The Jordan Curve Theorem

1. Preparations

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Introduction

In this two-part article we will consider one of the classical theorems of mathematics, the *Jordan curve theorem.* It states that a *simple closed curve* (i.e., a closed curve which does not cross itself) always separates the plane, E^2, into two pieces. (For example, it is easy to see that the unit circle, $S^1 = \{x+iy \in \mathbf{C} : x^2+y^2 = 1\}$, separates the plane into two components, namely, $\{x+iy : x^2+y^2 > 1\}$ which is unbounded and the bounded component $\{x+iy : x^2+y^2 < 1\}$). While this statement appears intuitively obvious its proof is somewhat involved and actually has three parts:

1. *Jordan separation theorem:* A simple closed curve in E^2 separates it into at least two components.
2. *The nonseparation theorem:* An arc does not separate E^2.
3. *Jordan Curve theorem:* A simple closed curve separates E^2 into precisely two components.

This theorem first appeared in Jordan's *Cours d'Analyse* (1887), but his proof was faulty. The first rigorous proof was given by Veblen in 1905. The purpose of this note is to give a elementary (new?) proof of the theorem.

Preliminaries

We begin with some definitions.

1. An *arc* is a space homeomorphic to the unit interval [0,1].
2. A *simple closed curve* is a space homeomorphic to the circle S^1.

3. Given points $x, y \in E^2$, a *path* in E^2 from x to y is a continuous map $f : [0,1] \to E^2$ such that $f(0) = x, f(1) = y$.

4. An arc is said to be *piece-wise circular* if it consists of a finite number of straight line segments and circular arcs.

Similarly we can define piece-wise circular paths and piece-wise circular simple closed curves. Next we introduce some notations.

Notation: Suppose γ is an arc and $p, q \in \gamma$, then $\gamma_{[p,q]}$ denotes the part of γ joining p and q. So, $\gamma_{[p,q]} = \gamma_{[q,p]}$ and define $\gamma_{[p,q)} := \gamma_{[p,q]} - \{q\}$. Similarly we define $\gamma_{(p,q]}$ and $\gamma_{(p,q)}$.

Definition: Suppose γ is an arc and C is a circle (in E^2). We say that γ crosses C at $p \in C$ if there are points $q, r \in \gamma$ such that $\gamma_{[q,p)}$ lies inside of C and $\gamma_{(p,r]}$ lies outside of C.

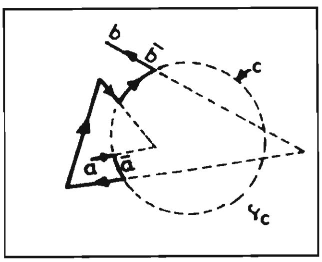
Now suppose γ is a polygonal arc and C is a circle, such that γ crosses C at least once and the end points, a and b, of γ lie 'outside' C. If we start 'moving' from a towards b along γ, let $\bar{a}$ be the first crossing and $\bar{b}$ the last. At every point p in $C - \{\bar{a}, \bar{b}\}$ we attach a direction (clockwise or anticlockwise) given by the orientation of the arc $\bar{a}p\bar{b}$.

Definition of γ_C and γ_C^*: We 'travel' from a to b, through $\gamma \cup C$ as follows. We start at a and go to $\bar{a}$. At $\bar{a}$ we have two choices we can take either of the two circular arcs towards $\bar{b}$. After we choose one of the arcs the rest of the path is determined by the following rules:

(i) while travelling on $C - \{\bar{a}, \bar{b}\}$ move towards $\bar{b}$ such that at every point $p \in C - \{\bar{a}, \bar{b}\}$ we move in the direction attached with p, namely, $\bar{a}p\bar{b}$.

(ii) At crossings change track, i.e., if we were moving on C we switch to γ and vice-versa such that whenever we change to γ we take the arc 'moving out' from C.

Figure 1.



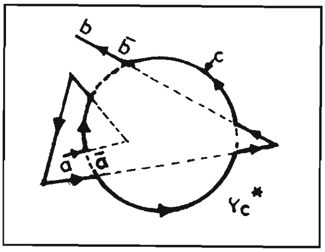


Figure 2.

