PLANAR ALGEBRAS
AND THE OCNEANU-SZYMAŃSKI THEOREM

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ABSTRACT. We give a very simple 'planar algebra' proof of the part of the Ocneanu-Szymański theorem which asserts that for a finite index, depth two, irreducible II_1-subfactor $N \subset M$, the relative commutants $N' \cap M_1$ and $M' \cap M_2$ admit mutually dual Kac algebra structures. In the hyperfinite case, the same techniques also prove the other part, which asserts that $N' \cap M_1$ acts on M with invariants N.

The Ocneanu-Szymański theorem [Cnn, Szy] is a basic result that has inspired interesting generalisations of Hopf and Kac algebras [BhmNllSzl, NksVnr]. Given the plethora of algebraic structures inherent in the planar algebras of [Jns], it is natural to expect relationships between the planar algebra and the Kac algebra structures. These are what we elucidate in this paper for the case of an irreducible, depth two subfactor. The reducible case is treated in [Das].

1. PLANAR ALGEBRAS

We begin with a brief summary of planar algebras. For details, the reader is referred to [Jns] or to [KdyLndSnd]. The basic structure that underlies planar algebras is an action by the 'coloured operad of planar tangles', which concept we will now explain. Consider the set $Col = \{0_+, 0_-, 1, 2, \cdots\}$, elements of which we call colours. We will not define a tangle here, but several examples are shown in Figure 1. We point out some of their features.

1.1. **Tangles.** Each tangle has an external box, denoted D_0, and some (possibly 0) ordered internal boxes denoted D_1, D_2, $\cdots$. Each box has an even number of points marked on its boundary (again possibly 0)—a box with $2k$ points on its boundary is called a k-box.

If a box has at least one point marked on its boundary, one of them is distinguished and marked with a '∗'. There is also given a collection of disjoint curves, each of which is either closed or joins a marked point on one of the boxes to another such point. The whole picture is to be planar, and each marked point on a box must be the end-point of one of the curves. Finally, there is given a black-and-white shading of the regions, such that moving away from (resp. towards) the ∗ on one of the internal boxes (resp. the external box) along the curve of which it is the end-point, a black region is to the right. A 0-box is said to be a 0_+-box if the

