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Effect of boundary conditions on the invariant density of noisy maps at fully-developed chaos

V BALAKRISHNAN, G NICOLIS* and C NICOLIS†
Department of Physics, Indian Institute of Technology, Madras 600 036, India
*Center for Nonlinear Phenomena and Complex Systems, Université Libre de Bruxelles, C.P. 231, 1050 Bruxelles, Belgium
†Institut Royal Météorologique de Belgique, 1180 Bruxelles, Belgium

Abstract. The invariant density of one-dimensional maps in the regime of fully-developed chaos with uncorrelated additive noise is considered. Boundary conditions are shown to play a significant role in determining the precise form of the invariant density, via the manner in which they handle the spill-over, caused by the noise, of orbits beyond the interval. The known case of periodic boundary conditions is briefly recapitulated. Analytic solutions for the invariant density that are possible under certain conditions are presented with applications to specific well-known maps. The case of 'sticky' boundaries is generalized to 're-injection at the nearest boundary', and the exact functional equations determining the invariant density are derived. Interesting boundary layer effects are shown to occur, that lead to significant modifications of the invariant density corresponding to the unperturbed (noise-free) case, even when the latter is a constant – as illustrated by an application of the formalism to the noisy tent map. All our results are non-perturbative, and hold good for any noise amplitude in the interval.

Keywords. One-dimensional maps; fully developed chaos; invariant density; boundary conditions; noise.

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1. Introduction

Random noise is almost inevitably present in systems of physical interest. Its effects on the evolution of these systems in time have long been recognized as worthy of investigation, and chaotic dynamics is no exception. From the earliest days of the modern phase in the study of deterministic chaos, there has been interest in analyzing the consequences of the inclusion of noise in chaotic flows and maps. The broad features of the resulting dynamics are understood reasonably well (see [1]–[5] and references therein).

In this paper, we re-examine the problem of one-dimensional, fully chaotic discrete dynamics with additive noise, with the aim of highlighting an aspect that has not, in our opinion, received the attention it deserves: the "spill-over" of the state variable beyond its original domain (interval) caused by the additive noise – equivalently, the effect of different boundary conditions. Studies of noisy maps have for the most part focussed on the regimes preceding fully-developed chaos, where this spill-over is perhaps not so significant, as the corresponding attractors do not comprise the entire interval. In contrast, we restrict our attention to the case of fully-developed chaos – in particular, the effect of specific boundary conditions on the invariant densities of noisy maps. In general, it is found that different boundary conditions lead to significant differences in the form of the invariant density. As the latter quantity is a central one in the statistical description of the systems concerned, we are led to the conclusion that the conjunction of noise and boundary conditions must be handled with care.

In § 2, we recapitulate the formulation of the problem along the lines best suited for what is to follow. In § 3, we consider the case of periodic boundary conditions, the one most frequently adopted in the literature [5]–[9]. After a brief summary of the formalism we point out instances where a complete solution is possible, provided certain conditions are met. In § 4, we introduce what we believe is a realistic boundary condition, namely, re-injection at the nearest boundary, derive the exact integral equation obeyed by the invariant density, and analyze the structure of its solution. A numerical illustration is provided for the case of the tent map. In § 5, we consider the generalized dimension D_q corresponding to the noisy maps investigated. Section 6 contains our concluding remarks.

2. Frobenius–Perron equation for a noisy map

Consider the one-dimensional map $x_{n+1} = f(x_n)$, $x_0 \in [a, b]$. We assume that the map is onto, i.e., we consider the case of 'fully-developed' chaos. Well-known examples include the logistic map $f(x) = 4x(1-x)$ and the tent map $f(x) = 1 - |2x-1|$ in $[0, 1]$, the square-root cusp map $f(x) = 1 - 2|x|^{\frac{1}{2}}$ in $[-1, 1]$, and so on. The invariant density $\rho^0(x)$ of the unperturbed map f is given by the Frobenius–Perron equation [10]

$$\rho^0(x) = P_0[\rho^0(x)] \equiv \int_a^b \mathrm{d}y\, \delta(x - f(y))\rho^0(y). \tag{2.1}$$

We assume that the operator P_0 is known explicitly (via the functional equation to which (2.1) is reducible), as also the solution $\rho^0(x)$.

Now consider the noisy map

$$x_{n+1} = f(x_n) + \xi_n, \tag{2.2}$$

where $\langle \xi_n \rangle = 0$, $\langle \xi_n \xi_{n'} \rangle = \langle \xi^2 \rangle \delta_{nn'}$ (uncorrelated noise). We want to find an exact expression for the invariant density in this case, given the distribution $g(\xi)$ of the noise. Earlier treatments either ignore the complications due to the 'spill-over' of $f(x_n) + \xi_n$ in some realizations to values outside the interval $[a, b]$, or else assume [5]–[9] that $g(\xi)$ is a periodic function with fundamental interval $[a, b]$. ($g(\xi)$ is generally taken to be a Gaussian with zero mean, or an infinite sum of Gaussians to ensure periodicity.) Let us denote by P the perturbed Frobenius–Perron operator (i.e., the one corresponding to the noisy map), and by $\rho(x)$ the corresponding invariant density, averaged over the realizations of the additive noise. A formal expression for the kernel of the operator P is well known, and is obtained as follows. From (2.2), it follows that the noise-averaged density at time n evolves according to

$$\rho_{n+1} = \int \mathrm{d}\xi \int_a^b \mathrm{d}y\; \delta(x - f(y) - \xi)\;\; \rho_n(y)\; g(\xi). \tag{2.3}$$

This is an exact expression if the noise distribution is periodic, but imprecise if it is not, precisely because the spill-over across the boundaries a and b has been ignored (nor is it immediately clear what the effects of this neglect will be). Formally performing the integration over ξ using the δ-function, one gets for the invariant density the perturbed Frobenius–Perron [3, 6, 10]

$$\rho(x) = P[\rho(x)] \equiv \int_a^b \mathrm{d}y\; g(x - f(y))\; \rho(y). \tag{2.4}$$

Comparision with the noise-free case, (2.1), shows that the effect of the noise is to replace the δ-function kernel of P_0 with the noise distribution function g.

3. Periodic boundary conditions

3.1 *Fourier expansion method*

The convolution form of (2.4) suggests a solution in terms of a Fourier expansion, and this is what has been done in earlier treatments [6, 7], taking g to be a periodic function of its argument. As mentioned earlier, this is appropriate when periodic boundary conditions are assumed. We may then write

$$g(\xi) = (b-a)^{-1} \sum_{n=-\infty}^{\infty} \tilde{g}_n \exp(ik_n\xi), \tag{3.1}$$

where $k_n = 2\pi n/(b-a)$. The inversion formula is

$$\tilde{g}_n = \int_a^b \mathrm{d}\xi\, g(\xi) \exp(-ik_n\xi). \tag{3.2}$$

Expanding $\rho(x)$ in a similar fashion, i.e.,

$$\rho(\xi) = (b-a)^{-1} \sum_{n=-\infty}^{\infty} \tilde{\rho}_n \exp(ik_n\xi), \tag{3.3}$$

we obtain from (2.4) the relation

$$\tilde{\rho}_n = \sum_{m=-\infty}^{\infty} \tilde{g}_n S_{nm} \tilde{\rho}_m, \tag{3.4}$$

where

$$S_{nm} = (b-a)^{-1} \int_a^b \mathrm{d}y \exp i[k_m y - k_n f(y)]. \tag{3.5}$$

The effects of the noise are contained in $\{\tilde{g}_n\}$. The noise-free case is recovered by setting $\tilde{g}_n = 1$ for all n, corresponding to replacing $g(\xi)$ by $\delta(\xi)$. The normalization conditions on g and ρ imply that

$$\int_a^b g(\xi)\mathrm{d}\xi = \tilde{g}_0 = 1, \quad \int_a^b \rho(x)\mathrm{d}x = \tilde{\rho}_0 = 1. \tag{3.6}$$

