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On a simple derivation of master equations for diffusion processes driven by white noise and dichotomic Markov noise

V BALAKRISHNAN
Department of Physics, Indian Institute of Technology, Madras 600036, India

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Abstract. A very simple way is presented of deriving the partial differential equations (the master equations) satisfied by the probability density for certain kinds of diffusion processes in one dimension, in which the driving term is a Gaussian white noise, or a dichotomic noise, or a combination of the two. The method involves the use of certain 'formulas of differentiation' to derive the equations obeyed by the characteristic functions of the processes concerned, and thence the corresponding master equations. The examples presented cover a substantial number of diffusion processes that occur in physical modelling, including some master equations derived recently in the literature for generalizations of persistent diffusion.

Keywords. Diffusion processes; master equations; white noise; dichotomic noise.

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1. Introduction

A wide variety of physical problems involving the simultaneous action of deterministic and random 'forces' is amenable to description by diffusion processes and their generalizations. A very convenient way of studying such processes is to work with the partial differential equations satisfied by the corresponding probability densities. These equations (often called 'master equations') are linear, and are usually easier to handle mathematically than the stochastic equations obeyed by the random processes themselves. The latter may be driven by singular functions such as white noise, various kinds of jump processes, etc., making it necessary to properly take into account subtle technicalities of a mathematical nature. In contrast, given a partial differential equation, one can generally identify readily the initial conditions and boundary conditions under which the problem is well-posed, in the sense of possessing a unique, physically meaningful solution in a suitable space of functions. There are many other advantages to be gained by a master-equation approach—for instance, in the analysis of the role of noise in bifurcations from one class of stable states to another [1] and of noise-induced transitions in general [2], to name just one example. For a relatively recent review of these and related applications, see West and Lindenberg [3].

There is an extensive literature on the derivation of master equations from the corresponding stochastic differential equations—in particular, on the rigorous derivation of the Fokker–Planck (or forward Kolmogorov) equation from the "nonlinear" Langevin equation for a Markovian diffusion process. Much of this literature involves the mathematics of probability theory and stochastic processes at a fairly sophisticated level [3a–6]. Treatments addressed to physicists are quite numerous and relatively more accessible [7–9], but the clear identification of the essential inputs is often obscured in the interests of a reasonable degree of rigour and generality. Moreover, as already mentioned, the discussion is generally restricted to Markovian (mostly Fokker–Planck type) master equations. Given the ubiquitous nature of the stochastic models under consideration, it seems to be worthwhile to have a derivation of various master equations that is both pedagogically simple and comprehensive. This is the purpose of this paper. From a formal point of view, our approach is actually subsumed under the general "characteristic functional technique". However, it turns out that the use of certain special properties of the driving noise processes greatly simplifies the derivation of a large class of master equations. Accordingly, we present a simple method of derivation of master equations for a variety of processes driven by Gaussian white noise, or a dichotomic Markov noise, or combinations of the two. These two processes represent, respectively, prototypical continuous and discontinuous processes that are most frequently used as paradigms of random noise. Our procedure therefore covers a substantial part of the diverse class of diffusion processes that occur in physical modelling. In particular, we obtain in a direct and unified manner several master equations that are more comprehensive versions of equations derived earlier in different contexts [10, 11], thereby helping place these in proper perspective.

The method used is easily described. We consider one-dimensional driven processes $x(t)$ ($x \in \mathbb{R}$), with a (conditional) probability density $P(x,t)$. The characteristic function of x, defined as

$$\phi(k,t) = \int P(x,t)\exp(ikx)\,\mathrm{d}x, \tag{1}$$

may also be written as $\langle \exp ikx(t) \rangle$, where $\langle \cdots \rangle$ stands for the average over the realizations of the noise. If a formal solution for $x(t)$ can be obtained by explicitly integrating the corresponding stochastic differential equation, this solution may be substituted in eq. (1) for ϕ. "Formulas of differentiation" for the averages of functionals of the noise then enable us to find the equation obeyed by ϕ. Inversion of the Fourier transform then leads to the master equation for $P(x,t)$. We turn now to specific cases, beginning with a quick illustration on a well-known example.

