A myopic random walk on a finite chain

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Abstract. We solve analytically the problem of a biased random walk on a finite chain of 'sites' $(1, 2, \ldots, N)$ in discrete time, with 'myopic boundary conditions'—a walker at 1 (or N) at time n moves to 2 (or $N - 1$) with probability one at time $(n + 1)$. The Markov chain has period two; there is no unique stationary distribution, and the moments of the displacement of the walker oscillate about certain mean values as $n \to \infty$, with amplitudes proportional to $1/N$. In the continuous-time limit, the oscillating behaviour of the probability distribution disappears, but the stationary distribution is depleted at the terminal sites owing to the boundary conditions. In the limit of continuous space as well, the problem becomes identical to that of diffusion on a line segment with the standard reflecting boundary conditions. The first passage time problem is also solved, and the differences between the walks with myopic and reflecting boundaries are brought out.

Keywords. Random walk; myopic random walker; periodic Markov chain.

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1. Introduction

In an unbiased random walk on a regular lattice, the walker is usually assumed to jump from a site m to any of the q nearest-neighbour sites with an *a priori* probability $1/q$. An interesting situation arises in a random walk on a disordered lattice, such as a percolation cluster (the well-known 'ant-in-the-labyrinth' problem (de Gennes 1976; Straley 1980)). Owing to the disorder, the number of nearest-neighbour sites may vary randomly from site to site. A walker at site m may jump to any of the q_m *available* nearest-neighbour sites with a probability that continues to be $1/q$ (the 'blind'·ant), or with probability $1/q_m$ (the 'myopic' ant) (Mitescu and Roussenq 1983; Stauffer 1985). The latter situation may be expected to induce some extra correlations in the random walk. There is a considerable amount of literature on the subject, dealing especially with numerical simulations (Mitescu and Roussenq 1983; Pandey *et al* 1984). While there is evidence to show that the blind and myopic ants belong to the same universality class (Seifert and Suessenbach 1984; Stauffer 1985), the two random walks differ in detail in several respects.

By far the simplest situation in which one can examine analytically the differences between 'myopic' and 'ordinary' random walks is a random walk on a finite chain of sites $(1, 2, \ldots, N)$. At each non-terminal site m (i.e., $m = 2, \ldots, N - 1$), the walker jumps (at the next time step) to the neighbouring site $(m - 1)$ or $(m + 1)$, the total probability of a transition out of the site m being equal to unity. At site 1 (respectively, N), a

myopic walker would jump at the next transition to site 2 (respectively, $N - 1$) with probability one, as sites 1 and N have just a single nearest-neighbour each. In contrast, the reflecting boundary conditions one usually imposes (in 'probability-conserving' random walks without traps, absorbers, exit points, etc.) imply, in the present instance, that the walker has a non-zero probability of *remaining* at site 1 (or N) after the next time step. This difference in the behaviour of the walker at the terminal sites enables us to understand, in a simple model that can be solved analytically, at least some of the interesting features of a myopic random walk. In a slight abuse of language, we shall refer to myopic (as opposed to reflecting) boundaries or boundary conditions, although it is the walker who is myopic.

We solve the problem of a myopic random walk on a finite linear chain in discrete time, and in the presence of a uniform bias. There is no unique equilibrium distribution. We examine the asymptotic behaviour of the probability distribution, and also that of the mean displacement and the mean squared displacement of the walker. In contrast to the case of reflecting boundaries, these quantities oscillate asymptotically with amplitudes proportional to N^{-1}, the inverse of the system size. Although the oscillatory behaviour disappears in the continuous-time limit, and a unique equilibrium distribution is reached asymptotically, this distribution displays a depletion at the terminal sites 1 and N because of the boundary conditions. In the limit of continuous space as well, the differences between myopic and reflecting boundary conditions disappear altogether. The first passage time from one site to another is also studied, providing further insight into the role played by myopic boundary conditions in the random walk.