Equation (3.4) can be written [6, 7] as the following *inhomogeneous* equation for $\tilde{\rho}_n$, $n \neq 0$:

$$\tilde{\rho}_n = \tilde{g}_n S_{n0} + \sum_{m=-\infty}^{\infty}{}' \tilde{g}_n S_{nm} \tilde{\rho}_m, \quad (n \neq 0) \tag{3.7}$$

where $\sum'$ is over non-zero integers m. In principle, iteration of (3.7) yields the formal solution

$$\tilde{\rho}_n = \tilde{g}_n S_{n0} + \sum_m{}' \tilde{g}_n \tilde{g}_m S_{nm} S_{m0} + \sum_m{}' \sum_l{}' \tilde{g}_n \tilde{g}_m \tilde{g}_l S_{nm} S_{ml} S_{l0} + \cdots \tag{3.8}$$

for $\tilde{\rho}_n$, and hence for $\rho(x)$. Rapid convergence of the iteration, at least in the fully chaotic cases considered here, helps numerical calculations in specific instances [6, 7]. Clearly, it is rather difficult to extract analytical information regarding the dependence of $\rho(x)$ on the parameters of the noise distribution from the formal solution, except under special circumstances. To reconstruct $\rho(x)$ in closed form, one must find $\tilde{\rho}_n$ exactly and then sum the Fourier series. In fact, this procedure is complicated enough even in the noise-free case ($\tilde{g}_n = 1$) – e.g., for the logistic map $x_{n+1} = 4x_n(1 - x_n)$ on $[0, 1]$, S_{nm} is expressible [6] in terms of Fresnel integrals, but finding $\tilde{\rho}_n$ explicitly and summing the Fourier series hardly seems to be the easiest way to obtain the well-known result [11] $\rho^0(x) = [\pi^2 x(1 - x)]^{-\frac{1}{2}}$. However, there are two situations in which (3.4) (or (3.7)) can be solved easily. These are considered below.

3.2 $S_{n0} = 0$; *application to the Bernoulli and tent maps*

If the quantity

$$S_{n0} = (b - a)^{-1} \int_a^b \mathrm{d}y \exp[-ik_n f(y)] \tag{3.9}$$

happens to vanish for every nonzero integer n, then (3.7) reduces to a set of *homogeneous* equations for $\{\tilde{\rho}_n\}$, $n \neq 0$. The determinant of this set of equations will be nonzero in general, so that only the trivial solution $\tilde{\rho}_n = 0$, $n \neq 0$, exists. This means that

$$\rho(x) = \rho^0(x) = (b - a)^{-1} \tag{3.10}$$

in this case, for any (periodic) noise distribution $g(\xi)$. This is what happens for the general Bernoulli map

$$x_{n+1} = \begin{cases} \alpha x_n, & 0 \leq x_n \leq 1/\alpha \\ (\alpha x_n - 1)/(\alpha - 1), & 1/\alpha < x_n \leq 1, \end{cases} \tag{3.11}$$

where $\alpha > 1$. The same result holds good for the continuous counterpart of (3.11), namely, the tent map in which the second segment of the map reads $x_{n+1} = \alpha(1 - x_n)/(\alpha - 1)$. (For $\alpha = 2$, these maps are the usual Bernoulli map and the symmetric tent map, respectively). Another example, involving a map that is *not* piecewise linear, is the antisymmetric square-root cusp map in $[-1, 1]$ given by

$$x_{n+1} = \begin{cases} 1 - 2\sqrt{-x_n}, & -1 \leq x_n \leq 0 \\ 2\sqrt{x_n} - 1, & 0 < x_n \leq 1. \end{cases} \tag{3.12}$$

We find in this case

$$S_{nm} = (-1)^n \int_0^1 \mathrm{d}y \cos\ \pi\,(my - 2n\sqrt{y}), \tag{3.13}$$

so that $S_{n0} = 0$. This leads, as before, to the constant normalized density $\rho(x) = \rho^0(x) = 1/2$ for both the unperturbed map and the noisy map. In general: under periodic boundary conditions, additive white noise does not alter the unperturbed invariant density ρ^0 when the latter is a constant. We shall see later that this conclusion does *not* hold good if the boundary conditions are changed.

3.3 $S_{nm} = \pm S_{n,-m}$, $S_{n0} = \mp S_{-n,0}$: *application to the cusp map*

There is another situation in which an explicit solution for $\rho(x)$ is feasible in the above approach, even if $\rho^0(x)$ is not a constant (i.e., $S_{n0} \neq 0$). This occurs when (i) S_{nm} is even [odd] in the index m, (ii) S_{n0} is odd [even] in the index n, and further (iii) $\tilde{g}_n = \tilde{g}_{-n}$, i.e., $\tilde{g}_n$ is real (which happens when $g(\xi)$ is symmetric about $(b-a)/2$). It is evident that if conditions (i)–(iii) are satisfied, all terms beyond the first on the right-hand side of the iterative solution (3.8) vanish identically. Moreover, $\tilde{\rho}_n^0 = S_{n0}$ in this case (since the noise-free case corresponds to $\tilde{g}_n = 1$). We are therefore enabled to arrive at the solution

$$\tilde{\rho}_n = \tilde{g}_n\, S_{n0} = \tilde{g}_n\, \tilde{\rho}_n^0. \tag{3.14}$$

Therefore

$$\rho(x) = (b-a)^{-1} \sum_{n=-\infty}^{\infty} \tilde{g}_n\, S_{n0} \exp(ik_n x). \tag{3.15}$$

Inserting the definitions of $\tilde{g}_n$ and S_{n0}, and using the relation

$$(b-a)^{-1} \sum_{n=-\infty}^{\infty} \exp(ik_n z) = \delta(z), \tag{3.16}$$

we find the solution

$$\rho(x) = (b-a)^{-1} \int_a^b \mathrm{d}\xi \int_a^b \mathrm{d}y\ g(\xi)\ \delta(x - f(y) - \xi). \tag{3.17}$$

A concrete example of the situation just described is provided by the symmetric square-root cusp map

$$x_{n+1} = 1 - 2|x_n|^{\frac{1}{2}}, \quad x \in [-1, 1]. \tag{3.18}$$

This map is of importance as a model of intermittent chaos, arising from the marginal stability of the fixed point at the left boundary $x = -1$. This is reflected in its invariant density [12]

$$\rho^0(x) = (1-x)/2, \tag{3.19}$$

which shows how the probability is 'piled up' towards $x = -1$. We have in this case

$$S_{nm} = (-1)^n \int_0^1 \mathrm{d}y \cos(\pi m y) \exp(2\pi n i \sqrt{y}), \tag{3.20}$$

(which may be compared with (3.13) corresponding to the case of the antisymmetric counterpart (3.12) of the cusp map). Thus S_{nm} is even in m. Further,

$$S_{n0} = (-1)^n (i\pi n)^{-1}, \tag{3.21}$$

which is odd in n. Therefore the conditions just discussed are met, and we have $\tilde{\rho}_n^0 = S_{n0}$, for $n \neq 0$. Together with $\tilde{\rho}_0^0 = 1$, this leads on summing the Fourier series to the recovery of the known result (3.19) for the invariant density $\rho^0(x)$ of the unperturbed map. The point we now make is that this form of the invariant density can undergo significant modification in the presence of noise. For a symmetric noise distribution in $[-1, 1]$ (so that $\tilde{g}_n = \tilde{g}_{-n}$), using (3.6), (3.14) and (3.21) we get the exact result

$$\rho(x) = \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \, \tilde{g}_n \, \frac{\sin n\pi x}{n\pi}. \tag{3.22}$$