Thus we get two piece-wise circular paths from a to b through $\gamma \cup C$. We denote them by γ_C and γ_C^*, and call them *exterior paths* of γ w.r.t C. (In this note we will use the same symbol to represent the path and the corresponding set.) Now let $\delta_C := (\gamma_C - \gamma)$ and $\delta_C^* := (\gamma_C^* - \gamma)$, so δ_C and δ_C^* are subsets of C. Infact δ_C and δ_C^* are finite disjoint unions of open arcs of C. Let $I_1, .\ \ , I_n$ be the sequence in which the (open) arcs are visited while moving from $\bar{a}$ to $\bar{b}$; then $\delta_C = \sqcup_{k=1}^{n} I_k$. Similarly, $\delta_C^* = \sqcup_{l=1}^{m} J_l$.

At this point we digress a little bit and give a proof of JCT for simple polygons.

Theorem 1 *A simple polygon $\mathcal{P}$ separates E^2 into two components.*

Proof Let N be a *strip neighbourhood* of $\mathcal{P}$ in E^2, i.e., N is the set of all points which are at a distance less than ϵ from the points on the polygon for some $\epsilon > 0$. If we choose N sufficiently 'thin', then it is easy to show that $(N - \mathcal{P})$ has exactly two components. Observe that any point $p \in (E^2 - \mathcal{P})$ can be connected to one 'side' of $(N - \mathcal{P})$ by a line segment. Hence the open set $(E^2 - \mathcal{P})$ has at most two components.

Consider a family of parallel lines $\{l_\alpha\}_{\alpha \in R}$ in a direction different from that of any segment of $\mathcal{P}$. Intuitively, p is outside $\mathcal{P}$ if it lies on an unbounded segment of an $(l_\alpha - \mathcal{P})$, or in general if one crosses $\mathcal{P}$ an even number of times in order to reach p from an unbounded segment of an $(l_\alpha - \mathcal{P})$. The points $p \in (E^2 - \mathcal{P})$ with this property obviously constitute an open set O, and the points with the contrary property constitute another open set I. Since O and I are disjoint, by definition, $(E^2 - \mathcal{P})$ has atleast two components, and therefore exactly two. (Observe that I is nonempty. To prove this concentrate on a particular l_α.) □

Remark: A careful scrutiny of the above proof reveals that JCT holds for a larger class of simple closed curves, namely, all those curves for which such 'thin' strip neighbourhoods exist. (If a curve comes arbitrarily close to itself such strip neighbourhoods don't exist.) Since piece-wise circular simple closed curves also have strip neighbourhoods, we have actually proved

Theorem $1'$

(i) A piece-wise circular simple closed curve separates E^2 into two components.

(ii) Let Γ be a piece-wise circular simple closed curve, p, q points in $(E^2 - \Gamma)$. If the line segment joining p and q has direction different from that of any segment of Γ and crosses Γ an odd number of times then p and q are in different components.

Jordan Separation Theorem

We shall begin with the proof of Jordan Separation Theorem. It is perhaps the easiest of the three parts. We start with a lemma which is often used to reduce the general set up of JCT to polygonal set up.

Lemma 1 *If p, q are endpoints of an arc γ contained in an open set S then there exists a polygonal arc in S joining p and q.*

Proof Cover up γ with open discs S_α such that $S_\alpha \subset S$. As γ is compact, we get a finite covering, say $\{S_{\alpha_1}, \ldots, S_{\alpha_k}\}$ such that $\cup_{i=1}^{k} S_{\alpha_i}$ is connected. (γ acts as a thread connecting all S_{α_i}'s.) Without loss of generality we could take p to be the centre of S_{α_1} and q be the centre of S_{α_k}. We form a graph G as follows. The set of vertices of G is the set $\{c_{\alpha_i}\}$ where c_{α_i} is the centre of S_{α_i}. Join c_{α_i} and c_{α_j} by an edge if $S_{\alpha_i} \cap S_{\alpha_j} \neq \phi$ and note that such an edge is completely contained in $S_{\alpha_i} \cup S_{\alpha_j}$. Observe that G is connected and $p, q \in G$. Hence there exists a polygonal arc in S joining p and q. (Observe that $G \subset S$.) $\square$