2. The random walk

2.1 *The transition matrix*

Consider a random walk (RW) in discrete time $n(= 0, 1, 2, \ldots)$ on the sites $m = 1, 2, \ldots, N$ of a finite linear chain. The walker makes nearest-neighbour hops with probabilities $\Pr(m \to m + 1) = p$ for $m \geqslant 2$, $\Pr(m \to m - 1) = q$ for $m \leqslant N - 1$, where $p + q = 1$. Since the walker is myopic, $\Pr(1 \to 2) = \Pr(N \to N - 1) = 1$. The probability of the walker *remaining* at any site for more than a time step (sometimes called the sojourn probability at a site (Murthy and Kehr 1989)) is thus zero. Let $P_n(m)$ be the probability that the walker is at site m at time n, and let $\mathbf{P}_n$ denote the column vector with elements

$$(\mathbf{P}_n)_m = P_n(m), \quad m = 1, 2, \ldots, N. \tag{1}$$

Then the random walk is a Markov chain (Cox and Miller 1972; Feller 1972) with transition matrix M:

$$\mathbf{P}_{n+1} = M\mathbf{P}_n \quad (n = 0, 1, 2, \ldots) \tag{2}$$

where M is a tridiagonal matrix with

$$M_{m+1,m} = p \quad (m \neq 1), \quad M_{21} = 1,$$

$$M_{m,m+1} = q \quad (m \neq N - 1), \quad M_{N-1,N} = 1. \tag{3}$$

We note that, in contrast, the transition matrix for a (biased) RW on a linear chain with standard reflecting barriers at the ends has $M_{21} = p$, $M_{N-1,N} = q$, and moreover $M_{11} = (1-p) = q$, $M_{NN} = (1-q) = p$. These nonzero diagonal elements at the ends arise because the walker can remain at site 1 or site N after a time step, after 'bouncing back' from the reflecting barriers imagined to be at positions $1/2$ and $N + \frac{1}{2}$ respectively. They are mainly responsible for the differences between the walks with reflecting and myopic boundaries.

The stochastic matrix M is irreducible and is not symmetric even in the unbiased case $p = q$. Its eigenvalues are

$$\lambda_0 = 1, \lambda_r = 2(pq)^{1/2} \cos \theta_r, \quad (r = 1, \ldots, N-2), \lambda_{N-1} = -1, \tag{4}$$

where

$$\theta_r = r\pi/(N-1). \tag{5}$$

(In contrast, the eigenvalue spectrum for reflecting ends is given by $\lambda_0 = 1$, $\lambda_r = 2(pq)^{1/2} \cos(r\pi/N)$, $r = 1, \ldots, N-1$.) The Frobenius-Perron theorem (Gantmacher 1959; Cox and Miller 1972) applied to M leads at once to the following conclusions:

The Markov chain is positive recurrent: each site m is visited infinitely often by the walker, with a finite mean time between successive visits. The existence of two eigenvalues $(\lambda_0, \lambda_{N-1})$ with unit modulus makes the Markov chain periodic, with period two: if the walker is at any site at time n, a return to that site is possible only at times $n+2$, $n+4, \ldots$. The reason is that the probability of remaining at any site after a time step is zero. (In the reflecting-barrier case, the non-zero sojourn probability at a terminal site suffices to make the walk aperiodic.) Owing to the periodicity above, there is no unique equilibrium or stationary distribution given by $\lim_{n \to \infty} \mathbf{P}_n$. Instead, the walker alternates between even-numbered and odd-numbered sites in successive time steps, retaining in this sense some memory of the starting point. These remarks will be made more precise shortly.

2.2 Oscillatory asymptotic behaviour

The determination of the right and left eigenvectors ψ_r and $\phi_r^\dagger$ of the matrix M is sketched in the Appendices. In terms of these, the solution to (2) is

$$\mathbf{P}_n = M^n \mathbf{P}_0 = \sum_{r=0}^{N-1} \lambda_r^n (\phi_r^\dagger \mathbf{P}_0) \psi_r, \tag{6}$$

for an arbitrary initial distribution $\mathbf{P}_0$. Therefore, when n becomes very large,

$$\mathbf{P}_n \to (\phi_0^\dagger \mathbf{P}_0) \psi_0 + (-1)^n (\phi_{N-1}^\dagger \mathbf{P}_0) \psi_{N-1}, \tag{7}$$