The 'antisymmetry' of $\rho(x)$ about the value $\frac{1}{2}$ is at once evident, as is the fact that $\rho(0) = \frac{1}{2}$. Further, if $\rho(-1) = \rho(1)$, then each of these is also equal to $\frac{1}{2}$; else, their mean value is equal to $\frac{1}{2}$. In order to exhibit analytically the form of the solution, let us consider first the simple case of a rectangular or uniform noise, given by the density

$$g(\xi) = \frac{1}{2\eta} \theta(\eta - |\xi|) \tag{3.23}$$

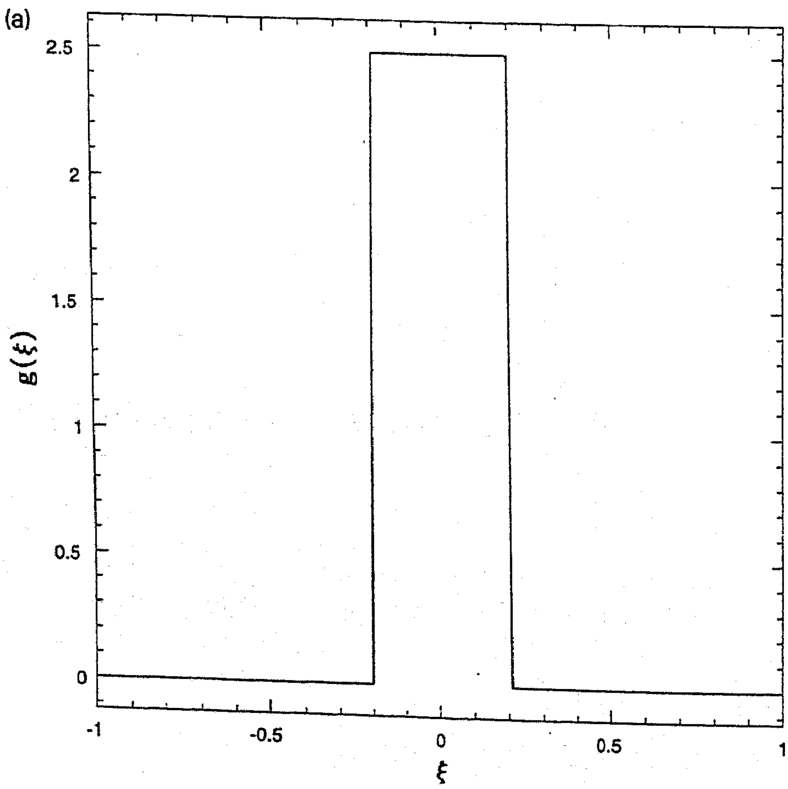


Figure 1(a). The rectangular (uniform) noise distribution $g(\xi) = (2\eta)^{-1}\theta(\eta - |\xi|)$, for $\eta = 0.2$.

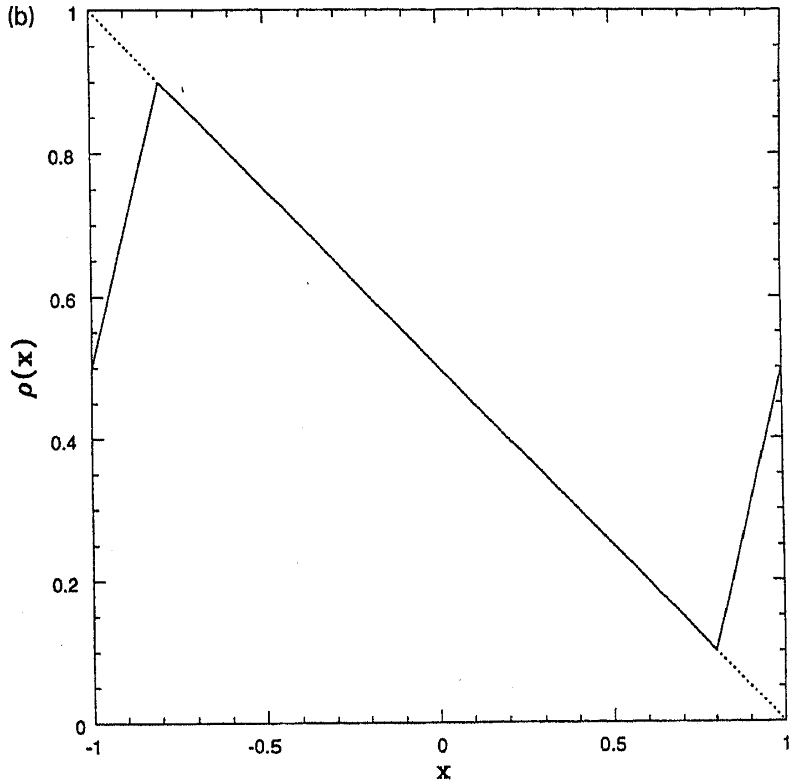


Figure 1(b). The invariant density $\rho(x)$ for the symmetric cusp map $f(x) = 1 - 2|x|^{\frac{1}{2}}$ in the presence of additive noise distributed as in figure 1(a), under periodic boundary conditions. The dotted line depicts the invariant density $(1 - x)/2$ of the unperturbed (noise-free) map.

in the fundamental interval $[-1, 1]$, where $0 < \eta < 1$. The amplitude of the noise is thus limited by the parameter η. Carrying out the calculations required, we arrive at the result

$$\rho(x) = \begin{cases} \frac{1}{2}[\eta^{-1} + (\eta^{-1} - 1)x], & -1 \le x \le -1 + \eta \\ \frac{1}{2}[1 - x], & -1 + \eta \le x \le 1 - \eta \\ \frac{1}{2}[(2 - \eta^{-1}) + (\eta^{-1} - 1)x], & 1 - \eta \le x \le 1 . \end{cases} \tag{3.24}$$

The noise distribution (3.23) and the invariant density are shown in figures 1(a) and (b) respectively. We note how the noise suppresses the effect of the marginally stable fixed point of the map at $x = -1$. Equation (3.24) is an exact result (under the periodic boundary conditions assumed) that is valid, in fact, for any value of the amplitude η of the noise. As $\eta \to 1$, $\rho(x)$ tends to the uniform distribution $\rho(x) = 1/2$, as one might expect under the circumstances.

The fact that $\rho(x)$ (3.24) continues to remain piecewise linear is actually an artefact of the uniform noise distribution (3.23). A smoother modification of the unperturbed density $(1 - x)/2$, but one that retains the general features of $\rho(x)$ as found above, is provided by the one-parameter family of unimodal, normalized noise densities

$$g(\xi; \mu) = \frac{\mu(\cos \mu\xi - \cos \mu)}{2(\sin \mu - \mu \cos \mu)}, \quad \xi \in [-1, 1], \tag{3.25}$$

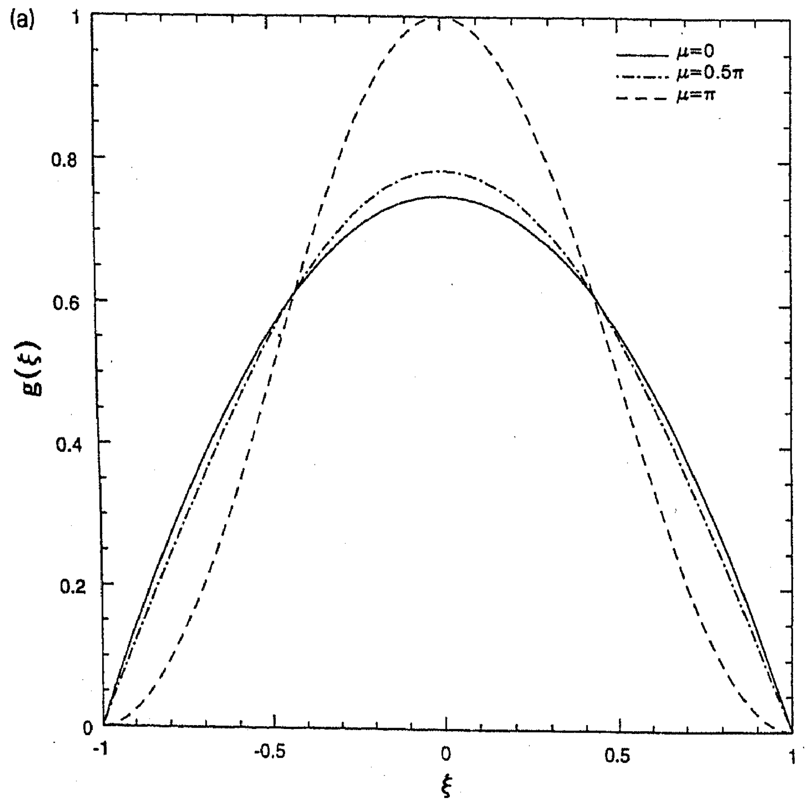


Figure 2(a). The noise distribution $g(\xi;\mu)$ of (3.25) in the limiting cases $\mu=\pi$ (3.27) and $\mu=0$ (3.29). Also shown is the case $\mu=\pi/2$, i.e., $g(\xi;\pi/2)=(\pi/4)\cos(\pi\xi/2)$.

