Iterated Functions and Intermittency

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Introduction

In the 'Think it Over' Section of Resonance, May 2000 (p. 90), S Shirali has posed a very interesting question: "Consider the 'n-fold' sine function

$$f_n(x) = \sin(\sin(\sin(\cdots(\sin x)\cdots))) \qquad (1)$$

representing n successive applications of the sine function. The dependence of the value of $f_n(x)$ upon x seems to diminish as n increases; and for fixed x and sufficiently large n, we have $f_n(x) \approx \sqrt{(3/n)}$. How may this phenomenon be explained?"

This question is interesting in its own right. It is also related to a more general one in the context of dynamical systems – in particular, to a phenomenon called intermittency that occurs in many physical situations. Experimental systems in which it has been observed include ring lasers, fluid layers undergoing convection, chemical reactions and nonlinear electronic circuits, to name a few. In fluid dynamics, the transition from the regular flow of a liquid to turbulent flow under certain conditions is believed to be related to the 'intermittency route to chaos'. What could be the connection between such seemingly unrelated things like the iteration of a function and the onset of turbulence? This is surely worth studying in a little more detail.

Stable and Unstable Fixed Points of a Map

Let $f(x)$ be a real, single-valued function of x (with some specified range of x and domain of f). For any given $x > 0$, let $x_1 = f(x_0)$; $x_2 = f(x_1)$, and so on. Thus $f_n(x_0)$ is obtained by the n-fold application of the function to the initial value x_0. We may write

$$x_n = f(x_{n-1}) = f(f(x_{n-2})) = \cdots = f^n(x_0). \quad (2)$$

The map f may be regarded as the rule of evolution of a dynamical system in which the integer variable n plays the role of (discrete) 'time'. In the example of (1), $f(x)$ is just $\sin x$. Asking for the value(s) of $f^n(x)$ as $n \to \infty$ is clearly equivalent to asking when $f(x)$ is x itself, i.e., to asking for the solution(s) of the equation $x = f(x)$. A solution $x = x^*$, if it exists, is called a fixed point (FP) of the map f. The well-known method of successive approximations deals with the solution of such equations. This is illustrated geometrically in Figure 1, for a case in which the map has two fixed points a and b, respectively. The staircase pattern shows pictorially what happens to three different initial values x_0 under iteration of the map. An initial value $x_0 < a$ approaches a as n increases. So does a value x_0 lying between a and b. An initial value $x_0 > b$ moves away from b towards ∞. One therefore says that a is an attracting fixed point, while b is a repelling fixed point.

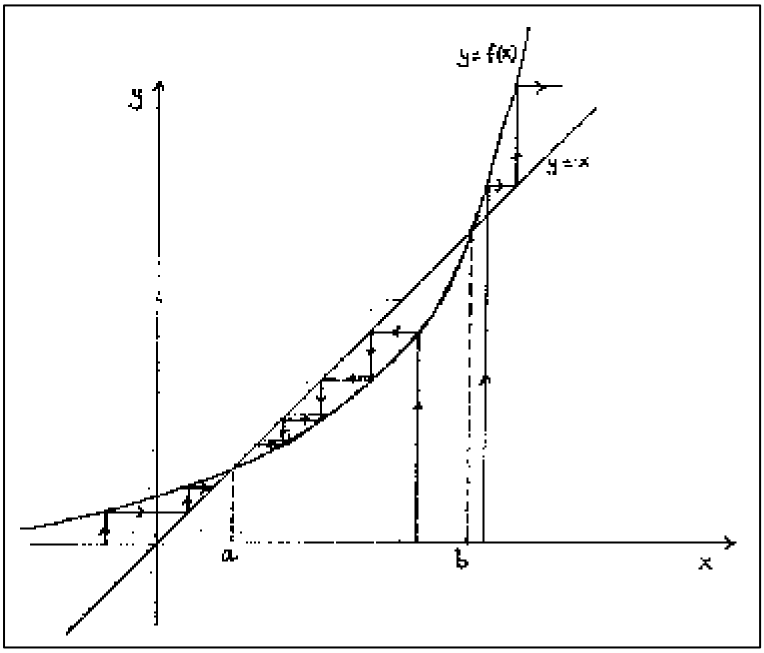


Figure 1.

