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Into isometries that preserve finite dimensional structure of the range

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Abstract. In this paper we study linear into isometries of non-reflexive spaces (embeddings) that preserve finite dimensional structure of the range space. We consider this for various aspects of the finite dimensional structure, covering the recent notion of an almost isometric ideals introduced by Abrahamsen et.al., the well studied notions of a M-ideal and that of an ideal. We show that if a separable non-reflexive Banach space X, in all embeddings into its bidual X^{**}, is an almost isometric ideal and if X^* is isometric to $L^1(\mu)$, for some positive measure μ, then X is the Gurariy space. For a fixed infinite compact Hausdorff space K, if every embedding of a separable space X into $C(K)$ is an almost isometric ideal and X^* is a non-separable space, then again X is the Gurariy space. We show that if a separable Banach space contains an isometric copy of c_0 and if it is a M-ideal in its bidual in the canonical embedding, then there is another embedding of the space in its bidual, in which it is not a M-ideal.

1. Introduction

Let X be a real non-reflexive Banach space and consider the canonical embedding of X in its bidual X^{**}. A well-known form of the principle of local reflexivity, states for any finite dimensional subspace $E \subset X^{**}$ and $\epsilon > 0$ there is a linear isomorphism $T : E \to X$ such that, $\frac{1}{1+\epsilon}\|e\| \leq \|T(e)\| \leq (1+\epsilon)\|e\|$ for all $e \in E$ and such that T is identity on $E \cap X$. Such a T is called an almost isometry. See the recent article [**9**] for historical remarks and variations on this theme. Thus there is lot of information on the structure of finite dimensional subspaces of a non-reflexive space X which can be gleaned from the canonical embedding of X into X^{**}. Hence a natural question is to study spaces X which exhibit 'similar' behavior in every embedding of X in its bidual. In this paper we consider 3 interpretations of similarity of finite dimensional structure that have been well studied in the literature. *Key common factors used in our proofs are that the properties considered are transitive and pass to an intermediate space, as well as the universality of certain classical Banach spaces.*

For a general embedding of X into a Banach space Y, denoted for simplicity by, $X \subset Y$, the authors of [**1**] call X an almost isometric ideal (*a.i*-ideal, for short),

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if the above consequence of the principle of local reflexivity is valid with Y in place of X^{**}. Thus an interesting question is to consider non-reflexive Banach spaces X which under every embedding into X^{**} are almost isometric ideals in X^{**}? We note that this in particular implies that in every self embedding of X, the range is an $a.i$-ideal in X.

We recall from [**3**] and [**1**], that a separable Banach space G is said to be a Gurariy space if for any finite dimensional spaces $E \subset F$ and for any $\epsilon > 0$, any linear isometry $T : E \to G$ has an almost isometric extension to F. Using this property of the Gurariy space, it was shown in [**1**] that G is an $a.i$-ideal in every embedding into a Banach space Y. In particular G is an $a.i$-ideal in every embedding of G in G^{**}.

It is known that Gurariy space G has the property that G^* is isometric to $L^1(\mu)$ for some positive measure μ. Such spaces are called L^1-preduals and were extensively studied in [**6**]. It was shown in [**8**] that Gurariy space is isometrically unique. See also [**12**]. We show that among separable L^1-preduals, the Gurariy space is determined by the property that it is an $a.i$-ideal in every embedding into its bidual.

Let K be an infinite compact Hausdorff space and let $C(K)$ be the space of real-valued continuous functions on K, equipped with the supremum norm. Since any Banach space embeds into a $C(K)$ space, a natural question is to consider for a fixed infinite compact set K, which Banach spaces X have the property that all of its embeddings into $C(K)$ are $a.i$-ideals in $C(K)$?

If X is an L^1-predual space then it is well-known that X^{**} is isometric to a $C(K)$ space. Thus the Gurariy space has the property that for a fixed $C(K)$ space, all its embeddings are $a.i$-ideal in $C(K)$. If X is a separable Banach space such that X^* is not separable and for a fixed infinite compact set K, all the embeddings of X are $a.i$-ideals in $C(K)$, then again X is the Gurariy space.

When X is canonically embedded in its bidual, the canonical projection $Q : X^{***} \to X^{***}$ is defined by $Q(\Lambda) = \Lambda|X$, for $\Lambda \in X^{***}$. One has, $\|Q\| = 1$, $ker(Q) = X^{\perp}$ and $range(Q) = X^*$. Our next comparison is to do with the notion of an M-embedded space X, i.e., when $\|\Lambda\| = \|Q(\Lambda)\| + \|\Lambda - Q(\Lambda)\|$ for all $\Lambda \in X^{***}$ (see [**4**], Chapter III). Q is then called a L-projection. Chapter III of [**4**] contains several results on the geometric structure of such an X. In particular a L-projection with the above properties, for this embedding is unique.Thus a natural question is, can a non-reflexive M-embedded space have the property that for all embedding of X in X^{**}, there is a L- projection $P : X^{***} \to X^{***}$ such that $ker(P) = X^{\perp}$? We show that for a separable space this does not happen if X contains an isometric copy of c_0.