where $0<\mu\leq\pi$. We then find the corresponding invariant density to be given by

$$\rho(x)=\frac{1}{2}\left[1+(sgn\ x)\frac{\sin\mu(1-|x|)-(1-|x|)\sin\mu}{(\mu\cos\mu-\sin\mu)}\right]. \tag{3.26}$$

Thus $\rho(-1)=\rho(0)=\rho(1)=1/2$. In the limiting case $\mu=\pi$, we have

$$g(\xi;\pi)=\tfrac{1}{2}(1+\cos\pi\xi), \tag{3.27}$$

the corresponding invariant density being

$$\rho(x)=\frac{1}{2}\left(1-\frac{\sin\pi x}{\pi}\right). \tag{3.28}$$

The limit $\mu=0$ is exceptional. It corresponds to the parabolic distribution

$$g(\xi;0)=\tfrac{3}{4}(1-\xi^2), \tag{3.29}$$

leading to the invariant density

$$\rho(x)=\tfrac{1}{2}(1-x)+\tfrac{3}{4}x(|x|-x^2). \tag{3.30}$$

In this case, $\rho(-1)$ and $\rho(1)$ remain equal to 1 and 0 respectively. The noise distributions of (3.27) and (3.29) are shown in figure 2(a), while the corresponding solutions for $\rho(x)$

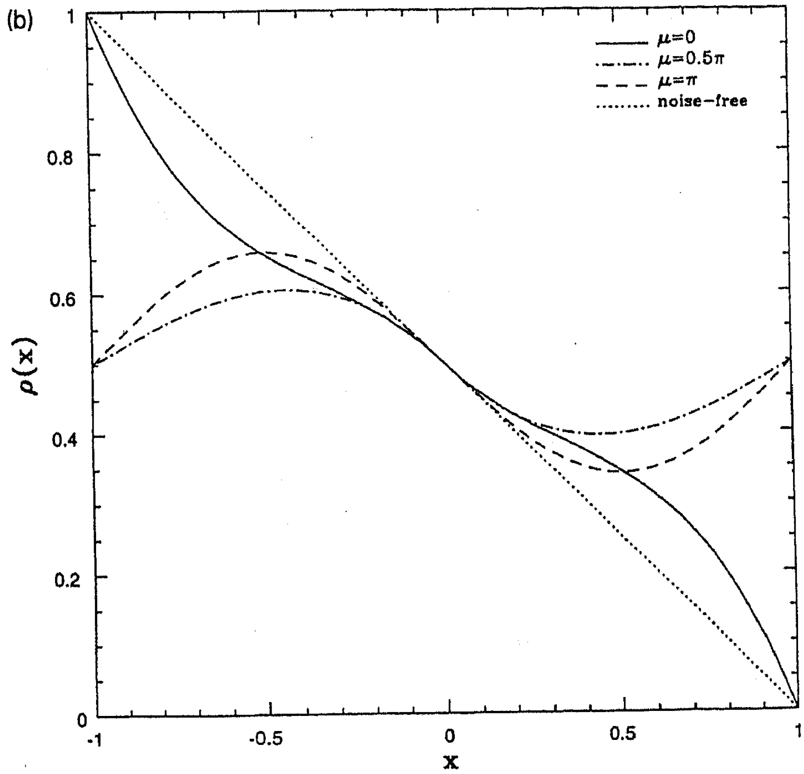


Figure 2(b). The corresponding invariant densities (3.26) for the noisy, symmetric cusp map, given by (3.28) for $\mu = \pi$ and by (3.30) for $\mu = 0$. Also shown is the case $\mu = \pi/2$, for which $\rho(x) = \frac{1}{2}(2 - x - \cos(\pi x/2))$ $(x > 0)$ and $\rho(x) = \frac{1}{2}(\cos(\pi x/2) - x)$ $(x < 0)$.

are depicted in figure 2(b). The 'evening out' effect of the noise is quite evident. It must be borne in mind, though, that these results correspond to relatively large stochastic perturbations of the original dynamics: the standard deviation of the noise for the distribution (3.29) is $1/\sqrt{5} \approx 0.45$, while that for (3.27) is $\sqrt{(1/3) - (2/\pi^2)} \approx 0.36$. The Lyapunov exponent λ, which is equal to 1/2 for the unperturbed map [12], is also altered in the presence of noise: we have $\ln |f'(x)| = -\frac{1}{2} \ln |x|$, so that the second term on the right in eq. (3.22) does not contribute to λ, being odd in x.

3.4 *Functional equation under uniformly distributed noise*

We present in brief a formalism that avoids Fourier series and leads to a functional equation for the invariant density, for an onto map in the fundamental interval $[a, b]$ perturbed by additive noise ξ, under periodic boundary conditions. This may be a useful approach in numerical calculations, especially in situations in which the noise is not just a small perturbation.

Returning to (2.3), we insert the factor $1 = \int_a^b du\ \delta(u - f(y))$ on the right-hand side:

$$\rho_{n+1}(x) = \int d\xi \int_a^b dy \int_a^b du\, \delta(x - f(y) - \xi)\delta(u - f(y))\ g(\xi)\ \rho_n(y)$$

$$= \int d\xi \int_a^b dy\, \delta(x-u-\xi) g(\xi) \int_a^b du\, \delta(u - f(y))\, \rho_n(y)$$
$$= \int d\xi \int_a^b du\, \delta(x-u-\xi)\, g(\xi)\, P_0[\rho_n(u)], \tag{3.31}$$

using the definition of the unperturbed Frobenius–Perron operator P_0. Therefore the invariant density is given by

$$\rho(x) = \int d\xi \int_a^b du\, \delta(x-u-\xi)\, g(\xi)\, P_0[\rho(u)]. \tag{3.32}$$

For brevity, let us introduce the notation $P_0[\rho(u)] = \bar{\rho}(u)$. (Of course, $\bar{\rho}^0(x) = \rho^0(x)$ itself). Thus

$$\rho(x) = \int d\xi \int_a^b du\, \delta(x-u-\xi)\, g(\xi)\bar{\rho}(u). \tag{3.33}$$

For the sake of definiteness (so as to be able to write all formulas explicitly), we consider henceforth the case of uniformly distributed noise, as in (3.23), with $0 < \eta < (b-a)/2$. For periodic boundary conditions, the support of the δ-function in (2.3) must be $(f(y) + \xi) \bmod (b-a)$, i.e., the support of $\delta(x-u-\xi)$ in (3.33) is $u = (x-\xi) \bmod (b-a)$. Writing $\delta^{\text{mod}}(x-u-\xi)$ to keep track of this fact, we have

$$\rho(x) = (2\eta)^{-1} \int d\xi\, \theta(\eta - |\xi|) \int_a^b du\, \delta^{\text{mod}}(x-u-\xi)\bar{\rho}(u). \tag{3.34}$$

Carrying out the integration over ξ, we find:

$$\rho(x) = \begin{cases} (2\eta)^{-1}\left[\int_a^{x+\eta} du\, \bar{\rho}(u) + \int_{x-\eta+b-a}^{b} du\, \bar{\rho}(u)\right], & \text{for } a \le x < a+\eta, \\ (2\eta)^{-1}\left[\int_{x-\eta}^{x+\eta} du\, \bar{\rho}(u)\right], & \text{for } a+\eta \le x \le b-\eta, \\ (2\eta)^{-1}\left[\int_{x-\eta}^{b} du\, \bar{\rho}(u) + \int_a^{x+\eta-(b-a)} du\, \bar{\rho}(u)\right], & \text{for } b-\eta < x \le b. \end{cases} \tag{3.35}$$

Equation (3.35) must be solved numerically to find $\rho(x)$.