2. Brownian motion, Ornstein–Uhlenbeck process

The most familiar stochastic equation one encounters in physics is of course the Langevin equation for the velocity of a Brownian particle,

$$\dot{x}(t) = -\gamma x(t) + (2D)^{1/2}\eta(t), \tag{2}$$

where γ and D are positive constants. The noise $\eta(t)$ is a stationary Gaussian–Markov process with zero mean and correlation $\langle \eta(t)\eta(t') \rangle = \delta(t-t')$. $x(t)$ is then the Ornstein–Uhlenbeck process. Solving for x with the initial condition $x(0)=0$, the characteristic function is given by

$$\phi(k,t) = \left\langle \exp\left(ik(2D)^{1/2}\int_0^t \exp[-\gamma(t-t')]\,\eta(t')\,\mathrm{d}t' \right)\right\rangle. \tag{3}$$

As $\eta(t)$ is a Gaussian process with zero mean, all its cumulants beyond the second cumulant vanish. It can then be shown that [7], for any sufficiently regular function

$f(t)$, we have the (well-known) identity

$$\left\langle \exp \int_0^t f(t')\eta(t')\mathrm{d}t' \right\rangle = \exp\left(-\frac{1}{2}\int_0^t f^2(t')\mathrm{d}t' \right). \tag{4}$$

Hence from (3) we obtain

$$\phi(k,t) = \exp[-(Dk^2/2\gamma)(1-\exp(-2\gamma t))], \tag{5}$$

so that $\phi_t = -Dk^2\phi\exp(-2\gamma t)$ (subscripts denote partial derivatives). To eliminate the explicit t-dependence on the right, we note that $\phi_k = -(Dk/\gamma)[1-\exp(-2\gamma t)]\phi$, so that $\phi_t = -\gamma k\phi_k - Dk^2\phi$. Inverting the Fourier transform, we obtain

$$P_t = \gamma(xP)_x + DP_{xx}, \tag{6}$$

the standard Fokker–Planck equation for the O–U process. The limit $\gamma = 0$ yields the conventional diffusion equation

$$P_t = DP_{xx} \tag{7}$$

for the Wiener process: the latter is "the integral of a δ-correlated stationary Gaussian–Markov process", a physical example being the velocity of a Brownian particle on time scales much larger than the velocity correlation time γ^{-1}. An even more familiar example is of course the position x of a Brownian particle.

3. Persistent diffusion, linear dichotomic flow

We turn now to the case of persistent diffusion or a dichotomic flow [12, 11], specified by the stochastic equation

$$\dot{x}(t) = \xi(t) \tag{8}$$

where $\xi(t)$ is a stationary dichotomic Markov process that jumps between two values c_1 and c_2 with mean rates $\nu(c_i \to c_j) = \nu_{ij}$ where $i,j = 1,2$ [13]. The process $x(t)$ thus describes the position of a particle that moves freely on a line, the velocity of the particle switching at random instants of time from the value c_1 to the value c_2, and vice versa. The mean value of the noise is

$$\langle \xi \rangle = (\nu_{12}c_2 + \nu_{21}c_1)/(2\bar{\nu}), \tag{9}$$

where $\bar{\nu} = (\nu_{12} + \nu_{21})/2$ is the mean transition rate. As this is non-zero in general, we are considering biased persistent diffusion: there is an overall drift in the position of the particle. The correlation function of the noise is given by

$$\langle \xi(t)\xi(t') \rangle = \langle \xi \rangle^2 + \frac{\nu_{12}\nu_{21}(c_1-c_2)^2}{4\bar{\nu}^2}\exp(-2\bar{\nu}|t-t'|). \tag{10}$$

Since ξ is not a white noise, $x(t)$ is no longer a Markov process. Writing

$$F[\xi] = \exp\left(ik\int_0^t \xi(t')\mathrm{d}t' \right), \tag{11}$$

the characteristic function of x is $\langle F\rangle$ where the average is over the realizations of ξ. Hence

$$\phi_t = ik\langle\xi(t)F[\xi]\rangle. \tag{12}$$

Using now the "formula of differentiation" [14]

$$\partial_t\langle\xi(t)F[\xi]\rangle = \langle\xi(t)\partial_t F\rangle + 2\bar{\nu}\{\langle\xi\rangle\langle F\rangle - \langle\xi F\rangle\} \tag{13}$$

for the average of such a functional of a dichotomic noise, we have, from (12),

$$\phi_{tt} = -k^2\langle\xi^2(t)F\rangle - 2\bar{\nu}\phi_t + 2\bar{\nu}ik\langle\xi\rangle\phi. \tag{14}$$

