How is a vector rotated?

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Introduction

In an earlier series of articles[1], we saw how very general arguments such as linearity, homogenity and symmetry could be used to derive a number of results in vector (and tensor) analysis. Moreover, generalizations of many of these results to an Euclidean space of an arbitrary number (n) of dimensions could be made quite readily. The special features of three dimensions (3D) were also brought out. Most notably, the cross-product of two vectors is again a vector only in 3D. This is connected to the fact that the number of mutually orthogonal *planes* through the origin, given by ${}^{n}C_2 = n(n-1)/2$, is equal to the number of independent *axes*, n, if and only if $n = 3$.

In this article, we shall see how this fact can be exploited to derive in a simple way an explicit formula for the action of an arbitrary rotation of the coordinate frame upon any given vector. Our arguments will again be of the "*What can the answer be?*" type. The only place where the argument needs to be supported by a more detailed calculation will be pin-pointed. Even this is instructive, as it highlights a very basic property of rotations. The entire argument is quite short, but it will be described somewhat elaborately for the sake of clarity and ease of understanding.

The formula to be derived is useful in practice. Numerous physical quantities (velocity, acceleration, force, electric field, angular momentum, $\cdots$) are vectors in the elementary sense of the term: like the position vector of a point in space, they have a magnitude and a direction. The numerical values of these, namely, the values of the *components* of a vector, naturally depend on the particular coordinate frame chosen. However, as explained in the earlier articles referred to (see especially Resonance, July 97, pp. 20-26), the whole point of writing relationships between physical quantities in the form of a vector (more generally, tensor) equations is as follows: These relations are independent of the particular coordinate frame one may choose; *their form is the same in all frames obtainable from each other by rotations of the set of axes by arbitrary amounts, in arbitrary*

[1] "What *can* the answer be? Parts 1 - 4, RESONANCE, August 96, October 96, May 96, July 97

directions. Therefore, a general formula that connects a vector in a given frame to its transformed version in a 'rotated' frame is likely to be rather useful. Moreover, the formula would be applicable to *any* vector quantity, regardless of its physical charecter or dimensional formula in terms of M, L and T.

Rotating a coordinate frame in 3D

Suppose S and $S^{'}$ are two coordinate frames related to each other by a rotation. Let $\mathbf{A}$ represent (the components of) a vector as described in S, and $\mathbf{A}^{'}$ its counterpart in $S^{'}$. Given $\mathbf{A}$, we want to find $\mathbf{A}^{'}$. For this, we need to specify the rotation that "takes" S to $S^{'}$ – namely, how to bring the axes of S into coincidence with those of $S^{'}$. It is customary to do this in terms of three "Euler angles" α, β and γ : (i) rotate the frame S abouts its z-axis by a cerain angle α ; (ii) rotate the resulting frame about its x-axis by another angle β ; (iii) finally, rotate *this* frame about *its* z-axis by a third angle γ . Any $S^{'}$, no matter what its orientation is with respect to S, is guaranteed to be reached in this manner. There are, of course, other Euler angle conventions, i.e., sequences of three rotations, that would do the same thing. In applications, however, one often has a rotation specified in a more 'physical' way: rotate S *about* such-and-such a direction, *by* such-and-such an angle, to get $S^{'}$. The *direction* is of course specified by a unit vector, say $\mathbf{n}$. The *amount* of rotation is specified by the angle, say ψ. [Note that the unit vector $\mathbf{n}$ is itself specified by two angles in S : a polar angle θ and an azimuthal angle ϕ. Therefore the rotation as a whole is again specified by three angles (θ, ϕ, ψ), instead of three Euler angles (α, β, γ).] Let us denote the corresponding *rotation operator* that is to act on vectors by $R(\mathbf{n}, \psi)$. In other words, R acts on any vector $\mathbf{A}$ (specified in S) to produce another vector $\mathbf{A}^{'}$, according to the formal equation

$$\mathbf{A}^{'} \;=\; R\,(\mathbf{n}, \psi)\,\mathbf{A}. \tag{1}$$

We would like to derive an explict formula for $\mathbf{A}^{'}$.

What *can* A$^{'}$possibly be?