It is easy to convince oneself (e.g., by drawing pictures as in Figure 1) that x^* is an attracting or stable FP if the magnitude of the slope of the map at x^* is less than unity, i.e., $|f'(x^*)| < 1$. On the other hand, x^* is a repelling or unstable FP if $|f'(x^*)| > 1$. Moreover, in these cases the iterates always approach or move away from the FP exponentially as a function of n. For instance, consider the map $f(x) = x/2$ for which $|f'(x)| = 1/2$ everywhere. Clearly the only FP is at $x^* = 0$. Since $x_n = f_n(x_0) = 2^{-n}x_0 = x_0 \exp(-n \ln 2)$, the approach is exponentially fast (as a function of n), with a characteristic rate equal to $\ln 2$. Similarly, the map $f(x) = 2x$ has a repellor at $x^* = 0$. Since $x_n = 2^n x_0 = x_0 \exp(n \ln 2)$ in this case, the repellor 'throws off' iterates on either side of itself with a rate equal to $\ln 2$. The exponents $-\ln 2$ and $+\ln 2$ in these two examples are called Lyapunov exponents.

Marginal Fixed Points

The most interesting case occurs when $|f'(x^*)|$ is exactly equal to unity: x^* is then a marginal FP. This is what happens in the problem posed in Eq. (1), in which $f(x) = \sin x$. The only root of $x = \sin x$ is at $x^* = 0$, and the slope of $\sin x$ at the origin is unity. It is clear from Figure 2a that $|f'(x)|$ has a maximum at $x = 0$. Therefore initial values x_0 on either side of 0 approach the FP under iteration of the map – it is a marginally stable FP. On the other hand, if the map has a shape as in Figure 2b, the origin is a marginally unstable FP. The third possibility is shown in Figure 2c, in which the marginal FP attracts iterates on one side and repels them on the other. (Exercise for the reader! Figures 2a–c have been sketched for the case in which the slope $f'(0)$ is $+1$ at the FP. The reader is invited to sketch the corresponding figures and check out what happens in the case $f'(0) = -1$.)

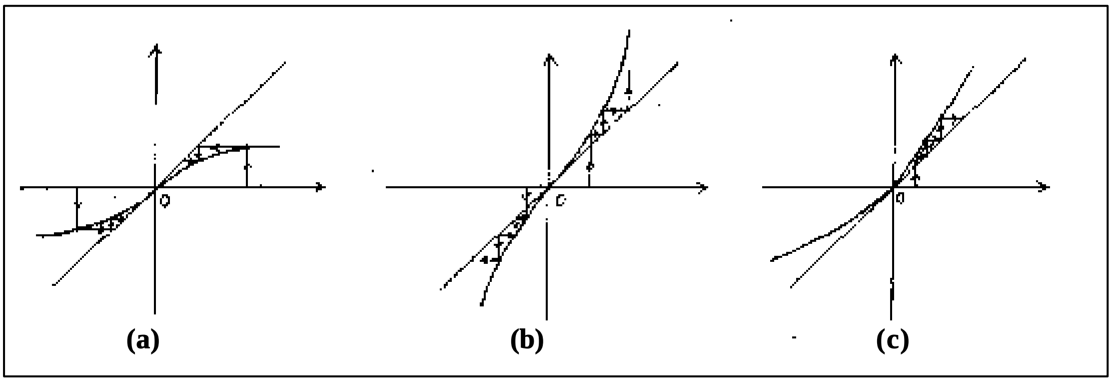


Figure 2.

The important point is the following: in all these cases, the approach towards the marginal FP (as $n \to \infty$ in the case of attraction, or as $n \to -\infty$ in the case of repulsion) is not exponential in n; rather, it is proportional to n raised to some power β. Given the map function f, we can easily find β (it is $-1/2$ for the sine map). We can also find the prefactor multiplying $|n|^{\beta}$ (it is $\sqrt{3}$ for the sine map). We shall derive these results heuristically, using a simple approximation. The results obtained can be confirmed by a more rigorous analysis.

Consider first the sine map (Figure 2a). When the iterates are very close to 0, $\sin x$ can be approximated by the first two terms in its power series expansion. Thus we have

$$x_n = \sin x_{n-1} \simeq x_{n-1} - \frac{x_{n-1}^3}{6}. \qquad (3)$$

Clearly, as $n \to \infty$ and $x_n \to 0$, the difference $(x_n - x_{n-1})$ becomes smaller and smaller, and so does $\Delta n = 1$ relative to n. Sufficiently close to the fixed point, therefore, we can approximate the difference equation above by a differential equation in continuous time, namely, $dx/dn \simeq -x^3/6$. This is easily integrated to give $1/x^2 = 3/n$, or, finally, $|x_n| \simeq \sqrt{(3/n)}$, as stated in the original problem.