However, we note that since ξ is a dichotomic variable, it satisfies the identity

$$\xi^2 = (c_1 + c_2)\xi - c_1c_2. \tag{15}$$

Using this in (14) and invoking (12), we obtain a linear relation between ϕ, ϕ_t and ϕ_{tt}. Inverting the Fourier transform in this relation, we obtain the required master equation for biased persistent diffusion:

$$P_{tt} + (c_1 + c_2)P_{xt} + 2\bar{\nu}P_t + (\nu_{12}c_2 + \nu_{21}c_1)P_x + c_1c_2P_{xx} = 0. \tag{16}$$

A number of special cases may be read off from this result. If $\langle\xi\rangle = 0$, so that the diffusion is unbiased, the $\partial P/\partial x$ term does not appear. If $c_1 = -c_2$, the mixed derivative term P_{xt} disappears. If $c_1 = -c_2$ and also $\nu_{12} = \nu_{21} = \nu$, we recover the well-known telegraph equation for the simplest version of persistent diffusion [15], namely,

$$P_{tt} + 2\nu P_t - c^2P_{xx} = 0. \tag{17}$$

In the further limit $\nu\to\infty$, $c\to\infty$ such that $c^2/(2\nu)\to D$ (a finite constant), the dichotomic noise reduces to a Gaussian white noise, and the ordinary diffusion equation (7) is obtained. Various other limits can be recovered as well—for instance, if $c_2\to\infty$, $\nu_{21}\to\infty$ such that the ratio $c_2/\nu_{21} = w$ is finite, the process x is driven by white shot noise. The master equation (16) then yields the Markovian master equation [10]

$$P_t + [c_1 + w\nu_{12}(1 + w\partial_x)^{-1}]P_x = 0. \tag{18}$$

Next, let us consider a linear dichotomic flow given by the stochastic equation

$$\dot{x}(t) = -\gamma x + \xi(t) \tag{19}$$

where γ is a positive constant and $\xi(t)$ is a dichotomic noise with the properties already described. A master equation for $P(x,t)$ that is an integro-differential equation (in t) is in fact already known [16] for the more general nonlinear dichotomic flow $\dot{x} = f(x) + g(x)\xi(t)$: This master equation is derived using the so-called "stochastic Liouville equation method" [17]. Here, we merely point out that the linear case of (19) can be easily treated by the method used to arrive at (16). Taking $c_1 = -c_2 = c$ and $\nu_{12} = \nu_{21} = \nu$ for simplicity, we obtain in this case the master equation

$$P_{tt} - 2\gamma xP_{xt} + (2\nu - 3\gamma)P_t + 2\gamma(2\gamma - \nu)xP_x + (\gamma^2x^2 - c^2)P_{xx} + 2\gamma(\gamma - \nu)P = 0. \tag{20}$$

4. Combinations of white noise and dichotomic noise

Many physical applications entail forced diffusion of an interesting kind: while the diffusion is going on, the velocity of the diffusing species switches randomly between two or more values. Taylor dispersion is a classic example [18, 19, 7, 20]. The simplest instance of this situation is that of forced dichotomic diffusion, given by the stochastic equation [21]

$$\dot{x}(t) = \xi(t) + (2D)^{1/2}\eta(t) \tag{21}$$

where the dichotomic noise ξ and the white noise η are independent random processes, with properties as defined in the earlier sections. Since we can independently average over the realizations of ξ and η, the characteristic function of x is given by

$$\phi(k,t) = \left\langle \exp\left(ik \int_0^t \xi(t')\mathrm{d}t' \right) \right\rangle \exp(-Dk^2 t), \tag{22}$$

after the average over η is done. On going through the same steps as in the case of the dichotomic flow, we get from (22) the following master equation:

$$P_{tt} - 2DP_{xxt} + (c_1 + c_2)P_{xt} + 2\bar{v}P_t + (v_{12}c_2 + v_{21}c_1)P_x + (c_1 c_2 - 2\bar{v}D)P_{xx} - D(c_1 + c_2)P_{xxx} + D^2 P_{xxxx} = 0. \tag{23}$$