Observe that there are only two vectors on which $\mathbf{A}^{'}$can depend, namely, $\mathbf{A}$ and $\mathbf{n}$. Thus $\mathbf{A}^{'}$might have components along each of these. Moreover, $\mathbf{A}$ and $\mathbf{n}$ together determine a certain plane, and the vector $(\mathbf{A} \times \mathbf{n})$ is normal to this plane. $\mathbf{A}^{'}$might also have a component in this direction. In other words $\mathbf{A}$, $\mathbf{n}$ and $(\mathbf{A} \times \mathbf{n})$ form a triad in general, i.e., a set of oblique axes, along which $\mathbf{A}^{'}$can be resolved. Thus $\mathbf{A}^{'}$*must* be a linear combination of the three vectors $\mathbf{A}$, $\mathbf{n}$ and $(\mathbf{A} \times \mathbf{n})$. Since each term in the expansion must be linear in $\mathbf{A}$ (equivalently, in physical terms: each term must have the same physical dimensions as $\mathbf{A}$), the part along $\mathbf{n}$ must actually be proportional to $(\mathbf{n} \cdot \mathbf{A})\ \mathbf{n}$, which is the portion of $\mathbf{A}$ along $\mathbf{n}$. Therefore $\mathbf{A}^{'}$*must* be of the form

$$\mathbf{A}^{'} \;=\; f\,\mathbf{A} + g\,(\mathbf{n} \cdot \mathbf{A})\,\mathbf{n} + h\,(\mathbf{A} \times \mathbf{n})\ , \tag{2}$$

where f, g and h are *scalar* quantities, yet to be determined.

At this stage, it is helpful to draw a figure (Fig. 1). Now, the effect on a vector $\mathbf{A}$ of rotating the coordinate *frame* S about the axis $\mathbf{n}$ through an angle ψ is the same as that of leaving the *frame* fixed, but rotating the *vector* about the same axis through an angle equal to *minus* ψ. Thus tips of the vector $\mathbf{A}$ amd $\mathbf{A}^{'}$ lie on the rim of a cone whose axis is along $\mathbf{n}$. It is obvious from Fig. 1 that the magnitude of $\mathbf{A}$ and $\mathbf{A}^{'}$ are equal ($|\mathbf{A}| = A = |\mathbf{A}^{'}|$),and that the angle between $\mathbf{n}$ and $\mathbf{A}$ = the angle between $\mathbf{n}$ and $\mathbf{A}^{'}$ = the half-angle of the cone. (Hence $\mathbf{n}\cdot\mathbf{A} = \mathbf{n}\cdot\mathbf{A}^{'}$.). This angle and the angle ψ are the only two angles in the problem. The scalars f, g, h can only depend on these angles. However, a rotation is a *linear* transformation (as implied by Eq.(1)). This means that f, g and h cannot themselves have any sort of dependence on any particular vector $\mathbf{A}$ on which the rotation acts. Hence f, g, h can *only* depend on ψ, and Eq.(2) becomes

$$\mathbf{A}^{'} \quad = \quad f(\psi)\,\mathbf{A} + \, g(\psi)\,(\mathbf{n}\cdot\mathbf{A})\,\mathbf{n} + h(\psi)\,(\mathbf{A}\times\mathbf{n})\;. \tag{3}$$

Further, a rotation about any axis by a multiple of 2π brings S back to S, so that $\mathbf{A}^{'}$must coincide with $\mathbf{A}$ in this case. Hence f, g, h must be *periodic* functions of ψ with a period of 2π, i.e., they must actually be functions of $\cos\psi$ and $\sin\psi$.

Determination of f, g and h

We note first that $\mathbf{A}$ will remain unaltered if the axis of rotation is collinear with $\mathbf{A}$ itself, whatever be the value of ψ. (Any vector along the axis of rotation is obviously left unchanged by rotation.) In this situation $(\mathbf{n}\cdot\mathbf{A})\,\mathbf{n} = \mathbf{A}$ itself, and $\mathbf{n}\times\mathbf{A} = 0$. Therefore we must have

$$\mathbf{A} = [f(\psi) + g(\psi)]\,\mathbf{A} \tag{4}$$

for every vector $\mathbf{A}$, i.e., $f(\psi) + g(\psi) = 1$ in *general.* Hence Eq.(13) reduces to

$$\mathbf{A}^{'} \quad = \quad f(\psi)\,\mathbf{A} + \,[1 - f(\psi)]\,(\mathbf{n}\cdot\mathbf{A})\,\mathbf{n} + \,h(\psi)\,(\mathbf{A}\times\mathbf{n})\;. \tag{5}$$

On the other hand, whatever be $\mathbf{n}$, $\mathbf{A}$ must remain unchanged if $\psi = 0$ (no rotation occurs at all). Hence

$$\mathbf{A} \quad = \quad f(0)\,\mathbf{A} + \,[1 - f(0)]\,(\mathbf{n}\cdot\mathbf{A})\,\mathbf{n} + h(0)\,(\mathbf{A}\times\mathbf{n})\;. \tag{6}$$