$R = R_2^2$　　　　　$C = C_{2,2}^2$　　　　　$B = B_{2,3}^{0+}$

$W = W_{2,2}^3$　　　　　$F = F_2^3$　　　　　$M_{2,2}^2$

$L = L_2^{0+}$　　　　　I_2^2　　　　　1^2　　　　　1^3
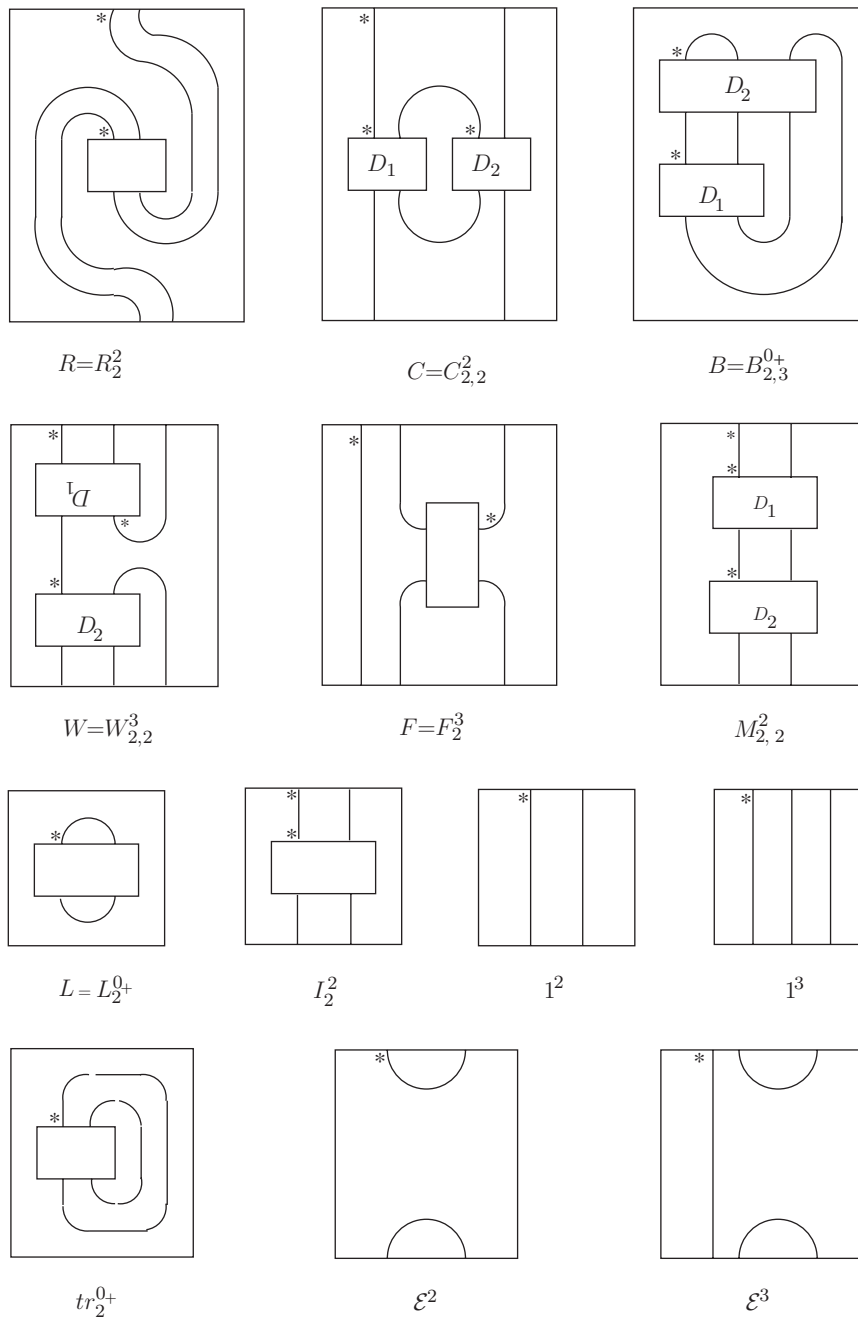
tr_2^{0+}　　　　　$\mathcal{E}^2$　　　　　$\mathcal{E}^3$

FIGURE 1. Some useful tangles

region touching its boundary is white and a 0_--box otherwise. A tangle is said to be a k-tangle if its external box is of colour k. As a matter of notation, tangles will be given names, including subscripts and a superscript. The subscripts indicate

the colours of the internal boxes and the superscript indicates the colour of the tangle. Given the superscript, the entire shading scheme is determined and will not be indicated. We will not distinguish between tangles that can be obtained from each other by a planar isotopy preserving the $*$'s, the shading and the ordering of the internal boxes.

1.2. **Operad of planar tangles.** The basic operation that one can perform on tangles is substitution of one into a box of another. If T is a tangle that has some internal boxes $D_{i_1}, \cdots, D_{i_j}$ of colours $k_{i_1}, \cdots, k_{i_j}$ and if $S_1, \cdots, S_j$ are arbitrary tangles of colours $k_{i_1}, \cdots, k_{i_j}$, then we may substitute S_t into the box D_{i_t} of T for each t, such that the '$*$'s match', to get a new tangle that will be denoted $T \circ_{(D_{i_1}, \cdots, D_{i_j})} (S_1, \cdots, S_j)$. If T has a single internal box, the subscript to $\circ$ will be omitted and indeed, the box D_1 itself will not be named, as is clear from Figure 1. The collection of tangles along with the substitution operation is called the coloured operad of planar tangles.

1.3. **Planar algebras.** A planar algebra P is an algebra over the coloured operad of planar tangles. By this we mean the following: P is a collection $\{P_k\}$ of vector spaces for $k \in Col$ and maps $Z_T : P_{k_1} \otimes P_{k_2} \otimes \cdots \otimes P_{k_b} \rightarrow P_{k_0}$ for each k_0-tangle T with internal boxes of colours $k_1, k_2, \cdots, k_b$. The collection of maps is to be 'compatible with substitution of tangles and renumbering of internal boxes' in an obvious manner. Further, planar algebras are required to be non-degenerate in the sense that for each $k \in Col$, the map $Z_{I_k^k}$ is the identity map of P_k. A pleasant verification then shows that each P_k has the structure of an associative, unital algebra, where the multiplication is given by $Z_{M_{k,k}^k}$ and the unit by $Z_{1^k}(1)$, which we will denote by $\mathbb{1}_k$. The tangles $I_k^k, M_{k,k}^k$ and 1^k are the straightforward generalisations to k-tangles of the tangles $I_2^2, M_{2,2}^2$ and 1^2 that appear in Figure 1.