because $\lambda_r^n \to 0$ for the remaining $(N-2)$ eigenvalues. As shown in Appendix A, the elements of $\phi_0^\dagger$ and $\phi_{N-1}^\dagger$ are given by

$$\phi_0^\dagger(m) = 1, \quad \phi_{N-1}^\dagger(m) = (-1)^{m-1}. \tag{8}$$

Therefore $\phi_0^\dagger \mathbf{P}_0 = P_0(1) + \cdots + P_0(N) = 1$. It is convenient to define the 'cyclic subsets' of sites $C = (1, 3, 5, \ldots)$ and $C' = (2, 4, 6, \ldots)$, and call

$$\sum_{m \in C} P_n(m) = P_n(C), \quad \sum_{m \in C'} P_n(m) = P_n(C'), \tag{9}$$

with of course $P_n(C) + P_n(C') = 1$ for each n. Further, we find (see Appendix A) that the elements of ψ_0 and ψ_{N-1} are given by

$$\psi_0(m) = \frac{1}{2}\left(\frac{p-q}{p^{N-1} - q^{N-1}}\right) q^{N-m-1} p^{m-2} \eta_m, \tag{10}$$

$$\psi_{N-1}(m) = (-1)^{m-1} \psi_0(m), \tag{11}$$

where

$$\eta_1 = p, \quad \eta_m = 1 \ (2 \leqslant m \leqslant N-1), \quad \eta_N = q. \tag{12}$$

Using the foregoing, the asymptotic behaviour of the probability is as follows:

$$\lim_{n \to \infty} P_{2n}(m) = \begin{cases} 2P_0(C)\psi_0(m), & m \in C \\ 2P_0(C')\psi_0(m), & m \in C', \end{cases} \tag{13}$$

$$\lim_{n \to \infty} P_{2n+1}(m) = \begin{cases} 2P_0(C')\psi_0(m), & m \in C \\ 2P_0(C)\psi_0(m), & m \in C'. \end{cases} \tag{14}$$

In particular, if the random walk starts from a definite site m_0, $P_0(m) = \delta_{m,m_0}$. Then the conditional probability $P_n(m|m_0)$ has the asymptotic behaviour

$$P_n(m|m_0) \to [1 + (-1)^{n+m+m_0}]\psi_0(m). \tag{15}$$

It is easily verified that $\Sigma\psi_0(m \in C) = \Sigma\psi_0(m \in C') = 1/2$, so that the normalization of $P_n(m|m_0)$ is retained in (15). The oscillatory behaviour implied by (15) is smoothed out by taking a time average: from the explicit solution for $P_n(m|m_0)$ to be presented further on, we can show that

$$\lim_{n \to \infty} (1/n) \sum_{n'=1}^{n} P_{n'}(m|m_0) = \psi_0(m). \tag{16}$$

The oscillatory behaviour of $\mathbf{P}_n$ is seen most clearly from the relationship

$$P_n(C) = P_{n+1}(C') = P_{n+2}(C) \tag{17}$$

which is valid for all n. This relation is evidently true because the sojourn probability at any site (including the end points) is zero, and there are no absorbing barriers. Equation (17) can also be established formally from the solution for $\mathbf{P}_n$, eq. (6): writing $\psi_r(C) = \Sigma\psi_r(m \in C)$ and $\psi_r(C') = \Sigma\psi_r(m \in C')$, the relation $M\psi_r = \lambda_r \psi_r$ can be used to show that

$$\lambda_r \psi_r(C) = \psi_r(C'), \quad \lambda_r \psi_r(C') = \psi_r(C) \tag{18}$$

for *each* eigenvalue. (We do not need to use the explicit solution for ψ_r in doing so, merely the relation $p + q = 1$.) Equation (17) follows at once.

2.3 *Exact solution for* $P_n(m|m_0)$

For a random walker starting from a specific site m_0 at time 0, the conditional probability to be at site m at time n is given by .

$$P_n(m|m_0) = \sum_{r=0}^{N-1} \lambda_r^n \phi_r^\dagger(m_0)\psi_r(m). \tag{19}$$

It is convenient to separate the $r = 0$ and $r = N - 1$ contributions to the sum. Using the relations (8) and (11), we have

$$P_n(m|m_0) = [1 + (-1)^{n+m+m_0}]\psi_0(m) + \sum_{r=1}^{N-2} \lambda_r^n \phi_r^\dagger(m_0)\psi_r(m). \tag{20}$$

We see that the limit quoted in (16) follows because $|\lambda_r| < 1$ for $1 \leqslant r \leqslant N - 2$.

