry, the natural way to specify a metric is by a 'Lipschitz seminorm'. Connes suggested this idea in [2], and developed it further in [3]. He pointed out that one may obtain an ordinary metric on the state space of a C^*-algebra in a simple way from a Lipschitz seminorm. A natural question in this context is whether this metric topology coincides with the weak* topology. Rieffel [7, 8, 10] identified a larger class of spaces, namely order unit spaces, in his search for an answer to this question. He introduced the concept of compact quantum metric spaces as a generalization of compact metric spaces, and in [10] used this new concept for the rigorous study of convergence questions of algebras in the spirit of Gromov–Hausdorff convergence. A natural question in this regard is whether there are many such spaces.

Rieffel [7, 8] gave some general principles for constructing compact quantum metric spaces. In [1], we used one of his principles to construct examples thereof. In fact, Rieffel [9] has shown that there are indeed many examples. But in concrete C^*-algebras one would like to have a more explicit description of these structures.

Our objective here is to construct compact quantum metric spaces out of quantum SU(2) and Podleś spheres. To do this, we develop a more general construction and produce compact quantum metric spaces starting from C^*-algebra extensions.

This paper is organized as follows. In the next section we recall the basics of these spaces. In Section 3 the basic construction is described. In the final section we employ the principle developed in Section 3 to special cases.

2. Compact quantum metric spaces: preliminaries

We recall some of the definitions from [10].

Definition 2.1. An order unit space is a real partially ordered vector space A with a distinguished element e, the order unit, with the following properties.

(i) For each $a \in A$, there is $r \in \mathbb{R}$ such that $a \leq re$ (order unit property).
(ii) If $a \in A$ and if $a \leq re$ for all $r \in \mathbb{R}$ with $r \geq 0$, then $a \leq 0$ (Archimedean property).

Remark 2.2. We may define a norm on an order unit space as follows:

$$\|a\| = \inf\{r \in \mathbb{R} : -re \leq a \leq re\}.$$

Definition 2.3. By a state of an order unit space (A, e) we mean an element $\mu \in A'$, the dual of $(A, \|\cdot\|)$, such that $\mu(e) = 1 = \|\mu\|'$. Here $\|\cdot\|'$ stands for the dual norm on A'. The collection of states on (A, e) is denoted by $S(A)$.

Remark 2.4. States are automatically positive.

Example 2.5. The motivating example for this concept is the real subspace of self-adjoint elements in a C^*-algebra with the order structure inherited from the algebra.

Definition 2.6. Let (A, e) be an order unit space. By a Lip-norm on A we mean a seminorm L on A with the following properties.

(i) If $a \in A$, then $L(a) = 0$ if and only if $a \in \mathbb{R}e$.
(ii) The topology on $S(A)$ coming from the metric

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : L(a) \leq 1\}$$

is the weak* topology.

Definition 2.7. A compact quantum metric space is a pair (A, L) consisting of an order unit space A and a Lip-norm L defined on it.

The following theorem of Rieffel will be of crucial importance.

Theorem 2.8 [10, Theorem 4.5]. *Let L be a seminorm on the order unit space A such that $L(a) = 0$ if and only if $a \in \mathbb{R}e$. Then ρ_L gives $S(A)$ the weak* topology exactly when both the following conditions hold.*

(i) *(A, L) has finite radius, that is, $\rho_L(\mu, \nu) \leq C$ for all $\mu, \nu \in S(A)$ for some constant C.*
(ii) *The set $\mathcal{B}_1 = \{a : L(a) \leq 1, \|a\| \leq 1\}$ is totally bounded in A for $\|\cdot\|$.*

