# Statistics of statistical anisotropy measures



2014 J. Phys.: Conf. Ser. 484 012046

(http://iopscience.iop.org/1742-6596/484/1/012046)

# Statistics of statistical anisotropy measures

**N Pant, A Rotti and T Souradeep**

IUCAA, Post Bag 4, Pune University Campus, Ganeshkhind, Pune 411007, India

E-mail: nidhip@iucaa.ernet.in, aditya@iucaa.ernet.in, tarun@iucaa.ernet.in

**Abstract.** Cosmic Microwave Background (CMB) is a Gaussian random field to a sufficient approximation, and its statistics is completely specified by the 2-point correlation function, which, most generally, can be expanded in Bipolar Spherical Harmonic (BipoSH) basis. Statistical Isotropy (SI) of 2-point correlation function is a common assumption in cosmology, which needs to be tested. Any SI violating signal can be searched in the expansion BipoSH coefficients. We have analytically evaluated the moments and the distribution of these coefficients using characteristic function approach. We have found that coefficients with $M = 0$ have an exact analytical form for any order moment. For the remaining BipoSH coefficients with $M \neq 0$, the moments have to be supplemented with a correction term. We have verified our results with measurements of BipoSH coefficients on numerically simulated statistically isotropic CMB maps.

## 1. Introduction

Assuming Gaussianity of CMB, BipoSH forms an orthonormal basis on $\mathbf{S}^2 \times \mathbf{S}^2$, making them a natural choice of basis to expand 2-point correlation function [1, 2, 4, 3, 5, 6]. These expansion coefficients were adopted by the WMAP team [7] to search for violations of isotropy in the WMAP data. We have found expressions for the moments of the distribution of the BipoSH coefficients analytically using the characteristic function approach. BipoSH coefficients can be expressed as linear combination of off-diagonal elements of the harmonic space covariance matrix [1]. Every term in the linear combination of BipoSH coefficients with $M = 0$, is found to be independent of every other term in summation, which allows us to use characteristic function method to find exact expressions for moments. BipoSH coefficients with $M \neq 0$ can have non-linear correlations among terms due to which we have corrected the expressions for moments derived using characteristic function method upto kurtosis with an extra additive term. We have tested these analytical results against simulations, and simulated the CMB maps using the widely used HEALPix [8] package.

In Section 2, we have briefly discussed the characteristic function approach, and in Section 3, we have discussed the BipoSH formalism and the statistics of the BipoSH coefficients. We have concluded with a discussion of our results in Section 4.

## 2. Characteristic function method

The characteristic function of any random variable is defined in the following manner [9]:

$$\varphi_X(t) = E[e^{itX}], \qquad t \in \Re\,. \tag{1}$$

Consider a random variable given by:

$$Z_n = \sum_{i=1}^{n} a_i X_i \ , \tag{2}$$

where $a_i$'s are constants and $X_i$'s are independent random variables, which are not necessarily identically distributed. Statistical independence of two random variables necessarily requires any form of correlations between them to vanish. The characteristic function of $Z_n$ will be the product of the characteristic function of the individual terms contributing to the linear sum given as:

$$\varphi_{Z_n}(t) = \varphi_{X_1}(a_1 t)\varphi_{X_2}(a_2 t)...\varphi_{X_n}(a_n t) \, . \tag{3}$$

Cumulant generating function is defined as the logarithm of the characteristic function as:

$$g_Z(t) = \log[\varphi_Z(t)] \, . \tag{4}$$

Cumulants of the random variable $Z$ can be obtained by taking the derivative of the cumulant generating function, and evaluating them at zero as:

$$K_n = i^n g_Z^n(t)|_{t=0} \, . \tag{5}$$

The explicit relationships between cumulants and central moments, till the fourth central moments are:

$$\begin{aligned}
\mu_1 &= K_1 && \text{Mean,} \\
\mu_2 &= K_2 && \text{Variance,} \\
\mu_3 &= K_3 && \text{Skewness,} \quad \textit{and} \\
\mu_4 &= K_4 + 3K_2^2 && \text{Kurtosis.}
\end{aligned} \tag{6}$$

Note that in Figures 1 and 2, we have plotted the moments normalized with standard deviation.