Unbiased random walk: In this case, $p = q = 1/2$ and $\lambda_r = \cos\theta_r = \cos(r\pi/(N-1))$. Using the expressions found in the Appendices for the eigenvectors of M, we obtain

$$P_n(m|m_0) = \frac{\delta_m}{(N-1)}\left[1 + (-1)^{n+m+m_0} \right.$$
$$\left. + 2\sum_{r=1}^{N-2} \cos^n\theta_r \cos(m_0-1)\theta_r \cos(m-1)\theta_r \right], \tag{21}$$

where

$$1 \leqslant m, m_0 \leqslant N$$

and

$$\delta_1 = \delta_N = 1/2, \quad \delta_m = 1 \text{ for } 2 \leqslant m \leqslant N - 1. \tag{22}$$

We note that $P_n(m|m_0)$ is not identically equal to $P_n(m_0|m)$ in general on a finite chain, even in the absence of bias. On the other hand, since the chain has no preferred end (in the unbiased case), we must have the symmetry

$$P_n(m|m_0) = P_n(m'|m'_0) \tag{23}$$

where

$$m' = N + 1 - m, \quad m'_0 = N + 1 - m_0. \tag{24}$$

This follows from our solution on observing that $\cos(m'-1)\theta_r = (-1)^r \cos(m-1)\theta_r$ and $\delta_{m'} = \delta_m$, for $1 \leqslant m \leqslant N$.

Biased random walk: Using the eigenvectors found in the Appendices, we get

$$P_n(m|m_0) = \frac{1}{2}\eta_m \left(\frac{p-q}{p^{N-1}-q^{N-1}} \right) p^{m-2} q^{N-m-1} [1 + (-1)^{n+m+m_0}]$$
$$+ \frac{2}{(N-1)}\eta_m (\sqrt{p/q})^{m-m_0} \sum_{r=1}^{N-2} \frac{(2\sqrt{pq}\cos\theta_r)^n}{(1-4pq\cos^2\theta_r)} A(m_0)A(m) \tag{25}$$

where η_m is as defined in (12), and

$$A(m) = p\sin m\theta_r - q\sin(m-2)\theta_r, \tag{26}$$

or, in a form that displays the correction to the unbiased case more clearly,

$$A(m) = (\sin\theta_r)[\cos(m-1)\theta_r + (p-q)\cot\theta_r \sin(m-1)\theta_r]. \tag{27}$$

It is easy to verify that $P_n(m|m_0)$ is unchanged on letting $m \to m'$, $m_0 \to m'_0$ and simultaneously interchanging p and q.

2.4 *The mean displacement*

The proper definition of the mean displacement of the random walker in n steps is as follows: Starting from a *uniformly* distributed initial point m_0, we require the mean value of $j = m - m_0$. Thus

$$\langle j(n) \rangle = (1/N) \sum_{m,m_0} (m - m_0) P_n(m|m_0), \tag{28}$$

which reduces to

$$\langle j(n) \rangle = (1/N) \sum_{m,m_0} m P_n(m|m_0) - (N+1)/2. \tag{29}$$

The factor $1/N$ in (28) represents the distribution of m_0 for both biased and unbiased walks. It is clear intuitively that $\langle j(n) \rangle$ must vanish identically in the absence of bias. This is established quite simply from (28) by changing variables to m' and m'_0 in the summations, and using the symmetry expressed in (23).

For the biased walk, we can write down an explicit expression for $\langle j(n) \rangle$. What is of interest is its asymptotic (large n) behaviour. Using (15) for the asymptotic behaviour of $P_n(m|m_0)$ and eq. (10) for $\psi_0(m)$, we find that

$$\langle j(n) \rangle \to \frac{Np^{N-1} - q^{N-1}}{p^{N-1} - q^{N-1}} - \frac{1}{2(p-q)} - \frac{(N+1)}{2} + \frac{[1-(-1)^N](-1)^n(p-q)}{4N}. \tag{30}$$

Thus for *odd* values of N, $\langle j(n) \rangle$ oscillates asymptotically about a mean value in successive time steps, with an amplitude proportional to the bias and inversely proportional to the size N of the chain.