It is evident that $\rho(x)$ is continuous at all points in the interval, including the points $a+\eta$ and $b-\eta$. A convenient way of writing the functional equation (3.35) is in terms of the derivative of $\rho(x)$: we have

$$2\eta\rho'(x) = \begin{cases} \bar{\rho}(x+\eta) - \bar{\rho}(x-\eta+b-a), & a \le x < a+\eta, \\ \bar{\rho}(x+\eta) - \bar{\rho}(x-\eta), & a+\eta \le x \le b-\eta, \\ \bar{\rho}(x-\eta) - \bar{\rho}(x+\eta-b+a), & b-\eta < x \le b. \end{cases} \tag{3.36}$$

The discontinuity of $\rho'(x)$ at $x = a+\eta$ and $x = b-\eta$ is given by

$$\text{disc}\, \rho'(a+\eta) = -\text{disc}\, \rho'(b-\eta) = (2\eta)^{-1}[\bar{\rho}(b) - \bar{\rho}(a)]. \tag{3.37}$$

This relation may be verified for the cusp map using the solution (3.24) already found for the invariant density.

4. Sticky boundaries: Re-injection at the nearest boundary

4.1 General equation for the invariant density

While periodic boundary conditions constitute a convenient way of handling the spill-over problem in a noisy map, other solutions are also possible [5]. These include reflecting boundary conditions, or working in the infinite domain $(-\infty, \infty)$ so that natural boundary conditions can be employed. Yet another approach is to tailor the noise distribution by making it state-dependent, such that the probability of an escape of the dynamical variable out of the interval is zero. This means that the noise is actually multiplicative rather than additive, with a density $g(\xi, x)$ satisfying $g(\xi, a) = g(\xi, b) = 0$. One way of implementing this is to assume $g(\xi, x)$ to be of the form [13]

$$g(\xi, x) = \begin{cases} g(\xi) / \int_a^b g(\xi' - x) \mathrm{d}\xi', & x + \xi \in [a, b] \\ 0, & \text{otherwise}. \end{cases} \tag{4.1}$$

While it is conceivable that this and other such prescriptions might be justified on physical grounds in certain instances, the very modelling of the stochastic forces on a system by means of multiplicative noise may be quite inappropriate in others, particularly if the noise arises from sources that are quite decoupled from the system dynamics.

A model that appears to be rather more plausible on physical grounds is provided by 'sticky' boundaries: once the dynamical system reaches a boundary value, it stays put at that value, until the noise moves it away. More generally, instead of escape out of the interval owing to the addition of noise to the evolution, we may assume re-injection at the nearest boundary. This avoids the arbitrary clipping of the orbit of x_n when $f(x_n) + \xi$ goes out of $[a, b]$ by re-defining the noisy map as follows. Denoting $f(x_n) + \xi$ by $F(x_n, \xi_n)$, we have

$$x_{n+1} = \begin{cases} f(x_n) + \xi_n & \text{if } a \le F(x_n, \xi_n) \le b, \\ a & \text{if } F(x_n, \xi_n) \le a, \\ b & \text{if } F(x_n, \xi_n) \ge b. \end{cases} \tag{4.2}$$

Therefore if the representative point overshoots the boundary at a or at b, it is replaced at that boundary, remaining trapped there until it is re-injected into the phase space by the noisy map. (This is so even if a or b is a fixed point of the unperturbed map.) The end points thus act as temporary traps that prevent any loss of 'measure' due to overflow.

As before, we are interested in the invariant density $\rho(x)$, rather than the time-dependent density $\rho_n(x)$. The counterpart of (2.3) for the map (4.2) is, in the limit $n \to \infty$,

$$\rho(x) = \int \mathrm{d}\xi \int_a^b [\delta(x - F(y, \xi)) + \delta(x - a)\theta(a - F(y, \xi)) + \delta(x - b)\theta(F(y, \xi) - b)]\rho(y) g(\xi). \tag{4.3}$$

We note that the re-injection prescription in (4.2) leads, in general, to δ-functions in the density with support at the boundary points. Hence $\rho(x)$ is of the form

$$\rho(x) = \alpha\delta(x-a) + \beta\delta(x-b) + \rho_*(x), \tag{4.4}$$

where α and β are (dimensionless) constants, and $\rho_*(x)$ is the 'regular' part of the invariant density. Substituting (4.4) in (4.3), we get the coupled equations

$$\alpha = \int \mathrm{d}\xi\, g(\xi)\Big[\alpha\theta(a-F(a,\xi)) + \beta\theta(a-F(b,\xi)) + \int_a^b \mathrm{d}y\,\theta(a-F(y,\xi))\rho_*(y)\Big], \tag{4.5}$$

$$\beta = \int \mathrm{d}\xi\, g(\xi)\Big[\alpha\theta(F(a,\xi)-b) + \beta\theta(F(b,\xi)-b) + \int_a^b \mathrm{d}y\,\theta(F(y,\xi)-b)\rho_*(y)\Big], \tag{4.6}$$

$$\rho_*(x) = \int \mathrm{d}\xi\, g(\xi)\Big[\alpha\delta(x-F(a,\xi)) + \beta\delta(x-F(b,\xi)) + \int_a^b \mathrm{d}y\,\delta(x-F(y,\xi))\rho_*(y)\Big]. \tag{4.7}$$

As before, in order to be specific, we shall take $g(\xi)$ to be the amplitude-limited (uniform) noise distribution of (3.23). Further, let us assume that the map f is a continuous, unimodal, onto map, with $f(y)$ increasing monotonically from $y=a$ to $y=y_m$, and decreasing monotonically from y_m to b; $f(a)=f(b)=a$, $f(y_m)=b$. (The modifications required in other cases can be worked out.) Let $y_{1,2}(z)$ denote the two roots of $f(y)=z\in[a,b]$. We have $y_1(a)=a$, $y_2(a)=b$, $y_{1,2}(b)=y_m$. Then, after some simplification (the details are described in the Appendix), we arrive at the following equations for α, β and $\rho_*(x)$:

$$\alpha = \beta + \frac{1}{\eta}\left\{\int_a^{y_1(a+\eta)} \mathrm{d}y(a+\eta-f(y))\rho_*(y) + \int_{y_2(a+\eta)}^b \mathrm{d}y(a+\eta-f(y))\rho_*(y)\right\}, \tag{4.8}$$

$$\beta = \frac{1}{2\eta}\int_{y_1(b-\eta)}^{y_2(b-\eta)} \mathrm{d}y(\eta-b+f(y))\,\rho_*(y), \tag{4.9}$$

$$\rho_*(x) = \left(\frac{\alpha+\beta}{2\eta}\right)[\theta(x-a)-\theta(x-a-\eta)] + \frac{1}{2\eta}\int_{-\eta}^{\eta}\mathrm{d}\xi\int_a^b \mathrm{d}y\,\delta(x-f(y)-\xi)\rho_*(y). \tag{4.10}$$

In the last integral on the right, the factor $1=\int_a^b \mathrm{d}u\,\delta(u-f(y))$ may be inserted as before, to lead to

$$\rho_*(x) = \left(\frac{\alpha+\beta}{2\eta}\right)[\theta(x-a)-\theta(x-a-\eta)] + \frac{1}{2\eta}\int_{\max(a,x-\eta)}^{\min(b,x+\eta)} \mathrm{d}u\,\bar{\rho}_*(u). \tag{4.11}$$