## 3. Statistics of BipoSH coefficients

BipoSH expansion of 2-point correlation function is given as:

$$C(\hat{n}_1,\hat{n}_2) = \sum_{l_1,l_2,L,M} A^{LM}_{l_1 l_2} \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}, \tag{7}$$

where $A^{LM}_{l_1 l_2}$ are BipoSH coefficients and $\{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}$ are bipolar spherical harmonics [10]. BipoSH functions can be expressed as:

$$\{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM} = \sum_{m_1 m_2} C^{LM}_{l_1 m_1 l_2 m_2} Y_{l_1 m_1}(\hat{n}_1) \, Y_{l_2 m_2}(\hat{n}_2), \tag{8}$$

where $C^{LM}_{l_1 m_1 l_2 m_2}$ are Clebsch-Gordon coefficients. The indices of these coefficients satisfy the triangularity conditions $|l_1 - l_2| \le L \le l_1 + l_2$, and $m_1 + m_2 = M$. BipoSH coefficients are a measure of off-diagonal elements of covariance matrix:

$$A^{LM}_{l_1 l_2} = \sum_{m_1 m_2} a_{l_1 m_1} a_{l_2 m_2} C^{LM}_{l_1 m_1 l_2 m_2} \ , \tag{9}$$

where $a_{lm}$'s are the spherical harmonic coefficients of the CMB maps. SI of CMB implies that the 2-point correlation function is rotationally invariant $C(\hat{n}_1, \hat{n}_2) \equiv C(\hat{n}_1 \cdot \hat{n}_2) = \sum_l \frac{(2l+1)}{4\pi} C_l P_l(\hat{n_1} \cdot \hat{n_2})$, where $\hat{n} = (\theta, \phi)$ is a unit vector on the sphere, and $C_l$ is well known angular power spectrum. Under this symmetry, only non-vanishing coefficients of expansion are $A^{00}_{ll} = (-1)^l C_l \sqrt{2l+1}$ [1].

BipoSH coefficients are linear combination of random variables given in Eq. (9). Therefore, to arrive at the distribution of a given BipoSH coefficient, we have begun with finding out the characteristic function of individual terms involved in the linear combination. Assuming the terms in linear sum to be independent of each other, the characteristic function of the BipoSH coefficient can then be derived from Eq. (3).

*Case (a): Bipolar coefficient with $M = 0$*

SI violating signal detection has been seen in these coefficients in WMAP-7 [7]. Assumption of independence among terms holds true in this case. We have found out that these coefficients have an asymmetric distribution for $l_1 = l_2$, and symmetric for $l_1 \neq l_2$ [11], as seen in Figure 1. Subsets of this case are the coefficients with $L = 0$, which are nothing but angular power spectrum $A^{00}_{ll} = (-1)^l C_l \sqrt{2l+1}$, known to have $\chi^2$ distribution.

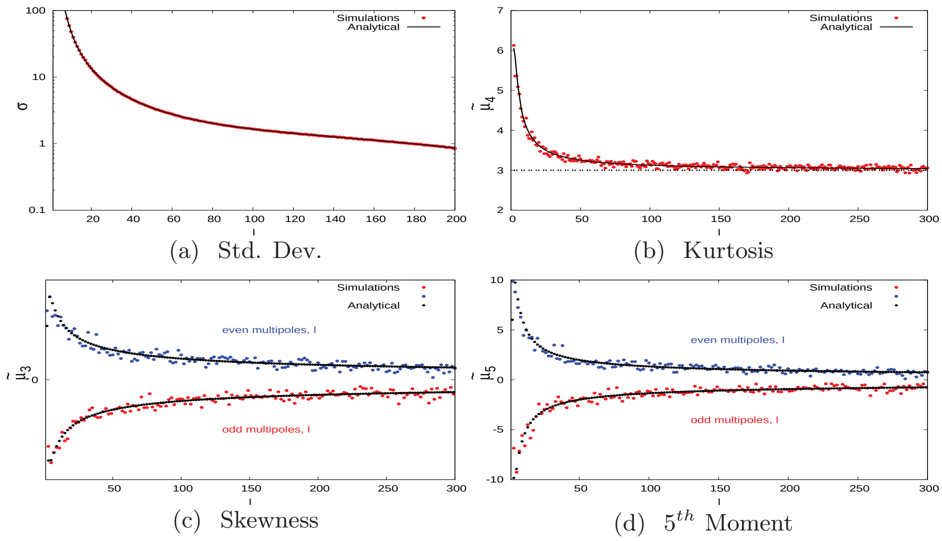
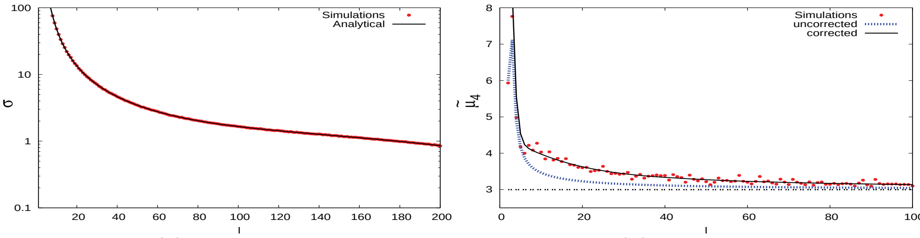