2.5 *The mean squared displacement*

This is defined in an analogous manner, namely,

$$\langle j^2(n) \rangle = (1/N) \sum_{m,m_0} (m - m_0)^2 P_n(m|m_0). \tag{31}$$

For simplicity, let us consider the unbiased case. We then have

$$\psi_0(1) = \psi_0(N) = 1/2(N-1), \quad \psi_0(m) = 1/(N-1) \quad (2 \leqslant m \leqslant N-1) \tag{32}$$

The asymptotic behaviour of $\langle j^2(n) \rangle$ is found to be

$$\langle j^2(n) \rangle \to \frac{1}{6}(N^2 - N + 1) - \frac{(-1)^n}{4N(N-1)}[(-1)^N(2N-1) + 1]. \tag{33}$$

The mean squared displacement in this random walk therefore oscillates eventually about the value $(N^2 - N + 1)/6$, with an amplitude $1/2(N-1)$ for even N and $1/(2N)$ for odd N. In contrast, for the usual reflecting boundary conditions one finds that (Khantha and Balakrishnan 1983)

$$\lim_{n \to \infty} \langle j^2(n) \rangle = \frac{1}{6}(N^2 - 1). \tag{34}$$

The generating function of $\langle j^2(n)\rangle$, given by $\Sigma_{n=0}^{\infty} z^n \langle j^2(n)\rangle$, can be computed from our solution for $P_n(m|m_0)$, and the discrete analog of a frequency-dependent diffusion coefficient on the chain can be determined. We do not give these expressions here, as they have no unexpected features.

3. The continuum limit

3.1 *Random walk in continuous time*

We find first the continuous-time limit of the random walk under consideration. This is done in the usual manner. We assume that the jumps of the walker are uncorrelated with each other, and are governed by a stationary Poisson process with a mean rate $2W$. The probability of a transition out of any site in an infinitesimal time interval Δt is then $2W\Delta t$, while that of no jump out of the site in the same interval is $(1 - 2W\Delta t)$. We note that a nonzero sojourn probability at a site thus appears in the limit of continuous time, even if $p + q = 1$. This fact is responsible for the disappearance of the oscillatory features of the discrete-time random walk that arise from the periodicity of the Markov chain, once we pass to the limit of continuous time.

Denoting $P_n(m)$ by $P(m, t)$ where $n\Delta t = t$, and taking the limit $n \to \infty$, $\Delta t \to 0$, we get the following rate equation for $\mathbf{P}(t) = (P(1, t), \ldots, P(N, t)))^T$:

$$\dot{\mathbf{P}}(t) = 2W(M - I)\mathbf{P}(t), \tag{35}$$

where I is the $(N \times N)$ unit matrix, and M is the same tridiagonal matrix as before (eq. (3)). The Laplace transform $\tilde{\mathbf{P}}(u)$ is then given by

$$\tilde{\mathbf{P}}(u) = (u + 2W - 2WM)^{-1}\mathbf{P}_0$$
$$= \sum_{r=0}^{N-1} [u + 2W(1 - \lambda_r)]^{-1}(\phi_r^{\dagger}\mathbf{P}_0)\psi_r. \tag{36}$$

As already mentioned, the oscillatory behaviour of the probability distribution is no longer present. The eigenvalue $\lambda_{N-1} = -1$ merely corresponds, now, to the fastest ($= 4W$) of the relaxation rates governing the approach of the probability distribution to a unique stationary value, given by

$$\lim_{t \to \infty} \mathbf{P}(t) = \operatorname*{Res}_{(u=0)} \tilde{\mathbf{P}}(u) = \psi_0. \tag{37}$$