1.4. **Subfactor planar algebras.** Among planar algebras, the ones that we will be interested in are the subfactor planar algebras. These are finite dimensional and connected in the sense that each P_k is a finite-dimensional vector space and $P_{0_\pm}$ are one dimensional. They have a positive modulus δ, meaning that closed loops in a tangle T contribute a multiplicative factor of δ in Z_T. They are spherical in that for a 0-tangle T, the function Z_T is not just planar isotopy invariant but also an isotopy invariant of the tangle regarded as embedded on the surface of the two sphere. Further, each P_k is a C^*-algebra in such a way that for a k_0-tangle T with internal boxes of colours $k_1, k_2, \cdots, k_b$ and $x_i \in P_{k_i}$,

$$Z_T(x_1 \otimes \cdots \otimes x_b)^* = Z_{T^*}(x_1^* \otimes \cdots \otimes x_b^*),$$

where T^* is the adjoint of the tangle T (which, by definition, is obtained from T by reflecting it and moving all reflected $*$'s clockwise by one position). Finally, the pictorial trace $\tau : P_k \rightarrow \mathbb{C} = P_{0_+}$ defined by

$$\tau(x) = \delta^{-k} Z_{tr_k^{0_+}}(x)$$

is postulated to be a faithful, positive, normalised trace. Here, $tr_k^{0_+}$ is the tangle with a single internal k-box generalising $tr_2^{0_+}$ of Figure 1.

We illustrate the process of adjunction for tangles in an example in Figure 2 for the tangle W. The tangle on the left is W and the dotted line is the line of

reflection. The tangles on the right are W^*. Note that it follows that $Z_W(a \otimes b)^* = Z_W(Z_R(b^*) \otimes Z_R(a^*))$ in any subfactor planar algebra.
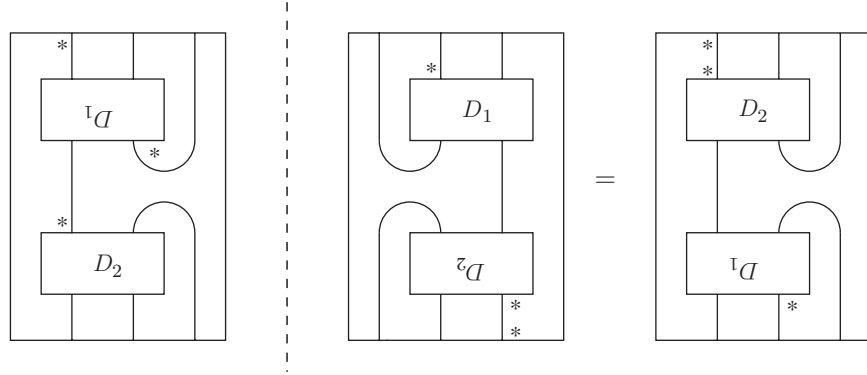
FIGURE 2. The adjoint of W

1.5. **Jones' theorem.** The fundamental theorem of Jones asserts that given an extremal II_1-subfactor $N \subset M$ of finite index δ^2 and basic construction tower $N \subset M(= M_0) \subset^{e_1} M_1 \subset^{e_2} \cdots \subset^{e_k} M_k \subset^{e_{k+1}} \cdots$, there is a subfactor planar algebra $P = P^{N \subset M}$, which is unique subject to certain conditions, such that $P_k = N' \cap M_{k-1}$ for $k \geq 1$, and that conversely all subfactor planar algebras arise in this manner. We will use the following four consequences:

(i) the pictorial trace on P_k agrees with the restriction to P_k of the canonical trace on M_{k-1},

