What can the Answer be?

1. Elementary Vector Analysis

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A very useful approach in tackling scientific problems is to ask what the answer could possibly be, under the constraints of the given problem. In the first part of the series, this approach is illustrated with some examples from elementary vector analysis.

Scientific problems are very often first solved by a combination of analogy, educated guesswork and elimination – in short, 'insight'. The refinements that come later do not make this earlier process less important. Rather, they generally serve to highlight its value.

There is no graded set of lessons by which one progressively gains insight. However, a profitable line of approach is to ask, at each stage, what the answer to a problem could possibly be, subject to the conditions involved. Techniques such as dimensional analysis, scaling arguments and order-of-magnitude estimates, as well as checks based on limiting values or limiting cases are part of the armoury in this mode of attack. In this set of three articles, I shall use a series of examples mainly in *elementary vector analysis* in an attempt to provide a flavour of this approach.

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An Example from Algebra

To set the stage, let us begin with an example in elementary algebra. Consider the determinant

$$\Delta(x_1, x_2, x_3) = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} \qquad (1)$$

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It is straightforward, of course, to find Δ explicitly by expanding the determinant. But the point I wish to make here is that Δ can be evaluated almost by inspection, if we note the following facts:

- Multiplying each of x_1, x_2 and x_3 by some number λ multiplies the value of Δ by λ^3. Thus Δ is a *homogeneous* function of degree 3.
- Δ vanishes if any two the x's are equal. Therefore, regarded as a function of x_1, Δ is quadaratic with factors $(x_1 - x_2)$ and $(x_1 - x_3)$; and similarly for x_2 and x_3.
- Δ changes sign if any two of the x's are interchanged.

Combining these points, we conclude that Δ *must* be given by

$$\Delta(x_1, x_2, x_3) = C(x_1 - x_2)(x_2 - x_3)(x_3 - x_1), \qquad (2)$$

where C is some numerical constant.

- To find the constant C, we have only to look at a simple case, e.g., $x_1 = 0, x_2 = 1, x_3 = 2$. This gives $C = 1$. (Alternatively, match the term $+x_2x_3^2$ obtained by multiplying together all the diagonal elements of the determinant with the corresponding term $+Cx_2x_3^2$ on the right in equation (2)). Finally, therefore, we have

$$\Delta(x_1, x_2, x_3) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2). \quad (3)$$

The factors have been written in such a way that *selecting first term in each bracket yields the product of the diagonal elements of the determinant with the correct sign.*

What is important is that our chain of reasoning permits us to *generalize* this result to the case of the $(n \times n)$ determinant (called the Vandermonde determinant)

$$\Delta(x_1, \ldots, x_n) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ . & . & \ldots & . \\ x_1^{(n-1)} & x_2^{(n-1)} & \ldots & x_n^{(n-1)} \end{vmatrix} \qquad (4)$$

We can now see that Δ must simply be a product of the $n(n-1)/2$ distinct factors $(x_j - x_k)$ that can be formed from the variables $x_1, \ldots, x_n$. The *diagonal element* $+x_2 x_3^2 x_4^3 \ldots x_n^{n-1}$ indicates that the sign of each term in Δ is taken care of if we always maintain $j > k$ in each factor $(x_j - x_k)$. Therefore we have the general result

$$\Delta(x_1, \ldots, x_n) = \prod_{1 \le k < j \le n} (x_j - x_k) \qquad (5)$$

without going through a tedious calculation. This is the spirit in which we shall approach the problems that follow.

Some Vector Identities

Let us now go on to vector analysis. As the first example, we consider the derviation of the identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \qquad (6)$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three arbitrary vectors (in the usual three dimensional space, say). We would like to avoid the 'brute force' method of writing out components, etc. in some particular coordinate system. We therefore proceed as follows. Let $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{d}$.

- If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three general non-planar vectors in three-dimensional space, any arbitrary vector can be uniquely written as a linear combination of these three vectors. (They serve to define a set of 'oblique' axes). But $\mathbf{d}$ cannot have any component along **a**, because, as is easily seen, $\mathbf{b} \times \mathbf{c}$ is perpendicular to **b**, **c** and **d** but not to **a**. Therefore, in general, **d** must be expressible as

$$\mathbf{d} = \beta\mathbf{b} + \gamma\mathbf{c} \tag{7}$$

- where β and γ are *scalars*. Note that this argument is valid even in the case of oblique axes, i.e. **b** and **c** are *not* required to be perpendicular to **a**.