As λ_r, $\phi_r^{\dagger}$ and ψ_r have been found already, we have an explicit solution for $P(m, t|m_0)$. We write it down for the unbiased case:

$$P(m, t|m_0) = \frac{\delta_m}{(N-1)} \Bigg[1 + (-1)^{m+m_0}e^{-4Wt} + 2 \sum_{r=1}^{N-2} \cos(m_0 - 1)\theta_r \cos(m - 1)\theta_r \exp\{-2Wt(1 - \cos\theta_r)\} \Bigg], \tag{38}$$

where $1 \leqslant m, m_0 \leqslant N$, and $\delta_1 = \delta_N = 1/2$, $\delta_m = 1$ for $2 \leqslant m \leqslant N - 1$. The asymptotic stationary distribution is given by $\delta_m/(N - 1)$ (as may be deduced directly from detailed

balance): while the oscillatory behaviour is no longer present, the relative 'depletion' of the end points 1 and N in the stationary state owing to the myopic boundary conditions continues to occur.

The summation (over r) can be performed to obtain closed-form expressions for the transform $\tilde{P}(m, u | m_0)$ in both the biased and unbiased cases. These are somewhat lengthy, and will not be given here. However, for the sake of explicit comparison with the case of reflecting boundaries, let us consider the special case $m_0 = 1$, $m = N$ for the unbiased walk – i.e., the transform of the probability of starting at one end of the chain and reaching the other at time t. Defining the variable

$$\xi = \cosh^{-1}(1 + u/2W), \tag{39}$$

we have found in earlier work (Khantha and Balakrishnan 1983) that, for reflecting barriers at the ends of the chain,

$$\tilde{P}(N, u | 1) = \sinh \xi / u \sinh N\xi. \tag{40}$$

In the present case (of 'myopic' boundary conditions),

$$\tilde{P}(N, u | 1) = 1/2W \sinh \xi \sinh(N - 1)\xi. \tag{41}$$

3.2 *Continuous space limit*

Next, we may pass to the limit of continuous space as well as continuous time: diffusion on a finite line segment $[0, L]$ with "myopic boundary conditions". Introducing the lattice spacing a, we set $m_0 a = x_0$, $ma = x$, $Na = L$ and $Wa^2 = D$ (the diffusion constant), and take the limit $W \to \infty$, $a \to 0$ such that x_0, x, L are finite. (For simplicity, we shall present here only the unbiased case.) The probability *density* $P(x, t | x_0) = \lim (1/a) P(m, t | m_0)$. From the fact that

$$\lim 2Wt(1 - \cos \theta_r) = r^2 \pi^2 Dt / L^2 \tag{42}$$

we find (using eq. (38)), in the case $2 \leqslant m, m_0 \leqslant N - 1$,

$$P(x, t | x_0) = (1/L) \sum_{r = -\infty}^{\infty} \cos(r\pi x_0 / L) \cos(r\pi x / L) \exp(-r^2 \pi^2 Dt / L^2) \tag{43}$$

for $x_0, x \in [0, L]$. The corresponding Laplace transform (the Green function for this diffusion problem) works out to

$$\tilde{P}(x, u | x_0) = \frac{\cosh[(L - x_>)\sqrt{u/D}] \cosh[x_< \sqrt{u/D}]}{\sqrt{uD} \sinh[L\sqrt{u/D}]}, \tag{44}$$

where $x_< = \min(x_0, x)$ and $x_> = \max(x_0, x)$.

The density found above is correctly normalized, and the stationary density is $1/L$, as expected. In fact, the solution found above is precisely that obtained (Khantha and Balakrishnan 1983a) in the case of unbiased diffusion on the line segment $[O, L]$ with reflecting boundary conditions, i.e., $\partial P/\partial x = 0$ at $x = 0$ and at $x = L$. In the limit of continuous time *and* space, therefore, the difference between myopic and reflecting boundaries vanishes altogether. This conclusion remains valid for biased diffusion as well.

4. First passage times

Finally, we examine the differences between reflecting and myopic ends with regard to the first passage time from m_0 to $m(>m_0)$. (The nature of the barrier at the other end, N, is evidently irrelevant in this context.) Considering the discrete-time random walk first, we note that general expressions are known (Murthy and Kehr 1989) for the mean (and higher moments) of the first passage time required. For the sake of illustration, let us consider first passage from one end of the chain to the other ($m_0 = 1$, $m = N$). Denoting the first passage time by t_{1N}, we find the following results for the mean $\langle t_{1N} \rangle$.