(ii) for $1 \leq k \leq n$, $P_{k,n} \overset{def}{=} ran(Z_{Q(k)_n^n})$ is equal to $M'_{k-1} \cap M_{n-1}$, where $Q(k)_n^n$ is the n-tangle in Figure 3, where there are k strands going straight through and k
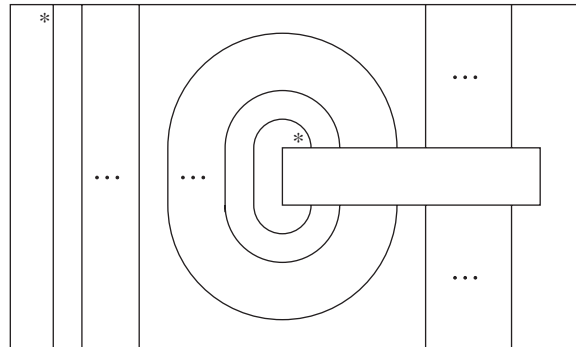
FIGURE 3. The tangle $Q(k)_n^n$

loops on the internal n-box.

(iii) $Z_{\mathcal{E}^{k+1}}(1) = \delta e_k$ for $k \geq 1$, where $\mathcal{E}^{k+1}$ is the $k+1$-tangle generalising $\mathcal{E}^2$ and $\mathcal{E}^3$ of Figure 1, and

(iv) $Z_{(E')_n^n}(x) = \delta E_{M'}(x)$ for all $x \in P_n$, where $E_{M'}$ is the N'-trace preserving conditional expectation of P_n onto $M' \cap M_{n-1}$, and $(E')_n^n$ is the tangle $Q(1)_n^n$.

2. Kac algebra structure on relative commutants

For the rest of this paper, we fix an irreducible, depth two subfactor $N \subset M$ and let $P = P^{N \subset M}$ so that $P_{0_\pm} = \mathbb{C}$ and $P_k = N' \cap M_{k-1}$ for $k \geq 1$. In particular, by irreducibility, $P_1 = \mathbb{C}$. We will show, in this section, that the relative commutants $N' \cap M_1$ and $M' \cap M_2$ are endowed with the structure of a pair of dual Kac algebras.

2.1. Hopf and Kac algebras.
We restrict ourselves to finite-dimensional algebras. Recall that a finite-dimensional complex vector space H equipped with maps $\mu : H \otimes H \to H$, $\eta : \mathbb{C} \to H$, $\Delta : H \to H \otimes H$, $\epsilon : H \to \mathbb{C}$ and $S : H \to H$ is said to be a Hopf algebra if

   (i) (H, μ, η) is an associative unital algebra,
   (ii) (H, Δ, ϵ) is a coassociative, counital coalgebra,
   (iii) the bialgebra conditions hold, i.e., Δ and ϵ are unital algebra maps and
   (iv) the antipode conditions hold, i.e., S (is the unique map that) satisfies $\mu \circ (S \otimes id_H) \circ \Delta = \eta \circ \epsilon = \mu \circ (id_H \otimes S) \circ \Delta$.

The dual vector space H^* admits a Hopf algebra structure for the dual maps $\Delta^*, \epsilon^*, \mu^*, \eta^*$ and S^*.

A Hopf algebra H is said to be a Kac algebra if it is, in addition to all the above, a C^*-algebra with Δ being a $*$-homomorphism. The dual vector space then acquires a Kac algebra structure for the $*$-structure defined by $f^*(a) = \overline{f(Sa^*)}$.

We will find it convenient to adopt (with some minor changes) Sweedler's notation for comultiplication and its iterates. In this notation it is usual to denote $\Delta(a)$ by $\sum_{(a)} a_{(1)} \otimes a_{(2)}$. We will omit the parentheses from the subscripts and the summation symbol and simply denote $\Delta(a)$ by $a_1 \otimes a_2$.

2.2. Definition of S, Δ and ϵ.
Let $S : P_2 \to P_2$ be Z_R. The tangle equations $R \circ R = I_2^2$ and $R^* = R$ imply that S is involutive (in particular, surjective) and commutes with $*$. The depth assumption implies that an arbitrary element of P_3 can be written as a linear combination of those of the form ae_2b for $a, b \in P_2$, and therefore also as a linear combination of $(Sa)e_2b = \delta^{-1} Z_W(a \otimes b)$, the equality following from (iii) above. This shows that $Z_W : P_2 \otimes P_2 \to P_3$ is surjective. Consideration of the graph invariants of $N \subset M$ in conjunction with the depth and irreducibility assumptions shows that the index δ^2 is an integer (which we denote n) and that the dimension of P_k is n^{k-1}. Hence Z_W is an isomorphism. The equations of Figure 4 hold for each $a \in P_2$ since P_1 is $\mathbb{C}$ and so an arbitrary element is a scalar multiple of its identity $\mathbb{1}_1$, where the scalar is just its trace. The last equality appeals, in addition, to the sphericality of P.