Equations (4.8)–(4.10) represent the exact integral equations for the invariant density under the prescription of "re-injection at the nearest boundary." The over-all scale of $\rho_*(x)$ is fixed by the normalization

$$\alpha+\beta+\int_a^b \rho_*(x)\mathrm{d}x = 1. \tag{4.12}$$

We note that $\alpha > \beta$. Moreover, in addition to the δ-function spikes at the boundaries, there occurs an extra boundary layer contribution on the left, in $(a, a+\eta)$. There is a discontinuity in $\rho(x)$ at $x = a+\eta$, and a discontinuous change in slope at $x = b-\eta$, with the respective discontinuities given by

$$\operatorname{disc}\rho(a+\eta) = -\left(\frac{\alpha+\beta}{2\eta}\right), \tag{4.13}$$

$$\operatorname{disc}\rho'(b-\eta) = -\bar{\rho}_*(b)/(2\eta) = -\frac{1}{4\eta^2}\int_{b-\eta}^b \mathrm{d}u\ \bar{\rho}_*(u). \tag{4.14}$$

The Lyapunov exponent is also altered, in general:

$$\lambda = \alpha\ln|f'(a)| + \beta\ln|f'(b)| + \int_a^b \rho_*(x)\ln|f'(x)|\,\mathrm{d}x. \tag{4.15}$$

We illustrate the foregoing in the case of the tent map at fully-developed chaos.

4.2 *Application to the tent map*

The map concerned is $f(x) = 1-|2x-1|$, $x\in[0,1]$. Thus $a=0$, $b=1$, $y_m=\frac{1}{2}$, $y_1(z) = z/2$, $y_2(z) = 1-(z/2)$. Further,

$$P_0[\rho(u)] \equiv \bar{\rho}(u) = \frac{1}{2}\left[\rho\left(\frac{u}{2}\right)+\rho\left(1-\frac{u}{2}\right)\right],$$

and the noise-free map has the constant invariant density $\rho^0(x) = 1$. The noisy map has an invariant density of the form

$$\rho(x) = \alpha\delta(x)+\beta\delta(x-1)+\rho_*(x), \tag{4.16}$$

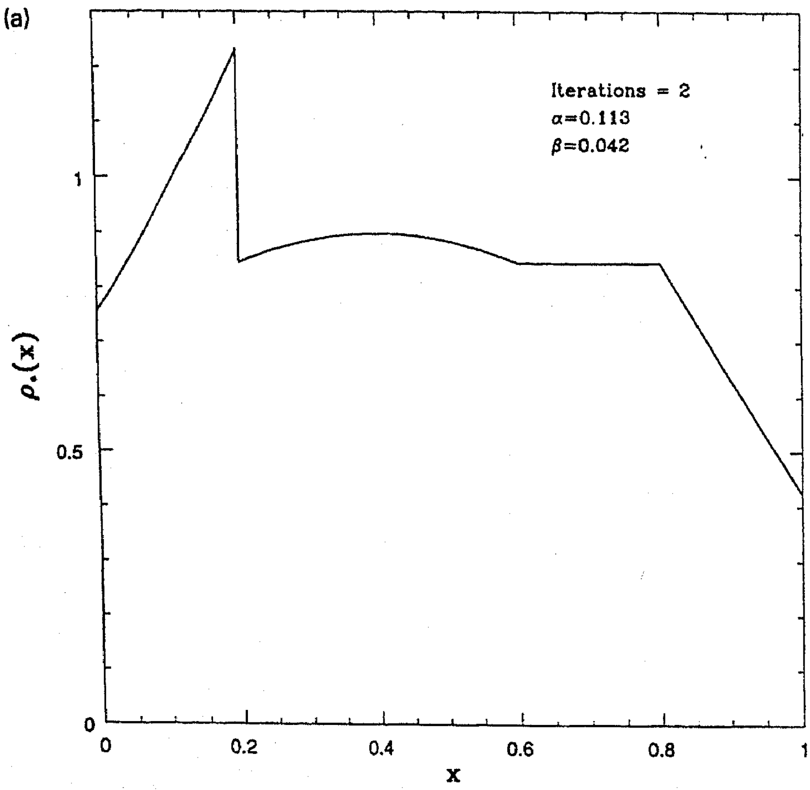
where

$$\alpha = \beta+\frac{1}{\eta}\left\{\int_0^{\eta/2}\mathrm{d}y(\eta-2y)\rho_*(y)+\int_{(1-\eta/2)}^1 \mathrm{d}y\,(\eta-2+2y)\rho_*(y)\right\}, \tag{4.17}$$

$$\beta = \frac{1}{2\eta}\int_{(1-\eta)/2}^{(1+\eta)/2}\mathrm{d}y\,(\eta-|2y-1|)\ \rho_*(y), \tag{4.18}$$

and

$$\rho_*(x) = \left(\frac{\alpha+\beta}{2\eta}\right)[\theta(x)-\theta(x-\eta)]+\frac{1}{2\eta}\int_{\max(0,x-\eta)}^{\min(1,x+\eta)}\mathrm{d}y\,\bar{\rho}_*(y). \tag{4.19}$$



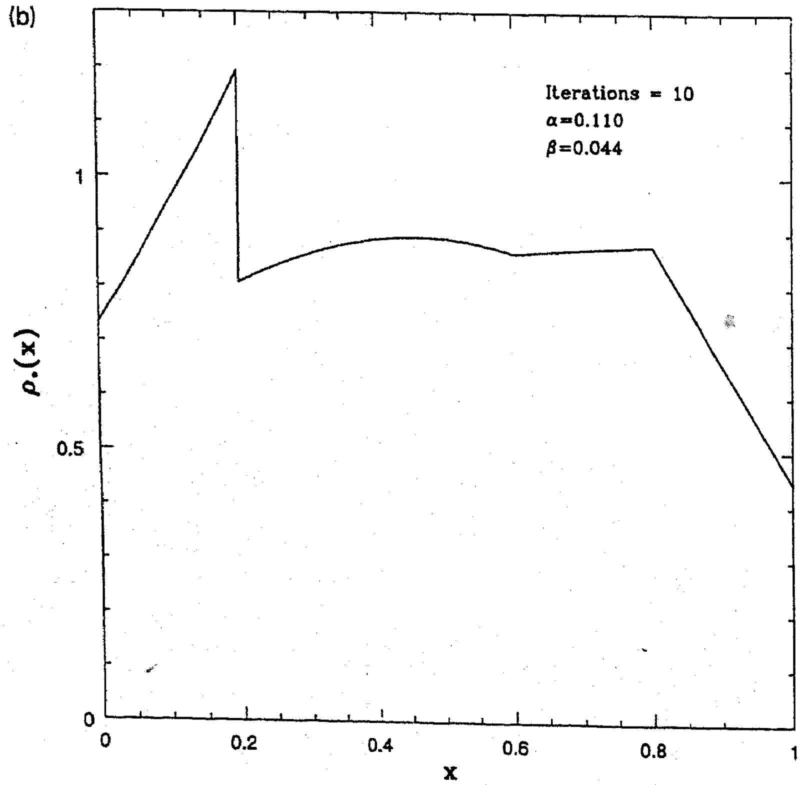


Figure 3(a–b). (*Continued*)