Once again, this must hold good for any arbitrary direction $\mathbf{n}$ and any vector $\mathbf{A}$, which is only possible if

$$f(0) = 1 \quad , \quad h(0) = 0. \tag{7}$$

Now square each side of Eq.(5), and use the fact that $|\mathbf{A}| = |\mathbf{A}^{'}| = A$. We get

$$A^2 \quad = \quad \{f^2(\psi) + h^2(\psi)\}\,A^2 + \,[1 - \{f^2(\psi) + h^2(\psi)\}]\,(\mathbf{n}\cdot\mathbf{A})^2. \tag{8}$$

This is possible for *arbitrary* $\mathbf{n}$ and $\mathbf{A}$ if and only if

$$\{f^2(\psi) + h^2(\psi)\} \quad = \quad 1 \tag{9}$$

Together with Eqs.(7), and the requirement that $f(\psi+2\pi) = f(\psi)$, $h(\psi+2\pi) = h(\psi)$, this suggests strongly that

$$f(\psi) = \cos\psi, \quad \text{while} \quad h(\psi) = \pm\sin\psi. \tag{10}$$

To decide whether $h(\psi) = +\sin\psi$ or $-\sin\psi$, it is convenient to look at the special case when $\mathbf{n}$ is perpendicular to $\mathbf{A}$ and, moreover, the angle of rotation is an infinitesimal one, $\delta\psi$. Then $\cos(\delta\psi) \approx 1$, $\sin(\delta\psi) \approx \delta\psi$. From Figure 2, we see that

$$\mathbf{A}' - \mathbf{A} \quad = \quad -(\mathbf{n}\times\mathbf{A})\,\delta\psi \;\approx\; -(\mathbf{n}\times\mathbf{A})\,\sin(\delta\psi). \tag{11}$$

¿From this we deduce that $h(\psi) = -\sin\psi$ in general (since a finite ψ is built up of a succession of incremental angles $\delta\psi$).

We are therefore ready, now, to write down the general formula for the transformation of an arbitrary vector $\mathbf{A}$ under rotation of the coordinate frame about an axis $\mathbf{n}$, through an angle ψ:

$$\mathbf{A}' \quad = \quad R(\mathbf{n},\psi)\,\mathbf{A} = (\cos\psi)\,\mathbf{A} + (1-\cos\psi)\,(\mathbf{n}\cdot\mathbf{A})\,\mathbf{n} - (\sin\psi)\,(\mathbf{n}\times\mathbf{A}) \tag{12}$$

The *rigorous* derivation of the results $f(\psi) = \cos\psi$, $h(\psi) = -\sin\psi$ depends on the only property of rotations that we have not *fully* exploited so far – namely, that a rotation through a finite angle ψ can be achieved by a succession of n rotations through infinitesimal angles $\delta\psi$ such that, in the limit $n\to\infty$ and $\delta\psi\to 0$, the product $n\,\delta\psi\to\psi$. (In technical language, "a proper rotation is continously connected to the identity transformation".) *All* other properties of rotations in 3D have been used in the simple step-by-step derivation just described: first and foremost, the existence of an axis of rotation, $\mathbf{n}$; next, the linear and homogeneous nature of the transformation (every term on the right in Eq.(2) is linear in $\mathbf{A}$); its distance-preserving nature ($|\mathbf{A}| = |\mathbf{A}'|$); and finally, the fact that a rotation by a multiple of 2π is equivalent to no rotation at all as far as scalars and vectors are concerned.

Remarks; rotations in dimensions other than 3

The formula in Eq.(12) gives an explicit representation of the rotation operator $R(\mathbf{n},\psi)$ in 3D in the form in which it acts on an arbitrary *vector*. Symbolically,

$$R(\mathbf{n},\psi) \quad = \quad (\cos\psi)\,\mathbf{1} + (1-\cos\psi)\,\mathbf{n}\,\mathbf{n}\cdot\, - (\sin\psi)\,\mathbf{n}\times \tag{13}$$

Any vector on which R acts is to be inserted to the right of the operator on the right-hand side. The symbol $\mathbf{1}$ stands for the unit operator (any vector on which it acts is left unchanged). Note the term $\mathbf{n}\,\mathbf{n}\cdot$ – when a vector $\mathbf{A}$ is inserted to its right, it yields $\mathbf{n}\,(\mathbf{n}\cdot\mathbf{A})$. (Objects such as $\mathbf{1}$ and $\mathbf{n}\,\mathbf{n}$ are sometimes referred to as *dyads*, especially in the older vector analysis literature.) Explicit representations such as this for the action of *finite* (as opposed to infinitesimal) transformations are obviously very useful, but not easy to obtain in most cases.