3. Extensions to compact quantum metric spaces

In this section we describe the general principle of construction of compact quantum metric spaces from certain C^*-algebra extensions. Let $\mathcal{A}$ be a unital C^*-algebra. Fix a faithful representation $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. Suppose that we have a dense order unit space $\mathrm{Lip}(\mathcal{A}) \subseteq \mathcal{A}_{\mathrm{s.a}}$, containing the unit $1_{\mathcal{A}}$ of $\mathcal{A}$, where $\mathcal{A}_{\mathrm{s.a}}$ denotes the real partially ordered subset of self-adjoint elements in $\mathcal{A}$. Let L be a Lip-norm on $\mathrm{Lip}(\mathcal{A})$ such that $((\mathrm{Lip}(\mathcal{A}), I), L)$ is a compact quantum metric space. Let ν be a state on $\mathcal{A}$, and define $\widetilde{\mathcal{A}}_\nu$ to be the collection of all $((a_{ij})) \in \mathcal{K}(L^2(\mathbb{N})) \otimes \mathcal{A}$ with the following properties:

(i) $a_{ij} \in \mathrm{Lip}(\mathcal{A})$;

(ii) $a_{ij} = a_{ji}$;

(iii) $\sup_{i\geq 1, j\geq 1} (i+j)^k (L(a_{ij}) + |\nu(a_{ij})|) < \infty$ for all k.

Clearly $\mathcal{A}_\nu := \widetilde{\mathcal{A}}_\nu \oplus \mathbb{R}I$, where I is the identity on $\mathcal{B}(L^2(\mathbb{N}) \otimes \mathcal{H})$, is an order unit space. Define $L_k : \mathcal{A}_\nu \to \mathbb{R}_+$ by $L_k(I) = 0$,

$$L_k((a_{ij})) = \sup_{i\geq 1, j\geq 1} (i+j)^k (L(a_{ij}) + |\nu(a_{ij})|).$$

Lemma 3.1. *Let d be the diameter of $((\mathrm{Lip}(\mathcal{A}), I), L)$, given by*

$$d = \sup\{\mu(a) - \mu'(a) : a \in \mathrm{Lip}(\mathcal{A}), L(a) \leq 1, \mu, \mu' \in S(\mathrm{Lip}(\mathcal{A}))\}.$$

Then, for all 'Lipschitz functions' $a \in \mathrm{Lip}(\mathcal{A})$,

$$\|a\| \leq (L(a) + |\nu(a)|)(1+d).$$

Proof. Let μ be an arbitrary state on $\mathcal{A}$. Since $\sup\{|\mu(a) - \nu(a)| : L(a) \leq 1\} \leq d$,

$$\begin{aligned} |\mu(a)| &\leq |\mu(a) - \nu(a)| + |\nu(a)| \\ &\leq L(a)d + |\nu(a)| \\ &\leq (L(a) + |\nu(a)|)(1+d), \end{aligned}$$

as required. □

Lemma 3.2. *There exists a constant $C > 0$ such that for all $((a_{ij})) \in \widetilde{\mathcal{A}}_\nu$,*

$$\|((a_{ij}))\| \leq C L_2((a_{ij})).$$

Proof. Let $\{e_i\}_{i\geq 1}$ be the canonical orthonormal basis for $L^2(\mathbb{N})$. Let $\sum_i \lambda_i e_i \otimes u_i$ and $\sum_i \mu_i e_i \otimes v_i$ be generic elements in $L^2(\mathbb{N}) \otimes \mathcal{H}$. Here $u_i, v_i \in \mathcal{H}$ are unit vectors. Then clearly

$$\Big\|\sum_i \lambda_i e_i \otimes u_i\Big\|^2 = \sum_i |\lambda_i|^2 \quad \text{and} \quad \Big\|\sum_i \mu_i e_i \otimes u_i\Big\|^2 = \sum_i |\mu_i|^2.$$

Now observe that

$$\begin{aligned}
&\Big|\Big\langle \sum_{i,j} \lambda_i e_i \otimes u_i, ((a_{ij})) \sum_{i,j} \mu_j e_j \otimes v_j \Big\rangle\Big| \\
&\quad \le \sum_{i,j} |\lambda_i||\mu_j||\langle u_i, a_{ij} v_j\rangle| \\
&\quad \le \sum_{i,j} |\lambda_i||\mu_j|(L(a_{ij}) + |\nu(a_{ij})|)(1+d) \\
&\quad \le (1+d) \sum_{i,j} |\lambda_i||\mu_j| \frac{L_2((a_{ij}))}{(i+j)^2} \\
&\quad \le (1+d) \sum_{i,j} |\lambda_i||\mu_j| \frac{L_2((a_{ij}))}{ij} \\
&\quad \le L_2((a_{ij}))(1+d) \sum_{n=1}^{\infty} \frac{1}{n^2} \Big(\sum_i |\lambda_i|^2\Big)^{1/2} \Big(\sum_i |\mu_i|^2\Big)^{1/2}.
\end{aligned}$$