For $a \in P_2$, define $\Delta(a) \in P_2 \otimes P_2$ and $\epsilon(a) \in \mathbb{C} = P_{0_+}$ by the equations

$$\Delta(a) = Z_W^{-1} \circ Z_F(a), \quad \epsilon(a) = \delta^{-1} Z_L(a).$$

Figure 5 is a pictorial depiction of these definitions.

We now show that P_2 endowed with the maps Δ, ϵ and S is a Kac algebra with dual Kac algebra $P_{1,3}$.

2.3. Δ and ϵ are unital algebra maps.
The multiplicativity of Δ follows from Figure 6 which shows that $Z_F(ab) = Z_W(a_1b_1 \otimes a_2b_2)$.

That Δ is unital is a consequence of the obvious tangle equation $F \circ 1^2 = W \circ_{(D_1, D_2)} (1^2, 1^2)$. Multiplicativity of ϵ follows from its definition and use of Figure 4 while $\epsilon(\mathbb{1}_2) = 1$ follows from the fact that the planar algebra P has modulus δ.
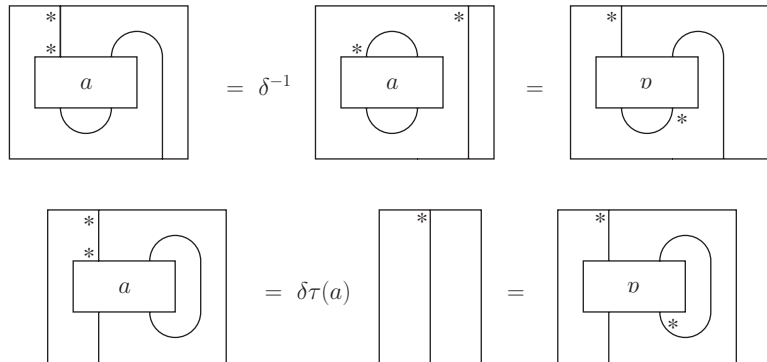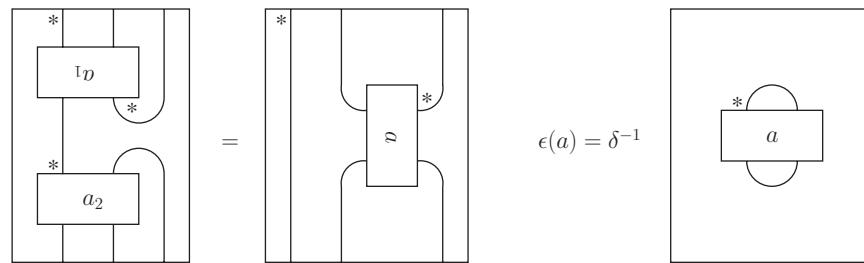
FIGURE 4. Pictorial equations for $a \in P_2$

FIGURE 5. Pictorial definitions of Δ and ϵ

2.4. **Coassociativity of Δ and counitality of ϵ.** We prove these by showing that with respect to a non-degenerate pairing between P_2 and $P_{1,3}$, Δ and ϵ are dual, respectively, to the multiplication and unit of $P_{1,3}$. The tangle B gives a map $Z_B : P_2 \otimes P_3 \to \mathbb{C}$. Since Z_F is known—and easily verified—to give an isomorphism from P_2 onto $P_{1,3}$ and $Z_B(a \otimes Z_F(b)) = \delta^2 \tau(ab)$, the non-degeneracy of τ implies that the pairing $\langle \cdot \mid \cdot \rangle : P_2 \otimes P_{1,3} \to \mathbb{C}$ defined by $\langle a|f \rangle = \delta^{-1} Z_B(a \otimes f)$ for $a \in P_2$ and $f \in P_{1,3}$ is a non-degenerate pairing satisfying $\langle a|Z_F(b) \rangle = \delta\tau(ab)$.