Let us generalise this result, taking the marginal FP of the map f to be located at 0 as before, for convenience.

Close to the FP, the map takes the form $x_n = x_{n-1} +$ terms of higher order in x_{n-1}. If $f(x)$ is regular at 0, these terms are generically of the form $c_2 x_{n-1}^2 + c_3 x_{n-1}^3 + \cdots$, unless something special occurs. (For instance, the fact that $\sin x$ is an odd function of x led to $c_2 = 0$ in that case.) Let us allow for an even more general possibility: $f(x)$ might be only once differentiable at $x = 0$. Therefore, sufficiently close to the origin, the map has the form

$$x_n = x_{n-1}(1 + c|x_{n-1}|^{\alpha}). \qquad (4)$$

Here α is any positive number, not necessarily an integer. We may regard it as the 'degree of tangency' of $f(x)$ at the fixed point with the 45° line. The sign of the coefficient c decides whether the FP is marginally stable as in Figure 2a ($c < 0$), or marginally unstable as in Figure 2b ($c > 0$). We now proceed exactly along the lines following (3), by going over from the difference equation to a differential equation. It is straightforward to show that the approach to the FP has the leading behaviour

$$|x_n| \simeq (\alpha|c|n)^{-1/\alpha} \qquad (5)$$

as $n \to \infty$ in the attracting case ($c < 0$), or as $n \to -\infty$ in the repelling case ($c > 0$). For the sine function we have $\alpha = 2$ and $c = -1/6$, and we recover the result quoted earlier. Similarly, repeated iteration of the function $\tan^{-1} x$ (taking the principal branch of the function lying between $-\pi/2$ and $\pi/2$) would lead to $|x_n| \simeq \sqrt{3/2n}$ for very large n.

An example of some importance in other contexts is that of the logistic map

$$x_n = \lambda x_{n-1}(1 - x_{n-1}); \quad x_n \in [0, 1] \qquad (6)$$

with $\lambda = 1$. At this value of the parameter λ, the fixed points of the map at 0 and $1 - \lambda^{-1}$ coalesce, and a so-called saddle-node bifurcation takes place. (Of course the most interesting features of this map lie at larger values of λ, following the first period-doubling cascade starting at $\lambda = 3$ and ending with the onset of chaos at $\lambda = 3.5699\cdots$. But this is not our concern here.) For $\lambda = 1$ the approach to the marginal FP at 0 can be read off from (5). Making the identifications $c = -1$ and $\alpha = 1$, we get $x_n \simeq 1/n$ for the leading behaviour of the approach of the iterates to zero in this case.

A remark is in order here. We obtained the expression given in (5) for the leading behaviour by the simple device of replacing the map by its continuum version, a differential equation. In general, this will not yield anything more than the leading term in the asymptotic behaviour of x_n. For higher order corrections, we must return to the original map and analyse it carefully. For instance, continuing with the example considered in the preceding paragraph, the asymptotic behaviour of the iterates of the logistic map at $\lambda = 1$ is given by

$$x_n = \frac{1}{n} - \frac{\ln n}{n^2} + \cdots \qquad (7)$$

Similarly, in the case of our original example (the sine map) itself, some work is required to show that

$$x_n = \sqrt{3/n}\left(1 - \frac{3\ln n}{10n} + \cdots\right). \qquad (8)$$

The '$+\cdots$' in these equations stand for higher order terms in the asymptotic expansions. Unlike the first two terms that we have written down explicitly in (7) and (8), these higher order terms turn out to depend, in general, on the initial value x_0. The reader is invited to establish these results for herself!