For instance, consider what happens in dimensions *other* than 3D. In 2D (i.e., in a plane), there is no *axis* about which the rotation of the coordinate frame occurs; rather, the rotation takes place about a *point* (the origin). The effect of a rotation of the coordinate axes by angle ψ is easily deduced. Using elementary trigonometry, we find that the position vector $\mathbf{r} = (x, y)$ of an arbitrary point is transformed to $\mathbf{r}^{'} = (x', y')$, where

$$x' = x\,\cos\phi + y\,\sin\phi \quad , \quad y' = -x\,\sin\phi + y\,\cos\phi. \tag{14}$$

We have already emphasised that a vector is *defined* as a quantity that transforms (under a rotation of the coordinate frame) in precisely the same way as the position coordinates do. Therefore for any *arbitrary* vector $\mathbf{A}$, Eqs.(14) specify the components (A'_x, A'_y) of the transformed vector $\mathbf{A}^{'}$in terms of the components (A_x, A_y) of the original vector $\mathbf{A}$. However, there is no way of writing $\mathbf{A}^{'}$in *vector* form in a manner analogous to Eq.(12). This is because in 2D there is no second vector (such as $\mathbf{n}$ in 3D) besides the original vector $\mathbf{A}$. The best we can do is to invoke the antisymmetric (Levi-Civita) symbol ϵ_{ij} (defined in Part 3 of the series already referred to), and express the *components* of $\mathbf{A}^{'}$in the form

$$A'_i \;=\; (\cos\psi)\,A_i + (\sin\psi)\,\epsilon_{ij}\,A_j \tag{15}$$

where the indices i, j run over the values 1 and 2, and a summation over the repeated index j is implied. The analogous formula in 3D is, from Eq.(12),

$$A'_i \;=\; (\cos\psi)A_i + (1-\cos\psi)\,n_i\,n_j\,A_j + (\sin\psi)\;\epsilon_{ijk}\,A_j\,n_k. \tag{16}$$

One might be tempted to jump to the following conclusion: In 2D, a rotation leaves a point (dimension = 0) invariant; in 3D, it leaves a line, or axis (dimension = 1) invariant. By extrapolation, one might imagine that in nD, an arbitrary rotation would leave a $(n-2)$-dimensional subspace invariant, thus affecting only some 'plane' in the space. *However, this is incorrect*, as the following argument shows.

Any rotation in nD can be expressed as the result of *successive* "planar" rotations, i.e., linear, homogeneous, distance-preserving transformations, each of which acts on just two of the coordinates, leaving the others unchanged. In n dimensions, there are $n(n-1)/2$ different orthogonal planes, so that a rotation in nD is specified by $n(n-1)/2$ angles (analogous to the three Euler angles in 3D). However, a general rotation in nD $(n \geq 4)$ *cannot* be reduced to a rotation of some single 'tilted' plane in space, leaving an $(n-2)$-dimensional subspace unaltered. It is easy to see why: If the latter were possible, a rotation in nD would be specifiable by just n parameters : namely, $(n-1)$ parameters to specify the unit vector normal to the plane concerned, together with 1 angle to specify the angle by which the rotation occurs in the plane. But when $n > 3$, this is *less* than the actual number of parameters $[n(n-1)/2]$ it takes to specify a general rotation in nD. Therefore a general rotation is not reducible to a rotation in some plane in the space. As a consequence, formulas as simple as those of Eqs.(15) and (16) are thus not available for these transformations in nD, when $n > 3$.

There is, however, an interesting fact that is worth pointing out. We may ask whether a general rotation in nD ($n \geq 3$) leaves at least a *direction* (a one-dimensional subspace) unchanged: in other words, can we associate an *axis* with *every* rotation? The answer reveals a deep and profound difference between *even* and *odd* dimensional spaces. It is 'no' if n is even, and 'yes' if n is odd. For example, let x_i ($i = 1, 2, 3, 4$) be the cartesian coordinates in 4D. Conside r a rotation in the $x_1 x_2$ plane by an angle α_1, followed by a rotation in the $x_3 x_4$ plane by an angle α_2. Then we are guaranteed that the net rotation leaves *no* nonzero vector unchanged, i.e., there is no invariant axis for t his particular rotation.

There is an easy way to see how this general result comes about. A rotation R in nD can be implemented (represented) by an $(n \times n)$ real orthogonal matrix with determinant equal to $+1$. All n eigenvalues of such a matrix lie on the unit circl e in the complex plane, in complex conjugate pairs. [In the example given above, the eigenvalues of R are $\exp(\pm i\alpha_1)$ and $\exp(\pm i\alpha_2)$.] If $n = 2k+1$, there are in general k such pairs $\exp(\pm i\alpha_1)$, …, $\exp(\pm i\alpha_k)$, and a final single eigenvalue equal to $+1$. This last fact implies that there is a direction, i.e., a non-zero vector $\mathbf{n}$ in the space such that $R\,\mathbf{n} = 1\,\mathbf{n}$. That is, there is an *axis* of rotation for *every* rotation in any odd-dimensional space.

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