Verifying that P_2 and $P_{1,3}$ are a dual pair of bialgebras amounts to checking that

$$\begin{aligned} \langle a_1|f \rangle \langle a_2|g \rangle &= \langle a|fg \rangle \quad \text{and} \\ \epsilon(a) &= \langle a|\mathbb{1}_3 \rangle, \end{aligned}$$

for all $a \in P_2$ and $f, g \in P_{1,3}$. We set $f = Z_F(x), g = Z_F(y)$ for $x, y \in P_2$. Observe that $Z_F(x)Z_F(y) = Z_{(F \circ C)}(x \otimes y)$ and therefore Figure 7 verifies the first equation. The second equation holds since $B \circ_{D_2} 1^3 = L$ and by definition of ϵ.

2.5. **The antipode condition.** To see that they are Hopf algebras, it suffices to check that id_{P_2} has a one-sided convolution inverse, for then the finite dimensionality implies that it is an antipode for P_2. The proof in Figure 8 shows that $S(a_1)a_2 = \epsilon(a)\mathbb{1}_2$ for $a \in P_2$. Note the second equality follows from the pictorial definition of Δ, while the third is a consequence of the pictorial definition of ϵ and Figure 4. Hence P_2 and $P_{1,3}$ are a dual pair of Hopf algebras. In particular, S is an anti-algebra and anti-coalgebra map.
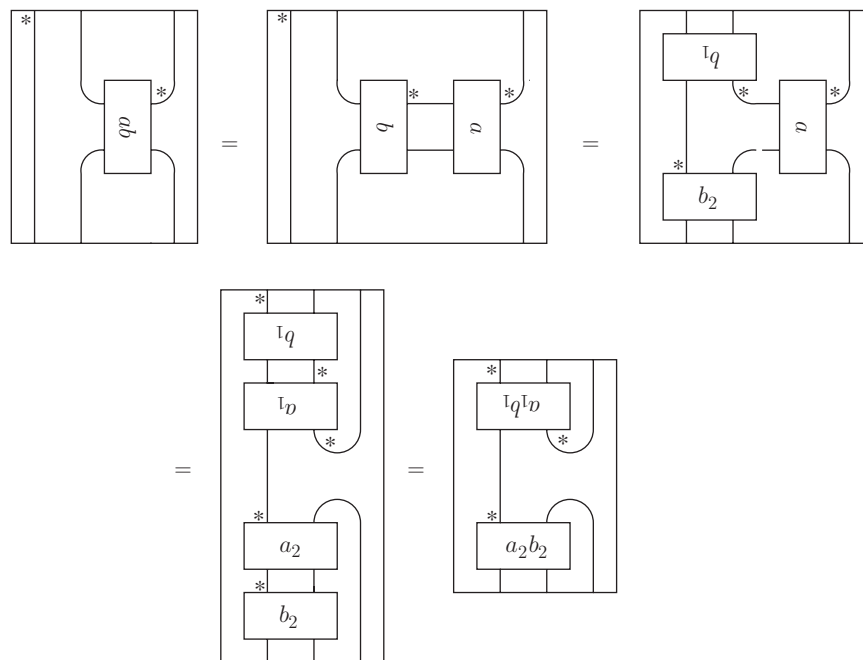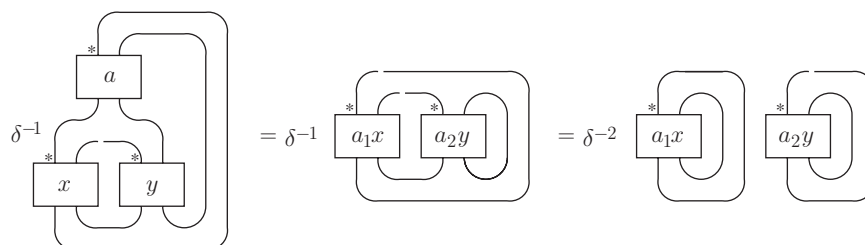
FIGURE 6. Multiplicativity of Δ
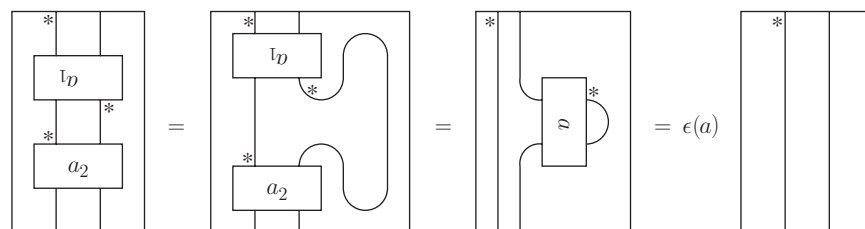