This proves the lemma with $C = (1+d)\sum_{n=1}^{\infty} n^{-2}$. □

Lemma 3.3. *The set $\mathcal{B}_1 = \{a \in \mathcal{A}_\nu : L_k(a) \le 1, \|a\| \le 1\}$ is totally bounded in norm if $k > 2$.*

Proof. Let $\epsilon > 0$, and choose N such that $N^{2-k} < \epsilon$. For $G = ((g_{ij})) \in \mathcal{A}_\nu$, define the element $P_N(G) \in \mathcal{K}(L^2(\mathbb{N})) \otimes \mathcal{A}$ by

$$P_N(G)_{ij} = \begin{cases} g_{ij} & \text{if } i, j \le N, \\ 0 & \text{otherwise.} \end{cases}$$

Now observe that

$$\begin{aligned}
L_k(G - P_N(G)) &= \sup\{(i+j)^k (L(g_{ij}) + |\nu(g_{ij})|) : i > N \text{ or } j > N\} \\
&\ge N^{k-2} \sup\{(i+j)^2 (L(g_{ij}) + |\nu(g_{ij})|) : i > N \text{ or } j > N\} \\
&= N^{k-2} L_2(G - P_N(G)).
\end{aligned}$$

Note that $L_k(G - P_N(G)) \le 1$ for all $G \in \mathcal{B}_1$, and therefore

$$\begin{aligned}
\|G - P_N(G)\| &\le C L_2(G - P_N(G)) \\
&\le C N^{-(k-2)} L_k(G - P_N(G)) < C\epsilon.
\end{aligned}$$

Here the constant C is that obtained in the previous lemma. Note that C does not depend on N. By Theorem 2.8, there exist $N \times N$ matrices $((a_{ij}^{(r)})) \in M_N(\mathcal{A})$, where $r = 1, \ldots, l$, such that for any $N \times N$ matrix $((a_{ij})) \in \mathcal{B}_1$, there exists r such that $\|((a_{ij})) - ((a_{ij}^{(r)}))\| < \epsilon$. Now for $G \in \mathcal{B}_1$, take $((a_{ij}^{(r)}))$ such that $\|P_N(G) - ((a_{ij}^{(r)}))\| < \epsilon$. Then

$$\|G - ((a_{ij}^{(r)}))\| \le \|G - P_N(G)\| + \epsilon \le (1+C)\epsilon.$$

This completes the proof. □

THEOREM 3.4. *$((\mathcal{A}_\nu, I), L_k)$ is a compact quantum metric space when $k > 2$.*

PROOF. Note that if $((a_{ij})) \in \widetilde{\mathcal{A}}_\nu$, then $L_k((a_{ij})) = 0$ implies that $L(a_{ij}) = 0$ and $\nu(a_{ij}) = 0$ for all i, j. As L is a Lip-norm, this implies that a_{ij} is a scalar. Since $\nu(a_{ij}) = 0$, this scalar must be zero. Hence $((a_{ij}))$ is the zero matrix. Therefore $L_k(a)$ is zero if and only if a is a scalar multiple of the identity. Now, in view of Theorem 2.8 and the previous lemma, we only have to show that $(\mathcal{A}_\nu, L_k)$ has finite radius. Take $\mu_1, \mu_2 \in S(\mathcal{A}_\nu)$ and $a \in \widetilde{\mathcal{A}}_\nu$ such that $L_k(a) \le 1$. By Lemma 3.2, $\|a\| \le C$, because $L_2(a) \le L_k(a)$. Hence $|\mu_1(a) - \mu_2(a)| \le 2C$, that is, $\operatorname{diam}(\mathcal{A}_\nu, L_k) \le 2C$. □