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2. Main Results

In this article we will be using standard terminology and results from the isometric theory of Banach spaces from [**5**]. We first need a Lemma that is a modification of Theorem 4.3 of [**1**]. We reproduce the part of the arguments that we will be needing for the sake of completeness. We recall that $X \subset Y$ is said to be an ideal (see [**2**]), if there is a linear projection $P : Y^* \to Y^*$ of norm one such that $ker(P) = X^{\perp}$. As remarked in [**1**], any $a.i$-ideal is an ideal.

LEMMA 1. *Let X be an infinite dimensional separable Banach space such that for all separable Banach spaces Y with $X \subset Y$, X is an $a.i$-ideal in Y. Then X is the Gurariy space.*

PROOF. We first note that the hypothesis implies that X is an L^1-predual space. Let K be any infinite compact Hausdorff space such that $X \subset C(K)$. For a $f \in C(K)$, let $Y = span\{f, X\}$. Now Y is separable and we have that $X \subset Y \subset C(K)$. By hypothesis X is an $a.i$-deal and in particular an ideal in Y. It now follows from the arguments given during the proof of Proposition 15 in [**10**], that X is an L^1-predual space.

By the universality of the Gurariy space G for separable L^1-predual spaces (see [**7**]), we have that $X \subset G$. Again by hypothesis we have that X is an $a.i$-ideal in G.

Now to show that X is the Gurariy space, we use the defining property of the Gurariy space mentioned in the introduction. Let $E \subset F$ be finite dimensional spaces and let $T : E \to X \subset G$ be an into isometry. Let $\epsilon,\ \delta > 0$ be such that $(1+\delta)^2 \leq 1+\epsilon$. Since G is the Gurariy space, there is a linear extension $T' : F \to G$ such that $\frac{1}{1+\delta}\|f\| \leq \|T(f)\| \leq (1+\delta)\|f\|$ for all $f \in F$. Now since X is an $a.i$-ideal in G, we get a linear map $S : T'(F) \to X$ such that $\frac{1}{1+\delta}\|T'(f)\| \leq \|S(T'(f))\| \leq (1+\delta)\|T'(f)\|$ for all $f \in F$ Further we have that $S = I$ on $T'(F) \cap X$. Thus considering $S \circ T' : F \to X$ we see that X satisfies the defining conditions of the Gurariy space. $\square$

In the proof of the following theorem we will be using results from the structure theory of separable L^1-predual spaces, whose dual is not separable. We recall that by Theorem 5 on page 226 of [**5**], X^* is isometric to $C([0,1])^*$. Also if a compact set K is not dispersed, then by Theorem 2 on page 29 of [**5**], there is a continuous surjection $\phi : K \to [0,1]$.

THEOREM 2. *Let X be a non-reflexive, separable L^1-predual space such that in every embedding of X into X^{**}, it is an $a.i$-ideal in X^{**}. Then X is the Gurariy space.*

PROOF. Suppose X satisfies the above hypothesis. To show that X is the Gurariy space, we apply the above Lemma and show that for any separable Banach space Y with $X \subset Y$, X is an $a.i.$-ideal in Y. Since $X^{**} = C(K)$ for an infinite compact set K, we see that if X^* is separable, then it is isometric to ℓ^1 so that K is homeomorphic to $\beta(N)$, which contains a perfect set. If X^* is not separable, by Theorem 5 on page 226 of [**5**], $X^* = C([0,1])^*$ and thus again K is not a dispersed space, i.e., K has a perfect set. Thus in either case by Theorem 2 on page 29 of [**5**], K can be continuously mapped onto $[0,1]$ so that by composition $C([0,1])$

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isometrically embeds into $C(K)$. Hence $C([0,1])$ isometrically embeds into X^{**}. Now as Y is separable, there is a canonical embedding of Y in $C([0,1])$. Thus for the embedding $X \subset Y \subset C([0,1]) \subset X^{**}$, we have by hypothesis that X is an $a.i$-ideal in X^{**}. It is easy to see that this implies that X is an $a.i$-ideal in Y. Therefore X is the Gurariy space. $\square$

We recall from [**7**] that the Gurariy space has non-separable dual.