FIGURE 7. Duality for Δ

FIGURE 8. S is an antipode

2.6. Δ **is a $*$-map.** To see that Δ commutes with $*$, use the definition of the adjoint tangle to take $*$ of both sides of $Z_W(a_1 \otimes a_2) = Z_F(a)$. This gives

$$Z_W(S((a_2)^*) \otimes S((a_1)^*)) = Z_F(S(a^*)).$$

Now use the commutativity of S and $*$ in all three places, replace Sa by a, and use that S is an anti-coalgebra map to read off that $a_1^* \otimes a_2^* = (a^*)_1 \otimes (a^*)_2$.

2.7. Kac algebra duality. Finally, to see that $\langle a|f^* \rangle = \overline{\langle Sa^*|f \rangle}$, suppose that $f = Z_F(b)$ for $b \in P_2$. It is easy to see that $f^* = Z_F(Sb^*)$. Then $\langle a|f^* \rangle = \delta\tau(aSb^*)$, while $\langle Sa^*|f \rangle = \delta\tau(Sa^*b)$. These are conjugates of each other since τ is positive, $\tau \circ S = \tau$ by sphericality of P, and S is a $*$-preserving anti-algebra involution. This establishes duality as Kac algebras.

3. ACTION ON M

We will now verify that—in the hyperfinite case—the Kac algebra $P_2 = N' \cap M_1$ 'acts on the factor M with invariants N'.

3.1. Definition of an anti-action. Begin by defining, for $a \in P_2$, a map $\alpha_a^{(n)} :$ $P_{1,n} \to P_{1,n}$ pictorially by the equation in Figure 9. Note that, by definition, the argument of $\alpha_a^{(n)}$ is an arbitrary element of $P_{1,n}$. It should be clear that the $\alpha_a^{(n)}$ are compatible with the inclusions of $P_{1,n}$ into $P_{1,n+1}$ and thus patch up to define a map, which we will denote α_a, from $\bigcup_n P_{1,n}$ to itself.
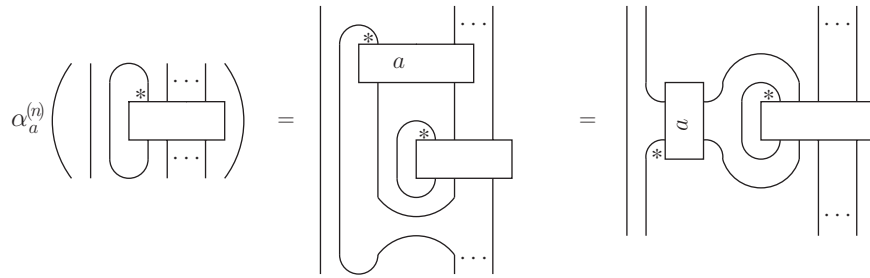

FIGURE 9. Definition of the action

3.2. Invariants of the anti-action. Easy pictorial proofs show that $\alpha_1 = id$, $\alpha_{ab} = \alpha_a \circ \alpha_b$ and that $\alpha_a(\mathbb{1}_k) = \epsilon(a).\mathbb{1}_k$ for all $a, b \in P_2$. The two other verifications needed to see that $a \mapsto \alpha_a$ defines an anti-action of P_2 on $\bigcup_n P_{1,n}$ are that $\alpha_a(xy) = \alpha_{a_2}(x)\alpha_{a_1}(y)$—recall the notation introduced in §2.1—and that $\alpha_a(x)^* = \alpha_{Sa^*}(x^*)$. The former follows from Figure 10 below, while the latter is an easy consequence of the definition of the adjoint tangle (and, of course, the fact that P is a subfactor planar algebra).