**Figure 1.** Standard deviation ($\sigma$), kurtosis ($\tilde{\mu}_4$), skewness ($\tilde{\mu}_3$), and $5^{th}$ moment ($\tilde{\mu}_5$) of $A^{20}_{ll}$, from 15000 simulations.

*Case (b): Bipolar coefficient with $M \neq 0$*

We have first calculated the moments of distribution for these coefficients using the characteristic function method, assuming that all terms in the linear combination are independent of each other. It is observed that only even ordered cumulants exist, implying that the distribution of these coefficients is symmetric as seen in Figure 2. The mismatch in simulations and analytically derived moments at low multipoles ($l$) are due to the assumed underlying independence of the terms contributing to the linear combination as seen in Figure 2. The moments calculated using the characteristic function method need to be supplemented with correction terms, which account for the higher order correlations among terms. We have found that variance does not




(a) Std. Dev. (b) Kurtosis

**Figure 2.** Standard deviation ($\sigma$) and kurtosis ($\tilde{\mu}_4$) of $A^{43}_{ll}$ derived from 15,000 simulations. The difference between corrected and uncorrected analytical moments is prominent at low values of multipole ($l$).

have any corrections due to the fact that the terms are linearly uncorrelated, and given by:

$$\tilde{\mu}_4 = \bar{\mu}_4 + \frac{3\Big[\sum_i^N (K_2^i)^2 + 2\sum_{i\neq j} E[X_i^2 X_j^2]\Big]}{\Big(\sum_i^N K_2^i\Big)^2}, \tag{10}$$

where second term is the correction term. In the above expression, $K_i$ is the cumulant of the $i^{th}$ term, and $X_i$ and $X_j$ are the $i^{th}$ and $j^{th}$ terms in the summation. The calculation for correction for higher order moments becomes very tedious, hence, we have restricted ourselves to calculating corrections for moments only upto kurtosis.

## 4. Conclusions

Statistics of BipoSH coefficients is important to understand, as signals of isotropy violation are being searched for in CMB data using these coefficients. Detection of SI violation has been claimed by WMAP-7 team in V-band and W-band maps. Due to the difference in significance of detection in the two frequency bands and its alignment with the ecliptic, these detections were not suspected to have cosmological origin. However, more recent work has attempted at explaining these observations by accounting for gravitational lensing modifications to the BipoSH coefficients [12]. Interestingly, in our study, we have found that the BipoSH coefficients showing SI violation detection have an asymmetric PDF. The WMAP team has used band power averaged BipoSH coefficients with the large bin sizes to reduce noise, however, with experiments like PLANCK, it might be possible to achieve similar signal to noise ratio for smaller bin sizes, with which the skewness of these coefficients might become considerable, and it will then become important to account for the non-Gaussian PDF of the BipoSH coefficients.

## References

[1] Hajian A and Souradeep T 2003 *Ap. J. Lett.* **597** L5
[2] Souradeep T and Hajian A 2004 *Pramana: J. Phys.* **62** 793
[3] Hajian A and Souradeep T 2004 [arXiv:astro-ph/0501001]
[4] Hajian A, Souradeep T and Cornish N 2004 *Ap. J.* **618** L63
[5] Basak S, Hajian A and Souradeep T 2006 *Phys. Rev. D* **74** 021301
[6] Hajian A and Souradeep T 2006 *Phys. Rev. D* **74** 123521
[7] Bennet C L *et al* 2011 *Ap. J. Supp. Ser.* **192** 19
[8] Gorski K M *et al* 2005 *Ap. J.* **622** 759
[9] Melvin D Springer 1979 *The Algebra of Random Variables* (John Wiley Sons Inc.)
[10] Varshalovich D A, Moskalev A N and Khersonskii V K 1988 *Quantum Theory of Angular Momentum* (Singapore: World Scientific)
[11] Joshi N, Rotti A and Souradeep T 2012 *Phys. Rev. D* **85** 043004
[12] Aditya Rotti, Moumita Aich and Tarun Souradeep 2011 [arXiv:1111.3357v1]