COROLLARY 3. *Let X be a separable L^1-predual space with X^* non-separable. If X is an a.i-ideal in every embedding of X into X, then X is the Gurariy space.*

PROOF. Let Y be a separable Banach space with $X \subset Y$. Since X^* is not separable, it follows from Theorem 4 on page 226 of [**5**] that, X contains an isometric copy of $C(\Delta)$, where Δ is the Cantor set. As Δ is not dispersed, as before we have the embeddings, $X \subset Y \subset C([0,1]) \subset C(\Delta) \subset X$. Hence by hypothesis, X is an $a.i$-ideal in Y. Therefore X is the Gurariy space. $\square$

THEOREM 4. *Let K be an infinite compact set. Let X be a separable Banach space with X^* non-separable and such that X is embedded into $C(K)$ and in all embeddings, X is an a.i-ideal in $C(K)$. Then X is the Gurariy space.*

PROOF. Let $X \subset C(K)$ satisfy the hypothesis of the theorem. As in the proof of Lemma, we have that X is an ideal in $span\{f, X\}$ for all $f \in C(K)$, as it is also an $a.i$-ideal in $span\{f, X\}$. Therefore X is an L^1-predual space. Let Y be any separable Banach space such that $X \subset Y$. Since X^* is not separable, by the arguments given during the proof of Theorem 2, we have the embeddings, $X \subset Y \subset C([0,1]) \subset C(K)$. Therefore by hypothesis X is an $a.i$-ideal in $C(K)$ and hence in Y. Thus by Lemma 1 again, X is the Gurariy space. $\square$

REMARK 5. *We do not know for the sequence spaces, if c_0 is an a.i-ideal in c, for all embeddings? Note that in the canonical embedding $c_0 \subset c \subset \ell^\infty$, we have that c_0 is an a.i-ideal in c. See* [**10**] *for some partial positive results on self embeddings of c_0.*

REMARK 6. *A variation on this theme is to consider small bound isomorphisms instead of embeddings. We do not know if X is the Gurariy space and Y is a Banach space such that the Banach-Mazur distance, $d(X,Y) = 1$, then Y is the Gurariy space?*

We next consider non-reflexive M-embedded spaces. It is known that (see [**4**], page 132) they contain an isomorphic copy of c_0, but as such spaces can be strictly convex (see Theorem III.4.6 in [**4**]), it need not have an isometric copy of c_0. We recall that $Y \subset X$ is said to be a M-ideal, if there is an L-projection $P : X^* \to X^*$ such that $ker(P) = Y^\perp$. The following theorem illustrates the role of the canonical projection Q in the study of M-embedded spaces.

THEOREM 7. *Let X be a separable M-embedded space with an isometric copy of c_0. There is an embedding of X in its bidual, where it is not a M-ideal.*

PROOF. Suppose X is a M-ideal in every embedding of X in X^{**}. Since X has an isometric copy of c_0, we have that X^{**} contains a copy of ℓ^∞. Thus as before can consider the embeddings, $c_0 \subset X \subset C([0,1]) \subset \ell^\infty \subset X^{**}$. Since X is a M-ideal in X^{**} in this embedding, we have that X is a M-ideal in $C([0,1])$. By the

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description of M-ideals in $C([0,1])$ given by Proposition I.1.8 in [**4**], we have that X is isometric to $\{f \in C([0,1]) : f(E) = 0\}$ for some closed set $E \subset [0,1]$. Since being a M-embedded space is an isometric property, it is easy to verify that for any closed set $E \subset [0,1]$, $\{f \in C([0,1]) : f(E) = 0\}$ is not an M-embedded space. This is a contradiction. $\square$

Remark 8. *It may seem that one needs only to assume that X embeds isometrically into ℓ^∞ in the above proof. However as any M-embedded space, by Theorem III.4. 6 of* [**4**]*, is weakly compactly generated, such an assumption implies separability of X.*

Remark 9. *It is well-known that in C^*-algebras, M-ideals are precisely closed two-sided ideals. They contain isometric copy of c_0. We do not know how to produce embeddings like the ones above which are also algebraic maps.*

We recall from [**2**] that a closed subspace $Y \subset X$ is an ideal if for any finite dimensional subspace $F \subset X$ and $\epsilon > 0$, there is a linear map $T : F \to Y$ such that $T = I$ on $F \cap Y$ and $\|T\| \leq 1 + \epsilon$. It was noted in [**11**] that any Banach space X such that X^* is isometric to $L^1(\mu)$ is an ideal in every embedding of X into a Banach space Z. If such an X is infinite dimensional, then X^{**} contains an isometric copy of ℓ^∞.

Remark 10. *More generally, let X be a separable Banach such that X^{**} has an isometric copy of ℓ^∞. Again consider the embedding $X \subset \ell^\infty \subset X^{**}$. If X is an ideal in X^{**} for this embedding, we again have that X is an ideal in ℓ^∞. Now applying a characterization of L^1-predual spaces using the binary intersection property, due to Lindenstrauss (see* [**5**] *page 212), it is easy to see that X^* is isometric to a $L^1(\mu)$-space.*

We do not know a general classification of non-reflexive spaces with the property in every embedding in the bidual or in every self embedding, it is an ideal?

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