This master equation is quite complicated, being of second order in time (a reflection of the dichotomic noise ξ) and fourth order in space (essentially two different diffusion processes interleaved). This kind of forced dichotomic diffusion can also be interpreted in the following way: The diffusing particle is in the velocity state c_i with an exponential waiting time distribution $\exp(-v_{ij}t)$, where $i,j = 1,2$. The probability density of a displacement x in a single sojourn in the velocity state c_i is the (normalized) displaced Gaussian $(4\pi Dt)^{-1/2}\exp[-(x - c_i t)^2/(4Dt)]$, corresponding to ordinary diffusion in a uniform field of force. As in (16) and (20), the master equations corresponding to a variety of special cases can be read off from (23). For instance, when $c_1 = -c_2 = c$ and $v_{12} = v_{21} = v$, equation (23) simplifies to

$$P_{tt} - 2DP_{xxt} + 2vP_t + (c^2 - 2vD)P_{xx} + D^2 P_{xxxx} = 0. \tag{24}$$

This master equation has recently been obtained [11] as a generalization of that for persistent diffusion, by first constructing (the Fourier–Laplace transform of) the probability density $P(x,t)$ with the help of the interpretation described above, and working backwards to write down the differential equation obeyed by P. On the other hand, our derivation identifies and starts with the appropriate simple-looking stochastic equation to which it corresponds (namely, eq. (21)), and also places (24) in the perspective of the more general (23).

Finally, let us consider dichotomic diffusion, i.e., Brownian motion in which the strength of the driving Gaussian white noise switches randomly between two values according to an independent dichotomic Markov process. The corresponding stochastic equation may be written as [21]

$$\dot{x}(t) = (2\Gamma)^{1/2}\xi(t)\eta(t), \quad (\Gamma > 0) \tag{25}$$

where ξ and η have their usual meaning. The diffusion coefficient (and not the velocity, which is of course infinite in this case!) thus flips randomly between the two values $D_1 = \Gamma c_1^2$ and $D_2 = \Gamma c_2^2$. We now have

$$\phi(k,t) = \left\langle \exp\left(ik(2\Gamma)^{1/2} \int_0^t \xi(t')\eta(t')\mathrm{d}t' \right) \right\rangle$$

$$= \left\langle \exp\left(-\Gamma k^2 \int_0^t \xi^2(t')\mathrm{d}t' \right) \right\rangle, \tag{26}$$

the final line following upon averaging over the Gaussian white noise η using (4): this is permissible because the random variable ξ is bounded and has only finite discontinuities. We now express ξ^2 as a linear function of ξ (cf. eq. (15)), use the formula of differentiation (13) suitably to obtain ϕ_{tt} in terms of ϕ and ϕ_t, and then invert the Fourier transform, as before. The following master equation is obtained, corresponding to the stochastic equation (25):

$$P_{tt} - (D_1 + D_2)P_{xxt} + 2\bar{\nu}P_t - (\nu_{12}D_2 + \nu_{21}D_1)P_{xx} + D_1 D_2 P_{xxxx} = 0, \tag{27}$$

where $\bar{\nu} = (\nu_{12} + \nu_{21})/2$ as before. Some insight into the structure of this equation is obtained by rewriting it in the form

$$(\partial_t - D_1\partial_x^2)(\partial_t - D_2\partial_x^2)P + 2\bar{\nu}(\partial_t - \bar{D}\partial_x^2)P = 0, \tag{28}$$

where $\bar{D}$ is the mean diffusion constant defined as

$$\bar{D} = (\nu_{12}D_2 + \nu_{21}D_1)/(\nu_{12} + \nu_{21}). \tag{29}$$

Formulas of differentiation like that of (13) for dichotomic noise are derivable [14] also for other exponentially correlated noise processes [22] driving the process $x(t)$. The procedure described above can then be used to obtain the corresponding master equation obeyed by $P(x,t)$ in such cases as well. Extensions to stochastic equations with space-dependent coefficients [23] and non-Markovian continuous time random walks with power-law waiting time distributions [24] are of considerable interest, and are under investigation.

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