PROPOSITION 3.5. *Let*

$$0 \longrightarrow A_0 \xrightarrow{i} A_1 \xrightarrow{\pi} A_2 \longrightarrow 0$$

be a short exact sequence of C^*-algebras, with A_1 and A_2 unital, and let $\sigma : A_2 \to A_1$ be a positive linear splitting. Let $\phi : A_1' \to A_0' \oplus A_2'$ and $\psi : A_0' \oplus A_2' \to A_1'$ be the bounded linear maps given by

$$\phi(\mu) = (\mu_1, \mu_2) \quad \text{where } \mu_1 = \mu|_{i(A_0)}, \mu_2 = \mu \circ \sigma,$$
$$\psi(\mu_1, \mu_2) = \mu \quad \text{where } \mu(a) = \mu_2(\pi(a)) + \mu_1(a - \sigma \circ \pi(a)).$$

Then ϕ and ψ are inverse to each other.

PROOF. Suppose that $\phi(\mu) = (\mu_1, \mu_2)$ and $\psi(\mu_1, \mu_2) = \mu'$. Then

$$\begin{aligned} \mu'(a) &= \mu_2(\pi(a)) + \mu_1(a - \sigma \circ \pi(a)) \\ &= \mu(\sigma \circ \pi(a)) + \mu(a - \sigma \circ \pi(a)) \\ &= \mu(a). \end{aligned}$$

Therefore $\psi \circ \phi = I_{A_1'}$. Similarly, one can show that the other composition is also the identity. □

Let $\mathcal{A}, \mathrm{Lip}(\mathcal{A}), L$ be as above. Suppose that we have a short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{A} \xrightarrow{i} \widetilde{\mathcal{A}}_1 \xrightarrow{\pi} \widetilde{\mathcal{A}}_2 \longrightarrow 0$$

with $\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2$ unital, and a positive unital linear splitting $\sigma : \widetilde{\mathcal{A}}_2 \to \widetilde{\mathcal{A}}_1$. Let $(\mathcal{A}_2, L_2)$ be a compact quantum metric space containing the unit of $\widetilde{\mathcal{A}}_2$ as its order unit, with $\mathcal{A}_2$ a dense subspace of self-adjoint elements of $\widetilde{\mathcal{A}}_2$. Define $\mathcal{A}_1 = i(\widetilde{\mathcal{A}}_\nu) \oplus \sigma(\mathcal{A}_2)$.

THEOREM 3.6. *In the setting above, $L_1 : \mathcal{A}_1 \to \mathbb{R}_+$, given by*

$$L_1(a) = L_2(\pi(a)) + L_k(a - \sigma \circ \pi(a))$$

is a Lip-norm for all $k > 2$.

PROOF. We break the proof down into several steps.

Step (i): $L_1(a) = 0$ if and only if $a \in \mathbb{R}1_{\mathcal{A}_1}$. The 'if' part is obvious, and for the 'only if' part note that if $L_1(a) = 0$ then $\pi(a) = \lambda 1_{\mathcal{A}_2}$ for some $\lambda \in \mathbb{R}$ and $L_k(a - \lambda 1_{\mathcal{A}_1}) = 0$. Hence $a = \lambda 1_{\mathcal{A}_1}$.

Step (ii): (A_1, L_1) has finite radius. Suppose that $(\mu_1, \mu_2) = \phi(\mu)$ and $(\lambda_1, \lambda_2) = \phi(\lambda)$, where $\mu, \lambda \in S(\mathcal{A}_1)$ and ϕ is as in Proposition 3.5. Then we have the norm estimates $\|\mu_i\|, \|\lambda_i\| \le 1$ for all $i = 1, 2$. This is because $\|\mu_i\| \le \|\mu\|$ and μ_2 is a positive unital linear functional and hence a state. Similar arguments hold for $\|\lambda_1\|$ and $\|\lambda_2\|$. Let $x \in \mathcal{A}_1$ with $L_1(x) \le 1$; then

$$\begin{aligned}|\mu(x) - \lambda(x)| &= |\mu_2(\pi(x)) + \mu_1(x - \sigma \circ \pi(x)) - \lambda_2(\pi(x)) - \lambda_1(x - \sigma \circ \pi(x))| \\ &\le |\mu_2(\pi(x)) - \lambda_2(\pi(x))| + |\mu_1(x - \sigma \circ \pi(x)) - \lambda_1(x - \sigma \circ \pi(x))| \\ &\le \operatorname{diam}(\mathcal{A}_2, L_2) + 2C,\end{aligned}$$

where C is the constant found in Lemma 3.2. This proves that $(\mathcal{A}_1, L_1)$ has finite radius.