- **d** is of first order in each of the vectors **a**, **b** and **c**: that is, multiplying any one of them by a constant multiplies the answer by the same constant; further, **d** vanishes if any of these three vectors is zero. Therefore β must be proportional to ($\mathbf{a}{\cdot}\mathbf{c}$ and γ to $(\mathbf{a} \cdot \mathbf{b})$, as these are the only first-order scalars that can be formed from $(\mathbf{a}, \mathbf{c})$ and $(\mathbf{a}, \mathbf{b})$ respectively. Hence

$$\mathbf{d} = \lambda(\mathbf{a} \cdot \mathbf{c})\mathbf{b} + \mu(\mathbf{a} \cdot \mathbf{b}))\mathbf{c}, \tag{8}$$

where λ and μ are absolute constants – dimensionless pure numbers – independent of **a**, **b** and **c**.

- But **d** changes sign if **b** and **c** are interchanged, because $\mathbf{c} \times \mathbf{b} = -\ \mathbf{b} \times \mathbf{c}$. Therefore

$$-\mathbf{d} = \lambda(\mathbf{a} \cdot \mathbf{b})\mathbf{c} + \mu(\mathbf{a} \cdot \mathbf{c})\mathbf{b}. \tag{9}$$

Comparison with equation (8) gives $\mu = -\lambda$, so that

$$\mathbf{d} = \lambda[(\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b}]. \tag{10}$$

- Having nearly solved the problem, we may *now* determine λ by looking at an appropriate simple special case because equation (10) is valid for *all* **a**, **b**, **c**. Thus, getting $\mathbf{a} = \mathbf{i}$, $\mathbf{b}$=$\mathbf{i}$, $\mathbf{c}$=$\mathbf{j}$, we get $\mathbf{d} = -\mathbf{j}$ by direct evaluation of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, while the right-hand side of equation (10) gives $-\lambda\mathbf{j}$. Hence $\lambda = 1$. We thus obtain the general formula quotes in equation (6).

The arguments used above can be repeated to tackle numerous other cases. Let us consider, for instance, the scalar product of $(\mathbf{a} \times \mathbf{b})$ and $(\mathbf{c} \times \mathbf{d})$, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are four arbitrary vectors. We again observe that

- the expression is linear (of first order) in each of the vectors, and
- the presence of $(\mathbf{a} \times \mathbf{b})$ and $(\mathbf{c} \times \mathbf{d})$ does not allow any contribution proportional to $(\mathbf{a} \cdot \mathbf{b})$ and $(\mathbf{c} \cdot \mathbf{d})$. Hence the answer *must* be of the form

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \lambda(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})+\mu(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \quad (11)$$

  where λ and μ are pure numbers.

- As before, since the answer changes sign if $\mathbf{a}$ and $\mathbf{b}$ are interchanged, we get $\lambda = -\mu$.
- Finally, the overall constant factor is fixed by looking at a special case, e.g., $\mathbf{a} = \mathbf{c} = \mathbf{i}$, $\mathbf{b} = \mathbf{d} = \mathbf{j}$. This gives $\lambda = 1$. We thus obtain the familiar formula

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}). \quad (12)$$

The formulae (6) and (12) are, of course, well known, and several different proofs of their validity can be given. My aim has been to bring out the fact that *general considerations of linearity, symmetry (or antisymmetry), dimensionality, homogeneity, etc. practically determine the final answer in such problems.* This is brought home even more convincingly in the example I came across sometime ago in an entrance test for admission to a research institute.

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Evaluation of Some Integrals

We will first evaluate the integral

$$I_4 = \int (\mathbf{e}_r \cdot \mathbf{a})(\mathbf{e}_r \cdot \mathbf{b})(\mathbf{e}_r \cdot \mathbf{c})(\mathbf{e}_r \cdot \mathbf{d})d\Omega, \qquad (13)$$

where the unit radial vector $\mathbf{e}_r$ varies over the surface of a sphere of unit radius centred at the origin. Here $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are four arbitrary constant vectors, which is why I have used the notation I_4. (Such integrals occur in several contexts in physical calculations – for example, in the theory of collisions of particles). A brute force approach to the evaluation of I_4 is a formidable task, but there is a very 'physical' way of tackling the problem. We may try to simplify the task by choosing spherical polar coordinates with the polar axis along one of the given vectors, say $\mathbf{a}$. But his does not help much, because there are *three* other vectors pointing in arbitrary directions. Instead, we note that I_4 (i) is a scalar, (ii) is of first order in *each* of the four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and vanishes if any one of them is zero, and (iii) is *totally symmetric* under the interchange of any of these vectors among themselves. Therefore I_4 *must* be of the form