Observe that for this anti-action of P_2 on $P_{1,n}$, the 'invariants' defined by $\{x \in P_{1,n} : \alpha_a(x) = \epsilon(a).x$ for all $a \in P_2\}$ are exactly the elements of $P_{2,n}$. That elements of $P_{2,n}$ are invariant follows from the definition of the action and Figure 4. For the other inclusion, observe that $\alpha_{e_1}(Z_{Q(1)_n^n}(y)) = Z_{Q(2)_n^n}(y)$ and therefore for $x \in P_{1,n}$, even $\alpha_{e_1}(x) = x$ implies that $x \in P_{2,n}$.

3.3. Conclusion of the proof. Using Jones' theorem to identify $N' \cap M_{n-1}$ with $P_{1,n}$ and $M' \cap M_{n-1}$ with $P_{2,n}$, we have thus defined an anti-action $x \mapsto \alpha_a(x)$ of P_2 on $\bigcup_n N' \cap M_{n-1}$ with invariants $\bigcup_n M' \cap M_{n-1}$. An appeal to consequence (iv) of Jones' theorem and the first equality in Figure 9 shows that $\alpha_a(x) = \delta^2 E_{M'}(axe_1)$ and that therefore α_a is continuous for the weak topology on N'. This implies that
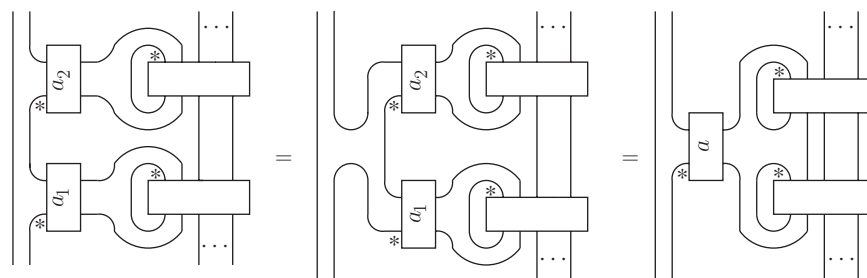
FIGURE 10. Verification of $\alpha_a(xy) = \alpha_{a_2}(x)\alpha_{a_1}(y)$

α_a admits a unique weakly continuous extension to the weak closure $M' \cap M_\infty$ which is still an anti-action and has invariants $M_1' \cap M_\infty$. Here M_∞ is the weak closure of $\cup M_n$ in its natural representation on the Hilbert space obtained by the GNS construction with respect to its unique trace. In the hyperfinite case, the fundamental result of Popa (see Theorem 5.5.1 of [Ppa]) shows that the subfactor $M_1' \cap M_\infty \subset M' \cap M_\infty$ is anti-isomorphic to $N \subset M$ and thus the anti-action of P_2 gets translated to an action on M with invariants N.

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REFERENCES

[BhmNllSzl] G.Bohm, F.Nill and K.Szlachanyi, *Weak Hopf algebras. I. Integral theory and C^*-structure*, J. Algebra 221 (1999), no. 2, 385–438 MR1726707 (2001a:16059)

[Das] Paramita Das, *Weak Hopf C^*-algebras and depth two subfactors*, J. Funct. Anal. 214 (2004), no. 1, 74-105 MR2079886

[Jns] V. F. R. Jones, *Planar Algebras I*, New Zealand Journal of Mathematics, To appear.

[KdyLndSnd] V. Kodiyalam, Zeph Landau and V. S. Sunder, *The planar algebra associated to a Kac algebra*, Proc. Ind. Acad. Sciences, 113 (2003), no. 1, 15-51 MR1971553 (2004d:46075)

[NksVnr] D.Nikshych, L.Vainerman, *A characterization of depth 2 subfactors of II_1 factors*, J. Funct. Anal. 171 (2000), no. 2, 278–307 MR1745634 (2000m:46129)

[Cnn] A.Ocneanu, *Quantized groups, string algebras and Galois theory for algebras*, Operator algebras and applications, Vol. 2, 119–172, London Math. Soc. Lecture Note Ser., 136, Cambridge Univ. Press, Cambridge, 1988. MR0996454 (91k:46068)

[Ppa] S. Popa, *Classification of amenable subfactors of type II*, Acta Math. 172 (1994), 163–255. MR1278111 (95f:46105)

[Szy] W.Szymański, *Finite index subfactors and Hopf algebra crossed products*, Proc. Amer. Math. Soc. 120 (1994), no. 2, 519–528. MR1186139 (94d:46061)

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