Intermittency

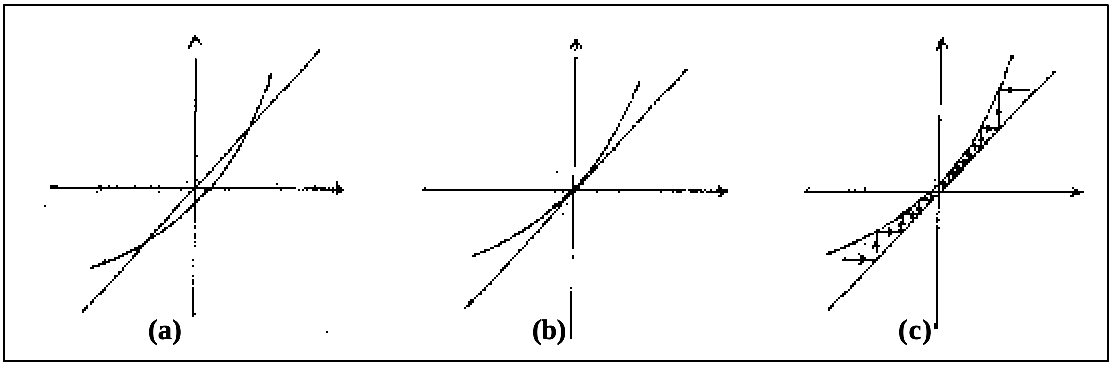
Let us now see how the foregoing is connected to the phenomenon of intermittency. This is the name given to a kind of dynamical behaviour that is intermediate between regular, periodic motion on the one hand, and irregular, chaotic motion on the other. The system displays fairly long intervals of approximate periodicity, called 'laminar phases'; these are separated by randomly intermittent 'bursts' of chaos, and hence the name. The phenomenon occurs quite commonly in so-called 'dissipative systems'. Intermittent chaos often precedes fully-developed chaos. One then speaks of 'the intermittency route to chaos'.

There are, in fact, several types of intermittency. These differ from each other in technical detail, having to do with the specific kinds of bifurcations that can occur in different cases. However, there is a simple model that helps us understand the basic mechanism by which chaotic motion can be interspersed with (possibly long) intervals of approximately periodic motion. Consider the discrete-time dynamical system

$$x_n = f(x_{n-1}) = \epsilon + x_{n-1} + x_{n-1}^2. \qquad (9)$$

Here $x \in \mathbb{R}$ and ϵ is a small parameter. For $\epsilon < 0$ (Figure 3a), the map has a stable FP at $x^* = -\sqrt{-\epsilon}$ and an unstable one at $x^* = +\sqrt{-\epsilon}$. When $\epsilon = 0$, these two FP's coalesce at 0, which becomes a marginal FP, as shown in Figure 3b. Finally, when $\epsilon > 0$ (Figure 3c), the map has no fixed point at all, since the roots of $x = f(x)$ become complex. The reverse progression from Figure 3c to Figure 3a as ϵ crosses zero from positive to negative

Figure 3.



values is the prototype of a phenomenon mentioned earlier, namely, a saddle-node bifurcation. The scenario at $\epsilon = 0$ is already familiar to us from the foregoing discussion on marginal FP's. (It corresponds to the case $\alpha = 1, c = 1$.)

Now consider the situation in Figure 3c, corresponding to $\epsilon > 0$. The staircase pattern suggests a long, approximately regular, sojourn across the narrow 'channel' in the vicinity of $x = 0$. In the simple model of (9), once the iterate escapes from the vicinity of this channel, it moves off to ∞. However, in truly chaotic systems (see the Box 1 for a very brief 'recap' of what we mean by chaos), the dynamical variable (here, x) remains bounded; and the map f representing the evolution is 'folded' in such a way that the variable is repeatedly injected back into the channel region, after wandering chaotically in other regions. This is essentially how intermittency arises in nonlinear dynamics. To be more precise, Figure 3c illustrates what is called 'Type I' intermittency. More detailed models involve maps in several variables, and more complicated scenarios occur than that shown here. But the basic mechanism in all types of intermittency is essentially the same: (i) near-regular sojourns in a channel-like region associated with a bifurcation that involves an FP or more complicated object like a limit cycle that is marginal; (ii) escape from such a region to more chaotic behaviour; (iii) re-injection into the channel; and so on. We have already mentioned several kinds of physical systems in which intermittency has been observed. Further details may be found in the references listed under Suggested Reading.