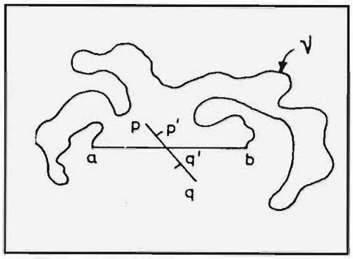


Figure 3.

Lemma 2 *Suppose γ is an arc with end points a, b and suppose that $\gamma \cup \overline{ab}$ forms a simple closed curve. Then $\gamma \cup \overline{ab}$ separates E^2.*

Proof Observe that we can choose points p, q sufficiently near to the midpoint of $\overline{ab}$, in such a way that $\overline{pq} \cap \gamma = \phi$ and $\overline{pq}$ intersects $\overline{ab}$ exactly at one point, say, r. If p, q lie in the same component of $E^2 - (\gamma \cup \overline{ab})$, then there exists a polygonal arc P joining p and q and not intersecting $\overline{ab} \cup \gamma$ (follows from Lemma 1 since an open connected set is path connected). Starting at p we 'move' along P towards q. Let

$$\beta := \inf\{x \in [0,1] \mid P(x) \in \overline{rq}\},\ \alpha := \sup\{x | P(x) \in \overline{rp}, x \leq \beta\}$$

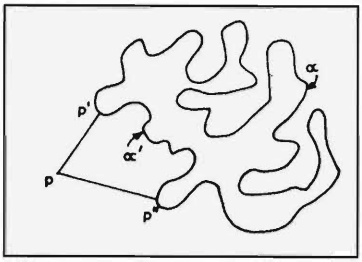
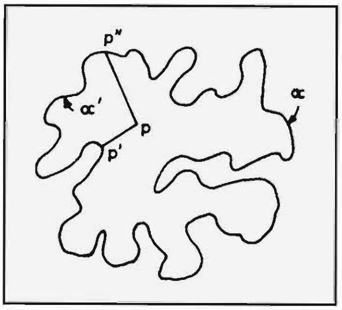


Figure 4.

As $r \in P$ (being a point in $\overline{ab} \cup \gamma$), $\alpha < \beta$. Let $P(\alpha) = p'$ and $P(\beta) = q'$. Then $\overline{p'q'} \cup P_{[p',q']}$ forms a polygonal simple closed curve, and $\overline{ab}$ intersects it (actually crosses it) exactly at one point r. Hence a, b are in different components of $E^2 - (\overline{p'q'} \cup P_{[p',q]})$. This is a contradiction since γ is an arc connecting a and b. □

Thus, given γ, a, b as above we can talk of *inside* and *outside* of $\gamma \cup \overline{ab}$, denoted by $I(\gamma \cup \overline{ab}), O(\gamma \cup \overline{ab})$, respectively, by defining the union of all bounded components to be the inside and the union of all unbounded components to be the outside. (Later it will be shown that there is one bounded and one unbounded component.) Observe that if p is a point on $\overline{ab}$ other than a, b and B_p a disc around p with sufficiently small radius then $\overline{ab}$ breaks B_p into two halves such that one lies inside $\gamma \cup \overline{ab}$ and the other outside.

Figure 5.



Proof (of the Separation Theorem): Suppose Γ is a simple closed curve in E^2. Take a point $p \in (E^2 - \Gamma)$. Then there are distinct points $p', p'' \in \Gamma$ such that the line segments $\overline{pp'}$ and $\overline{pp''}$ do not intersect Γ except at p', p'' respectively. Let α and α' denote the two arcs of Γ joining p' and p''. Then $\eta = \widehat{p'pp''} \cup \alpha$ and $\eta' = \widehat{p'pp''} \cup \alpha'$ form simple closed curves of the type considered in Lemma 2.