With a 'myopic' boundary at site 1, in the unbiased case we find

$$\langle t_{1N} \rangle = (N - 1)^2, \tag{45}$$

while a reflecting boundary gives

$$\langle t_{1N} \rangle = N(N - 1). \tag{46}$$

For a biased random walk, we find in the myopic case

$$\langle t_{1N} \rangle = \frac{(N - 1)}{(p - q)} + \frac{2pq}{(p - q)^2} [(q/p)^{N-1} - 1], \tag{47}$$

while a reflecting barrier leads to

$$\langle t_{1N} \rangle = \frac{N}{(p - q)} + \frac{p}{(p - q)^2} [(q/p)^N - 1]. \tag{48}$$

It is verified easily that $\langle t_{1N} \rangle_{\text{myopic}} < \langle t_{1N} \rangle_{\text{refl}}$, as we should expect on physical grounds.

In the continuous-time limit, we obtain exactly the same expressions as in (45)–(48), multiplied by the factor $(2W)^{-1}$ which is the natural unit of time in the problem. Moreover, a very compact expression can be derived for the Laplace transform of the probability density $Q(m, t|m_0)$ of the time of first passage from m_0 to m, using the renewal equation (Siegert 1951; Montroll and West 1979) $\tilde{P}(m_1, u|m_0) = \tilde{P}(m_1, u|m)$ $\tilde{Q}(m, u|m_0)(m_0 < m \leqslant m_1)$ that holds good for the Markovian random walk under consideration. Taking the unbiased case for illustration, we find the result (valid for $1 \leqslant m, m_0 \leqslant N$)

$$\tilde{Q}(m, u|m_0) = \cosh(m - 1)\xi / \cosh(m_0 - 1)\xi \tag{49}$$

where $\cosh \xi = 1 + u/2W$ as defined in (39). Given this closed-form expression for the Laplace transform (or generating function) of the first passage time density, the various moments of the first passage time can be computed in a straightforward manner. It is of interest to note, finally, that the corresponding expression in the case of a reflecting boundary is (Khantha and Balakrishnan 1983)

$$\tilde{Q}(m, u|m_0) = \cosh(m - \tfrac{1}{2})\xi / \cosh(m_0 - \tfrac{1}{2})\xi. \tag{50}$$

5. Concluding remarks

We have already described, in the Introduction and in the subsequent sections, some of the interesting features of the biased random walk of a myopic walker on a finite chain. We note that other problems involving a finite Markov chain of period two (i.e., $+1$ and -1 are the two eigenvalues of the transition matrix with modulus unity) have been well studied, the most notable one being the Ehrenfest urn model (Kac 1947; Cox and Miller 1972). In the language of random walks, this model corresponds to a myopic random walker on a finite chain of $(2N + 1)$ sites, $m = -N, \ldots, +N$, such that there is a symmetric, site-dependent bias towards the central site 0, simulating a harmonically bound random walker. The uniformly biased walk we have considered has no such symmetry, of course.

It is worth mentioning that the myopic boundary condition we have used may be regarded as a combination of the usual reflecting boundary condition together with an appropriate source term. At sites 1 and 2, for instance, we have $P_{n+1}(1) = qP_n(2)$, $P_{n+1}(2) = P_n(1) + qP_n(3)$. For the usual reflecting boundary (imagined to be implemented by placing a reflector at the position 1/2), we would have $P_{n+1}(1) = qP_n(1) + qP_n(2)$, while $P_{n+1}(2) = pP_n(1) + qP_n(3)$. Hence the behaviour at the terminal site can be taken into account by supplementing the standard equation $P_{n+1}(m) = pP_n(m - 1) + qP_n(m + 1)$, considered to hold good at $m = 1$ as well, with the *boundary condition* $qP_n(1) = pP_n(0)$. Clearly, this does not suffice for the myopic random walk. What is needed here is the boundary condition above *together* with a 'source' term $S_n = qP_n(1)$, to enable us to write the evolution equation as $P_{n+1}(m) = pP_n(m - 1) + qP_n(m + 1) + S_n(\delta_{m,2} - \delta_{m,1})$. This helps us see, too, why the extra contribution vanishes in the spatial continuum limit. On the other hand, the 'myopic boundary' we have considered is sometimes simply called a reflecting boundary (Chandrasekhar 1943; Cox and Miller 1972). (Of course, no confusion should arise once the transition matrix concerned is specified explicitly.) We see, however, that it is really a combination of a reflector and a source, in the sense described above. Another useful insight is afforded by a comparison of the first passage time distributions, eqs (49) and (50). The latter equation, valid for the reflecting barrier, shows explicitly how the physical situation may be regarded as a reflector placed at the position 1/2. In contrast, eq. (49) suggests that the myopic walker sees a reflector at the site 1 itself—in the sense that, having arrived at 1, an attempt to reach site 0 bounces the walker off the reflector (at 1) into the site 2. It is in *this* sense that the 'myopic' case we have considered can also be called a random walk in the presence of reflecting ends.