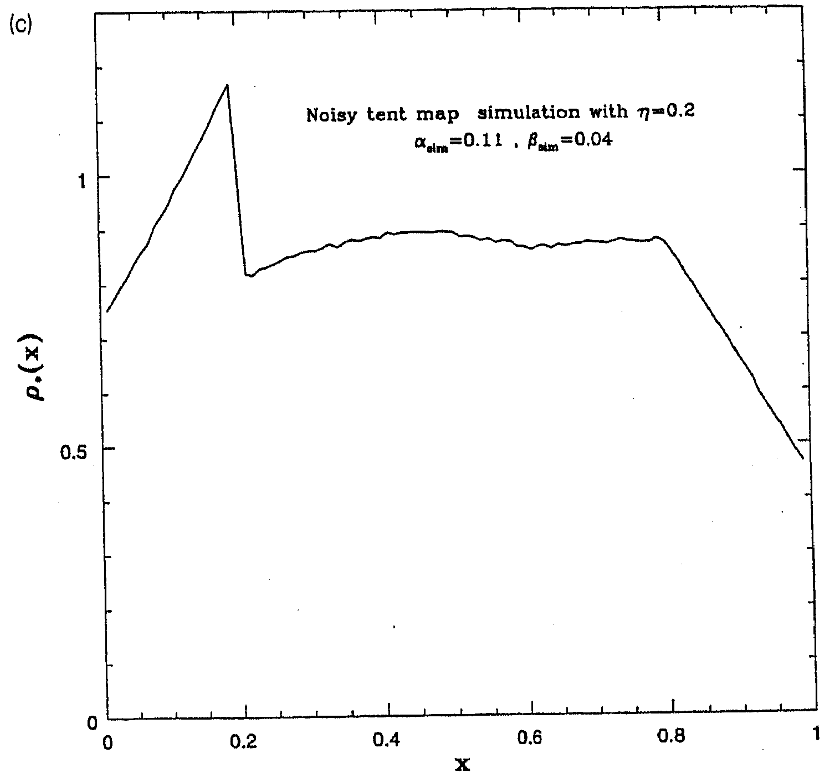


Figure 3. Numerical solution and comparision with simulation for the invariant density $\rho(x) = \alpha\delta(x) + \beta\delta(1-x) + \rho_*(x)$ of the noisy tent map with noise amplitude $\eta = 0.2$. **(a)** Solution after 2 iterations, starting with the zeroth approximation $\rho_*(x) = 1$. The solution essentially stablizes (correct to 3 decimal places) in 3 iterations. **(b)** Solution after 10 iterations. **(c)** The result of a numerical simulation on the noisy tent map with the re-injection prescription, (4.2). The result shown is an average over 10 simulations of 10^6 iterations each, with a bin size of 10^{-2}.

Equation (4.19) yields the following, for x in different ranges:

$$0 \le x < \eta: \; \rho_*(x) = \frac{\alpha+\beta}{2\eta} + \frac{1}{2\eta}\left[\int_0^{(x+\eta)/2} \mathrm{d}y \rho_*(y) + \int_{1-(x+\eta)/2}^{1} \mathrm{d}y\, \rho_*(y)\right], \tag{4.20}$$

$$\eta \le x < 1-\eta: \; \rho_*(x) = \frac{1}{2\eta}\left[\int_{(x-\eta)/2}^{(x+\eta)/2} \mathrm{d}y \rho_*(y) + \int_{1-(x+\eta)/2}^{1-(x-\eta)/2} \mathrm{d}y \rho_*(y)\right], \tag{4.21}$$

$$1-\eta \le x \le 1: \; \rho_*(x) = \frac{1}{2\eta}\int_{(x-\eta)/2}^{1-(x-\eta)/2} \mathrm{d}y\, \rho_*(y). \tag{4.22}$$

We note right away that $\rho(x) = 1$ or even $\rho_*(x) =$ constant is not a solution, although $\rho^0(x) = 1$. Equations (4.17)–(4.22) must be solved numerically. The broad features of the solution can be seen by adopting an iterative procedure: if we take $\rho_*(x) = 1$ in the zeroth approximation, we obtain the following as the first approximation (after normalization):

$$\alpha = \frac{3\eta}{4(1+\eta)}, \quad \beta = \frac{\eta}{4(1+\eta)}, \tag{4.23}$$

$$\rho_*(x) = \begin{cases} (1+\frac{x}{2\eta})/(1+\eta), & 0 \le x < \eta \\ 1/(1+\eta), & \eta \le x \le 1-\eta \\ (\frac{1}{2}+\frac{1-x}{2\eta})/(1+\eta), & 1-\eta \le x \le 1. \end{cases} \tag{4.24}$$

Successive iterations push the 'plateau' in $\rho_*(x)$ further to the right. The actual solution does not have any interval in which $\rho_*(x)$ is a constant, but rather a range $(\eta, 1-\eta)$ in which its variation with x is gentle. It varies linearly in the ranges $(0,\eta)$, $((1+\eta)/2, 1-\eta)$ and $(1-\eta, \eta)$. Figures 3(a) and 3(b) show how the solution for $\rho_*(x)$ stabilizes as the set (4.17)–(4.22) is solved iteratively, in the case $\eta = 0.2$. The discontinuity in $\rho_*(x)$ at $x = \eta = 0.2$ and that in $\rho'_*(x)$ at $x = 1-\eta = 0.8$ are also verified to satisfy (4.13) and (4.14) respectively. The foregoing form of the invariant density has been completely verified by direct numerical simulations of noisy tent-map dynamics with the prescription of re-injection at the nearest boundary as in (4.2), as shown in figure 3(c). The Lyapunov exponent remains equal to ln 2 in this case, by virtue of the fact that $|f'(x)|$ is a constant and the invariant density is normalized to unity according to (4.12).

5. Generalized dimension D_q

We have commented already on the manner in which the Lyapunov exponent λ is altered (or remains unaltered) for the various noisy maps considered. In the same vein, we may ask how the generalized dimension D_q is altered by the addition of noise. This quantity is defined as [11, 14]

$$D_q = \lim_{\epsilon \to 0} (q-1)^{-1} \ln\left(\sum_{j=1}^{N(\epsilon)} (\mu_j)^q\right) \Big/ \ln \epsilon, \tag{5.1}$$

where $-\infty < q < \infty$. In the present context, $N(\epsilon) = (b-a)/\epsilon$, while the measure of the jth cell is given by

$$\mu_j = \int_{a+(j-1)\epsilon}^{a+j\epsilon} \rho(x)\mathrm{d}x \approx \epsilon\rho(a+(j-\tfrac{1}{2})\epsilon). \tag{5.2}$$

Therefore

$$\sum_{j=1}^{N(\epsilon)} (\mu_j)^q = \epsilon^{q-1} \sum_{j=1}^{N(\epsilon)} \epsilon[\rho(a+(j-\tfrac{1}{2})\epsilon)]^q. \tag{5.3}$$

As the sum on the right-hand side tends to $\int_a^b [\rho(x)]^q \mathrm{d}x$, the behaviour of D_q is governed by the integrability or otherwise of $\rho^q(x)$, i.e., by the nature of the zeros and singularities of $\rho(x)$ [14]. It is at once evident that $D_q = D_0 = 1$ in all cases in which the invariant density is a constant. A simple non-trivial case is once again provided by the symmetric cusp map (3.18), for which $\rho^0(x) = (1-x)/2$ (3.19). Owing to the vanishing of $\rho^0(x)$ at $x = 1$, we have, for $q > -1$,

$$D_q \to \lim_{\epsilon\to 0} (q-1)^{-1} \ln \epsilon^{q-1}/\ln\epsilon = 1, \quad (q > -1). \tag{5.4}$$

For $q \le -1$, on the other hand, we find

$$D_q \to \lim_{\epsilon\to 0} (q-1)^{-1} \ln \epsilon^{2q}/\ln\epsilon = 2q/(q-1), \quad (q \le -1) \tag{5.5}$$

since $\mu_N \approx \epsilon\rho(-1 + (N-\frac{1}{2})\epsilon) = \epsilon^2/4$, on setting $a = -1$, $b = 1$, $N\epsilon = b - a = 2$. The question is: how is this result affected by the addition of noise?

In § 3.3, we have seen how the unperturbed density $\rho^0(x) = (1-x)/2$ is modified by various kinds of noise distributions $g(\xi)$ under periodic boundary conditions. For the purpose at hand, the relevant point is that the noise generically eliminates the zero of the density at $x = 1$, without introducing any fresh zeroes or singularities in the interval (cf. (3.24) and (3.26) for $\mu \neq 0$). Therefore D_q remains equal to 1 for all q, in contrast to the noise-free case. The case $\mu = 0$, corresponding to the noise distribution (3.29), is an exception: the corresponding density $\rho(x)$ (3.30) continues to have a simple zero at $x = 1$, so that D_q is given by (5.4) and (5.5), as in the noise-free case. The question can be analyzed for a general noise distribution $g(\xi)$ by examining the representation (3.22) for $\rho(x)$ in the vicinity of $x = 1$.