Power Laws in Intermittency

One of the most interesting features of intermittency is the occurrence of various power laws instead of the more usual exponential dependencies. We have already seen how the tangency at a marginal FP 'slows down'

Box 1. Fixed points, Periodic points and Chaos

As the main focus of this article is not on chaos, we do not go into the details of the latter here. The reader may refer to the sources in Suggested Reading, and also to earlier articles in *Resonance* – see, in particular, S Natarajan, Chaotic Dynamics on the Real Line, Parts 1 and 2, *Resonance*, Vol. 5, Nos. 4 and 5, 2000. However, as we have used the term 'chaos' in the present article without elaboration, it is useful to state a few relevant points briefly.

As explained in the text, a fixed point (or period-one point) of the map $f(x)$ is a solution of the equation $x = f(x)$. All such solutions obviously also satisfy the equation $x = f_n(x)$. But this equation may have additional solutions that are *not* fixed points of the map. Such solutions correspond to *period-n cycles*. More accurately, $(a_1, a_2, \ldots, a_n)$ is a period-n cycle of the map, provided each a_i $(i = 1, \ldots, n)$ is a fixed point of the map $f_n(x)$ but not of any $f_k(x)$ for all $k < n$. What happens is that $f(a_1) = a_2, f(a_2) = a_3, \cdots, f(a_n) = a_1$. Just as in the case of a fixed point of the map, the condition for the stability of the period-n cycle is given by $|f'(a_1) f'(a_2) \cdots f'(a_n)| < 1$.

It may so happen that the map has no *stable* periodic orbits of *any* finite period n. The iterates of 'most' initial values x_0 then wander 'randomly' over the interval concerned without settling down at any fixed point or periodic cycle as the iteration number $n \to \infty$; the map is then said to display chaos.

This is a rather loose description of how chaos occurs in one-dimensional maps. As we may expect, the case of higher dimensional maps is more involved. Also, there are substantial differences between dynamics described by differential equations (flows) as opposed to difference equations (maps). However, fairly general criteria for the occurrence of chaos are: (i) a bounded phase space; (ii) a dense set of unstable periodic points therein; and, most importantly, (iii) *sensitive dependence on initial conditions*. That is, the distance between two neighbouring initial points typically increases *exponentially* with the number of iterations, at least to start with.

Computer demonstrations (often with spectacular colour graphics) of complex dynamics including chaos are, by now, quite common. For a very modest home-grown version of some of the aspects relevant to us here, try the URL

http://hsb.iitm.ernet.in/~suresh/shaastra/

the dynamics in its vicinity and produces such a power-law behaviour. In the case of intermittency, the average time T between chaotic bursts is an important quantity. This can be related directly to the mean time taken by the system to traverse channel-like regions such as the one described above. Very interesting scaling relations emerge in this regard. Not surprisingly, it turns out that the parameter α, the degree of tangency introduced in (4), plays a crucial role here too.

Let us first consider the prototypical model of (9) for small positive ϵ. Going over to the differential form of the dynamics near $x = 0$, we now have $dx/dn \simeq \epsilon + x^2$. This implies that the time taken to cross the tunnel-like region is of the order of

$$\int_{-\infty}^{\infty} \frac{dx}{(\epsilon + x^2)} \sim \frac{1}{\sqrt{\epsilon}} \, ; \qquad (10)$$

i.e., it scales like $\epsilon^{-1/2}$ as $\epsilon \to 0$. In a more general setting, ϵ would represent the amount by which some 'control parameter' p differs from a threshold or critical value p_c, that is, $\epsilon = (p - p_c)$. The generic scaling exponent for this type of intermittency is therefore $-1/2$, as long as the leading nonlinearity in the map is quadratic (i.e., x_{n-1}^2, as in (9)).

Finally, let us extend this result to the case of the more general kind of tangency permitted by (4). This is easily done, and we find that the time taken to cross the tunnel-like region, and hence the mean time between chaotic bursts, scales like

$$T \sim (p - p_c)^{-\alpha/(\alpha+1)} \, . \qquad (11)$$

The scaling exponent corresponding to a degree of tangency α is thus found to be $-\alpha/(\alpha + 1)$. Other scaling relations associated with intermittency can be generalized in a similar fashion.

Suggested Reading

[1] P Berge, Y Pomeau and C Vidal, *Order Within Chaos*, Wiley, New York, 1984.

[2] J Argyris, C Faust and M Haase, *An Exploration of Chaos*, North-Holland, Amsterdam, 1994.

[3] N G de Bruijn, *Asymptotic Methods in Analysis*, Dover, New York, 1981.

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