Step (iii). It suffices to show that the set $\mathcal{B}_1 = \{a \in \mathcal{A}_1 : \|a\| \le 1, L_1(a) \le 1\}$ is totally bounded, in view of Theorem 2.8. Since $(\mathcal{A}_v, L_k)$ and $(\mathcal{A}_2, L_2)$ are compact quantum metric spaces, it follows that if we have a sequence $a_n \in \mathcal{B}_1$, then there exists a subsequence a_{n_k} such that both $\pi(a_{n_k})$ and $a_{n_k} - \sigma \circ \pi(a_{n_k})$ converge in norm. Hence a_{n_k} is Cauchy in norm, implying the required total boundedness. □

4. Examples

Example 4.1. This example is not an illustration of this construction but rather the motivating example of compact quantum metric spaces. In some of the following examples this is utilized implicitly. Let X be a compact metric space. Let A be the space of Lipschitz continuous functions with the associated Lipschitz seminorm L. Then (A, L) is a compact quantum metric space [8].

Example 4.2. Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary $\partial\Omega$ endowed with normalized surface measure. Let $H^2(\partial\Omega)$ be the closure in $L^2(\partial\Omega)$ of the space of boundary values of holomorphic functions that can be continuously extended to $\bar{\Omega}$. For $f \in C(\partial\Omega)$, let T_f be the associated Toeplitz operator, that is, the compression of the multiplication operator M_f on $L^2(\partial\Omega)$ on $H^2(\partial\Omega)$. Let $\mathfrak{T}(\partial\Omega)$ be the associated Toeplitz extension, that is, the C^*-algebra generated by the operators T_f along with the compact operators. Then [4, Definition 2.8.4] there is a short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K}(H^2(\partial\Omega)) \xrightarrow{i} \mathfrak{T}(\partial\Omega) \xrightarrow{\pi} C(\partial\Omega) \longrightarrow 0.$$

Since this sequence admits the positive unital splitting $f \mapsto T_f$, we get a compact quantum metric space structure on $\mathfrak{T}(\partial\Omega)$ by Theorem 3.6.

Example 4.3. The C^*-algebra of continuous functions on the quantum version of SU(2), which we denote by $C(\mathrm{SU}_q(2))$, is the universal C^*-algebra generated by two elements α and β satisfying the following relations:

$$\begin{gathered}\alpha^*\alpha + \beta^*\beta = I, \quad \alpha\alpha^* + q^2\beta\beta^* = I, \\ \alpha\beta - q\beta\alpha = 0, \quad \alpha\beta^* - q\beta^*\alpha = 0, \\ \beta^*\beta = \beta\beta^*.\end{gathered}$$

The C^*-algebra $C(\mathrm{SU}_q(2))$ introduced in [12] can be described more concretely as follows. Let $\{e_i\}_{i\geq 0}$ and $\{e_i\}_{i\in\mathbb{Z}}$ be the canonical orthonormal bases for $L_2(\mathbb{N}_0)$ and $L_2(\mathbb{Z})$ respectively. We denote by the same symbol N the operator $e_k \mapsto ke_k$ (where $k \geq 0$) on $L_2(\mathbb{N}_0)$ and $e_k \mapsto ke_k$ (where $k \in \mathbb{Z}$) on $L_2(\mathbb{Z})$. Similarly, denote by the same symbol ℓ the operator $e_k \mapsto e_{k-1}$ (where $k \geq 1$), $e_0 \mapsto 0$ on $L_2(\mathbb{N}_0)$, and the operator $e_k \mapsto e_{k-1}$ (where $k \in \mathbb{Z}$) on $L_2(\mathbb{Z})$. Now take $\mathcal{H}$ to be the Hilbert space $L_2(\mathbb{N}_0) \otimes L_2(\mathbb{Z})$, and define the representation π of $C(\mathrm{SU}_q(2))$ on $\mathcal{H}$ by

$$\pi(\alpha) = \ell\sqrt{I - q^{2N}} \otimes I, \quad \pi(\beta) = q^N \otimes \ell.$$