$$I_4 = \lambda(\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) + (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) + (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}), \quad (14)$$

where λ is a pure number. The plus signs and the overall constant λ for each term follow from (iii) above. (To be precise – and this will be relevant further on – we have also used the fact that $(\mathbf{a}\cdot\mathbf{b})$ $(\mathbf{c}\cdot\mathbf{d}) = (\mathbf{c}\cdot\,\mathbf{d})$ $(\mathbf{a}\cdot\mathbf{b})$, as well as $\mathbf{a}\cdot\mathbf{b}{=}\mathbf{b}\cdot\,\mathbf{a}$.) Likewise, combinations such as $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\,\mathbf{d})$ are not allowed by this symmetry, (iv) The constant λ is now determined by going over to the special case $\mathbf{a}{=}\ \mathbf{b}{=}\mathbf{c}{=}\ \mathbf{d}{=}\mathbf{k}$ (the unit vector along the polar or z-axis). In that case, since $\mathbf{e}_r\cdot k = \cos\theta$, I_4 reduces by direct evaluation to

$$I_4 = \int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\psi\cos^4\theta = \frac{4\pi}{5} \qquad (15)$$

on the one hand, while equation (14) gives $I_4 = 3\lambda$ on the other. Hence $\lambda = 4\pi/15$, completing the evaluation of I_4.

Generalization is again tempting and possible! We see at once that all the odd numbered integrals $I_1, I_2, I_5, \ldots$

must *vanish identically*, because there is no way that we can form a *scalar* from an *odd* number of vectors **a**, **b**, **c**, ... that satisfies both (ii) and (iii) listed above. What about the corresponding general integral of *even* order,

$$I_{2n} = \int d\Omega \prod_{i=1}^{2n} (\mathbf{e}_r \cdot \mathbf{a}_i) \tag{16}$$

involving the $2n$ arbitrary vectors $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_{2n}$? The arguments given earlier now yield

$$I_{2n} = \lambda \sum_{\{i_1\}} (\mathbf{a}_{i_1} \cdot \mathbf{a}_{i_2})(\mathbf{a}_{i_3} \cdot \mathbf{a}_{i_4}) \cdots (\mathbf{a}_{i_{2n-1}} \cdot \mathbf{a}_{i_{2n}}), \tag{17}$$

where λ is a constant, yet to be determined, and $(i_1, i_2, \ldots, i_{2n})$ is a permutation of $(1, 2, \ldots, 2n)$. An interesting little bit of combinatorics enters here. The summation in equation (17) is over all the possible permutations subject to the conditions that (i) $(\mathbf{a}_i \cdot \mathbf{a}_j)$ and $(\mathbf{a}_j \cdot \mathbf{a}_i)$ are not counted as two different combinations, and (ii) all the n permutations of each set of n scalar products $(\mathbf{a}_{i_1} \cdot \mathbf{a}_{i_2}) \cdots ((\mathbf{a}_{i_{2n-1}} \cdot \mathbf{a}_{i_{i_{2n}}})$ are counted only once. The number of distinct terms in the summation in 17) is therefore $(2n!)/(2^n n!)$. The special case of $\mathbf{a}_1 = \ldots = \mathbf{a}_{2n} = \mathbf{k}$ now gives

$$I_{2n} = 2\pi \int_{-1}^{1} d(\cos\theta)\cos^{2n}\theta = \frac{4\pi}{2n+1}, \tag{18}$$

while the right-hand side of equation (17) reduces to $\lambda(2n)!/2^n n!)$. The constant λ in equation (17) is therefore given by

$$\lambda = 2^{n+2\pi} \frac{n!}{(2n+1)!}. \tag{19}$$

This completes the evaluation of the general integral I_{2n}.

A *further* generalization of these results that suggests itself (and which may indeed occur in actual calculations) is the evaluation of such integrals in an arbitrary number d of *dimensions*. In other words, what is

$$I_{n,d} = \int d\Omega (\mathbf{e}_r \cdot \mathbf{a}_1) \ldots (\mathbf{e}_r \cdot \mathbf{a}_n), \tag{20}$$

where $\mathbf{e}_r$ varies over the surface of a unit sphere in d-dimensional (Euclidean) space? I leave the further exploration of this question to the reader.

Concluding Remarks

In the next part of the series we will see how, taking off from the simple vector identity in equation (6), we can understand concepts such as reciprocal bases, dual spaces, and bra and ket vectors. The concepts are extremely useful in many branches of physics including, among others, quantum mechanics and solid state physics.

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