We turn now to the effect on D_q of the other boundary condition considered in this paper, namely, that of re-injection at the nearest boundary. Again, taking the tent map $f(x) = 1 - |2x-1|$ in the unit interval as an illustration, we have an invariant density $\rho(x)$ of the form $\alpha\delta(x) + \beta\delta(1-x) + \rho_*(x)$ (cf. (4.16)). Breaking up the interval $[0, 1]$ into N parts with $N\epsilon = 1$, we now have

$$\mu_j = \epsilon\, \rho_*((j-\tfrac{1}{2})\epsilon), \quad 2 \le j \le (N-1), \tag{5.6}$$

while

$$\mu_1 = \alpha + \epsilon\rho_*\left(\frac{\epsilon}{2}\right), \quad \mu_N = \beta + \epsilon\rho_*((N-\tfrac{1}{2})\epsilon). \tag{5.7}$$

As $\alpha,\ \beta > 0$, this leads to

$$D_q \longrightarrow \lim_{\epsilon\to 0} (q-1)^{-1} \frac{\ln\left[\alpha^q + \beta^q + \epsilon^{q-1}\sum_{j=2}^{N-1} \epsilon\rho_*^q((j-\frac{1}{2})\epsilon)\right]}{\ln\epsilon}. \tag{5.8}$$

Although $\rho_*(x)$ has a finite discontinuity at $x = \eta$ and cusps at $x = (1+\eta)/2$ and $x = (1-\eta)$ (we consider uniformly distributed noise with maximum amplitude η), $\rho_*^q(x)$ has no zeros or infinities, and is integrable in $[0, 1]$ for all finite q. Therefore, for $q < 1$ the factor ϵ^{q-1} dominates, and $D_q = 1$. On the other hand, this factor tends to zero for $q > 1$, and $D_q = 0$. For $q = 1$, we have

$$D_1 = \lim_{\epsilon\to 0} \left(\sum_{j=1}^{N(\epsilon)} \mu_j \ln \mu_j\right) \bigg/ \ln\epsilon. \tag{5.9}$$

Using (5.6) and (5.7), we find

$$D_1 = \lim_{\epsilon \to 0} \sum_{j=1}^{N(\epsilon)} \epsilon\rho_*((j-\tfrac{1}{2})\epsilon) \ln [\epsilon\rho_*((j-\tfrac{1}{2})\epsilon)] / \ln \epsilon, \tag{5.10}$$

which reduces to

$$D_1 = \int_0^1 \mathrm{d}x \, \rho_*(x). \tag{5.11}$$

Collecting these results and using (4.12), we have

$$D_q = \begin{cases} 1, & q < 1, \\ 1 - \alpha - \beta, & q = 1, \\ 0, & q > 1. \end{cases} \tag{5.12}$$

The generalized dimensions of the attractor corresponding to the noisy tent map, under the prescription of re-injection at the nearest boundary, are thus modified considerably in comparision with the uniform value $D_q = 1$ for the unperturbed case.

6. Concluding remarks

We have shown that the form of the invariant density for noisy one-dimensional maps in the fully chaotic regime is quite sensitive to the boundary conditions imposed. The latter are necessary to prevent the arbitrary omission or clipping of orbits that overflow the interval when the noise is added. While it is well known that very small amplitude noise helps 'smoothen' the invariant density in general, our results are valid for arbitrary noise amplitude, and show that the invariant density at fully developed chaos can differ significantly from that in the noise-free case. The effect of a marginally stable fixed point can be diminished, as was shown in the case of the cusp map. On the other hand, an initially uniform invariant density can develop considerable structure, as in the case of the tent map with the re-injection prescription used in § 4: δ-function spikes occur at the end-points of the interval, and the unstable fixed point at $x = 0$ is the source of an enhanced boundary layer in its vicinity, as well. Similar effects occur in the case of other maps such as the cusp map and the logistic map. Our results suggest that care must be exercised in drawing conclusions from numerical studies on the aspects of chaotic dynamics considered here, as the combination of external noise and boundary conditions can, and do, lead to non-trivial and sizeable modifications of what one obtains in the absence of noise.

Appendix

We begin with equations (4.5) to (4.7) for α, β and $\rho_*(x)$. Using the rectangular distribution (3.23) for $g(\xi)$ and setting $f(a) = f(b) = a$, (4.5) for α yields

$$\alpha = \tfrac{1}{2}(\alpha + \beta) + \frac{1}{2\eta} \int_{-\eta}^{\eta} \mathrm{d}\xi \int_a^b \mathrm{d}y \, \theta(a - f(y) - \xi) \, \rho_*(y). \tag{A1}$$

The step function is non-zero only for $f(y) + \xi < a$; since $f(y) \geq a$, this is possible only for $\xi < 0$. Now the inequality $f(y) < z$ where $z \in [a, b]$ implies that $y \in [a, y_1(z)] \cup [y_2(z), b]$. Therefore

$$\alpha = \tfrac{1}{2}(\alpha + \beta) + \frac{1}{2\eta}\int_{-\eta}^{0} \mathrm{d}\xi \left[\int_{a}^{y_1(a-\xi)} \mathrm{d}y \rho_*(y) + \int_{y_2(a-\xi)}^{b} \mathrm{d}y \rho_*(y) \right]. \quad \text{(A2)}$$

Changing variables to $-\xi$ and interchanging the order of integration, we have

$$\alpha = \beta + \frac{1}{\eta}\int_{a}^{y_1(a+\eta)} \mathrm{d}y\, \rho_*(y) \int_{f_1(y)-a}^{\eta} \mathrm{d}\xi + \frac{1}{\eta}\int_{y_2(a+\eta)}^{b} \mathrm{d}y\, \rho_*(y) \int_{f_2(y)-a}^{\eta} \mathrm{d}\xi, \quad \text{(A3)}$$

where f_1 and f_2 are, respectively, the ascending and descending branches of f. Equation (4.8) for α follows immediately. The appropriate branch of $f(y)$ in the integrand of that equation is automatically selected by the range of integration.

Similarly, (4.6) yields

$$\beta = \frac{1}{2\eta}\int_{-\eta}^{\eta} \mathrm{d}\xi \left[(\alpha + \beta)\theta(a - b + \xi) + \int_{a}^{b} \mathrm{d}y \theta(f(y) + \xi - b)\rho_*(y) \right]. \quad \text{(A4)}$$

As η is certainly less than $b - a$, the first term on the right-hand side does not contribute anything. The second term reduces to

$$\beta = \frac{1}{2\eta}\int_{0}^{\eta} \mathrm{d}\xi \int_{y_1(b-\xi)}^{y_2(b-\xi)} \mathrm{d}y \rho_*(y). \quad \text{(A5)}$$

Interchanging the order of integration,

$$\beta = \frac{1}{2\eta}\left[\int_{y_1(b-\eta)}^{y_m} \mathrm{d}y\, \rho_*(y) \int_{b-f_1(y)}^{\eta} \mathrm{d}\xi + \int_{y_m}^{y_2(b-\eta)} \mathrm{d}y\, \rho_*(y) \int_{b-f_2(y)}^{\eta} \mathrm{d}\xi \right], \quad \text{(A6)}$$

where the peak in $f(y)$ occurs at $y = y_m$. Equation (4.9) for β follows immediately.

Finally, (4.7) becomes

$$\rho_*(x) = \frac{1}{2\eta}\int_{-\eta}^{\eta} \mathrm{d}\xi \left[(\alpha + \beta)\delta(x - a - \xi) + \int_{a}^{b} \mathrm{d}y \delta(x - f(y) - \xi)\rho_*(y) \right]. \quad \text{(A7)}$$

The first term in the integrand only contributes if x lies between a and $a + \eta$. Equation (4.10) therefore follows.

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