Then π is a faithful representation of $C(\mathrm{SU}_q(2))$, so that one can identify $C(\mathrm{SU}_q(2))$ with the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\pi(\alpha)$ and $\pi(\beta)$. The image of π contains $\mathcal{K} \otimes C(\mathbb{T})$ as an ideal with $C(\mathbb{T})$ as the quotient algebra, that is, we have a useful short exact sequence:

$$0 \longrightarrow \mathcal{K} \otimes C(\mathbb{T}) \xrightarrow{i} \mathcal{A} \xrightarrow{\sigma} C(\mathbb{T}) \longrightarrow 0. \tag{4.1}$$

The homomorphism σ is explicitly given by $\sigma(\alpha) = \ell$ and $\sigma(\beta) = 0$. It is easy to see that the above short exact sequence admits a positive splitting taking $z^n \in C(\mathbb{T})$ to $\ell^n \otimes I$ for all $n \geq 0$. Hence we get a compact quantum metric space structure on $C(\mathrm{SU}_q(2))$.

EXAMPLE 4.4. Podleś [6] introduced the quantum sphere. This is the universal C^*-algebra, denoted by $C(S^2_{qc})$, generated by two elements A and B subject to the following relations:

$$A^* = A, \quad B^*B = A - A^2 + cI,$$
$$BA = q^2AB, \quad BB^* = q^2A - q^4 + cI.$$

Here the deformation parameters q and c satisfy $|q| < 1$ and $c > 0$. We can write down two irreducible representations whose direct sum is faithful. Let $\mathcal{H}_+ = L^2(\mathbb{N}_0)$ and $\mathcal{H}_- = \mathcal{H}_+$. Define $\pi_\pm(A), \pi_\pm(B) : \mathcal{H}_\pm \to \mathcal{H}_\pm$ by

$$\pi_\pm(A)(e_n) = \lambda_\pm q^{2n} e_n \quad \text{where } \lambda_\pm = \tfrac{1}{2} \pm (c + \tfrac{1}{4})^{1/2},$$
$$\pi_\pm(B)(e_n) = c_\pm(n)^{1/2} e_{n-1} \quad \text{where } c_\pm(n) = \lambda_\pm q^{2n} - (\lambda_\pm q^{2n})^2 + c \text{ and } e_{-1} = 0.$$

Now $\pi = \pi_+ \oplus \pi_-$ is a faithful representation, so from [11],

$$C(S^2_{qc}) \cong C^*(\mathfrak{T}) \oplus_\sigma C^*(\mathfrak{T}) := \{(x, y) : x, y \in C^*(\mathfrak{T}), \sigma(x) = \sigma(y)\},$$

where $C^*(\mathfrak{T})$ is the Toeplitz algebra and $\sigma : C^*(\mathfrak{T}) \to C(\mathbb{T})$ is the symbol homomorphism. Further, we have a short exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} C(S^2_{qc}) \xrightarrow{\alpha} C^*(\mathfrak{T}) \longrightarrow 0. \tag{4.2}$$

As in the earlier case, this short exact sequence is also split exact. Here a positive splitting is given by $\ell \in C^*(\mathfrak{T}) \mapsto (\ell, \ell)$. To apply the basic theorem, note that, by the earlier example on Toeplitz extensions, we already have a Lip-norm on a dense subspace of $C^*(\mathfrak{T})$.

REMARK 4.5. These two examples were treated by Li in [5]. He produces compact quantum metric spaces using ergodic actions of compact quantum groups.

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PARTHA SARATHI CHAKRABORTY, The Institute of Mathematical Sciences,
C. I. T. Campus, Taramani, Chennai 600113, India
e-mail: parthac@imsc.res.in