Define $\overline{\alpha} := \alpha - \{p', p''\}$ and $\overline{\alpha'} := \alpha' - \{p', p''\}$. Suppose $\bar{\alpha} \subset I(\eta')$. Then

$$\begin{aligned}\bar{\alpha} \subset I(\eta') \quad &\Rightarrow \quad \eta \subset O(\eta')^c \Rightarrow I(\eta) \subset O(\eta')^c \\ &\Rightarrow \quad I(\eta) \subset I(\eta') \Rightarrow \bar{\alpha}' \subset O(\eta') \Rightarrow \bar{\alpha}' \subset O(\eta).\end{aligned}$$

Thus, either $\bar{\alpha} \subset O(\eta')$ or $\bar{\alpha}' \subset O(\eta)$. Assume $\bar{\alpha} \subset O(\eta')$. Then we have that either $I(\eta') \subset O(\eta)$ or $I(\eta') \subset I(\eta)$. For, if $I(\eta')$ intersects both $O(\eta)$ and $I(\eta)$, then $I(\eta') \cap \eta \neq \phi$ which implies that $\bar{\alpha} \subset I(\eta')$, a contradiction.

First, suppose $I(\eta') \subset O(\eta)$. Take $q \in I(\eta'), r \in O(\eta) \cup O(\eta')$. Observe that if $s \in \widehat{p'pp''} - \{p', p''\}$ and B_s is a sufficiently small ball around s then $\widehat{p'pp''}$ breaks B_s into two parts, i.e., $B_s - \widehat{p'pp''}$ has two components. One of them lies inside η' (and hence, outside η) and the other part lies outside η' (and hence, inside η).

Now, if q and r lie in the same component of $(E^2 - \Gamma)$, then there exists a polygonal arc P joining r and q and not intersecting Γ. Since $q \in I(\eta')$ and $r \in O(\eta')$, P must intersect η'. But it does not intersect Γ hence it must intersect $\widehat{p'pp''}$. Let $\alpha := \inf\{x | P(x) \in \widehat{p'pp''}\}$. Then for $\epsilon > 0$, sufficiently small, $P(\alpha - \epsilon) \in B_{P(\alpha)}$, for $P(\alpha) \neq p', p''$. Therefore, $P(\alpha - \epsilon) \in O(\eta') \Rightarrow P(\alpha - \epsilon) \in I(\eta)$ (above observation). But $r \in O(\eta)$. So while moving along P we have suddenly got inside η without intersecting η, a contradiction to Lemma 2. So, q and r are in different components of $(E^2 - \Gamma)$..

If $I(\eta') \subset I(\eta)$, then take $q \in I(\eta) - I(\eta')$. (Observe that $I(\eta) = I(\eta') \Rightarrow \eta = \eta'$ and hence q exists.) By a similar argument q and r are in different components of $(E^2 - \Gamma)$. $\square$

In the second part of the article, we shall prove the nonseparation theorem and conclude the proof of the Jordan curve theorem.

Suggested Reading

[1] J R Munkres, *Topology - A first course*, Prentice-Hall, Inc., 1983.

[2] John Stillwell, *Classical Topology and Combinatorial Group Theory*, Springer Verlag, (GTM - 72), 1993.

[3] M A Armstrong, *Basic Topology*, Springer Verlag, (UTM), 1983.

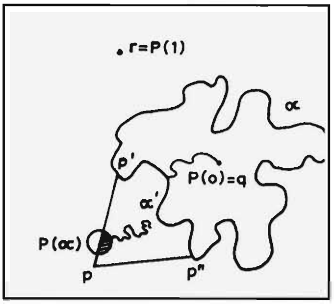


Figure 6.

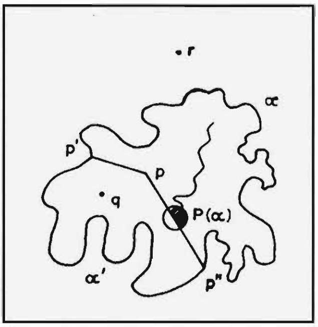


Figure 7.

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