Interesting new features emerge in myopic walks in higher dimensions. Results on such walks in two dimensions will be presented elsewhere.

Appendix A

Symmetrization of M and normalization of eigenvectors

The procedure we adopt is as follows. Let the (yet-to-be normalized) right eigenvector corresponding to the largest eigenvalue ($\lambda_0 = 1$) of M be the column vector

$$\psi_0' = (1, \psi_0'(2), \psi_0'(3), \ldots, \psi_0'(N))^T. \tag{A.1}$$

By the Perron-Frobenius theorem (Gantmacher 1959, Cox and Miller 1972), all the elements $\psi'_0(m)$ can be chosen to be positive quantities. Consider the hermitian, diagonal matrix S with elements

$$S_{mk} = [\psi'_0(m)]^{1/2} \delta_{mk}. \tag{A.2}$$

Then $V = S^{-1}MS$ is tridiagonal, real and symmetric, with elements

$$V_{mk} = [\psi'_0(k)/\psi'_0(m)]^{1/2} M_{mk}. \tag{A.3}$$

V and M have the same set of eigenvalues, and the eigenvectors of V can be used to form an orthonormal basis. If $V\chi_r = \lambda_r \chi_r$, then $S\chi_r = \psi_r$ and $\chi_r^{\dagger} S^{-1} = \phi_r^{\dagger}$ are the corresponding right and left eigenvectors of M. The normalization $\chi_r^{\dagger} \chi_r = 1$ implies $\phi_r^{\dagger} \psi_r = \psi_r^{\dagger} S^{-2} \psi_r = 1$, which can be used to normalize the eigenvectors of M (the latter thus form a bi-orthogonal set of vectors):
  Suppose

$$\psi'_r = (1, \psi'_r(2), \ldots, \psi'_r(N))^T \tag{A.4}$$

is the unnormalized right eigenvector corresponding to the eigenvalue $\lambda_r (r = 0, 1, \ldots, N - 1)$. Then the left eigenvector is

$$\phi_r^{\dagger} = \psi'^{\dagger}_r S^{-2} = (1, \psi'^{*}_r(2)/\psi'_0(2), \ldots, \psi'^{*}_r(N)/\psi'_0(N)). \tag{A.5}$$

Hence the inner product $\phi_r^{\dagger} \psi_r$ (no summation over r) is

$$\phi_r^{\dagger} \psi'_r = 1 + \sum_{l=2}^{N} |\psi'_r(l)|^2/\psi'_0(l) \tag{A.6}$$

$$= c_r, \text{ say.}$$

We then define the normalized eigenvector ψ_r by

$$\psi_r = \psi'_r/c_r, \tag{A.7}$$

so that

$$\phi_r^{\dagger} \psi_l = \delta_{r,l}. \tag{A.8}$$

Of course, the normalization constant c_r may be 'apportioned' between $\phi_r^{\dagger}$ and ψ_r in any manner we please, without affecting physical quantities—except that in the case when the eigenvalue $+1$ corresponds to a unique equilibrium distribution, the right eigenvector ψ_0 must correspond to the actual probability distribution P_n in the limit $n \to \infty$, which means that we must have $\Sigma_m \psi_0(m) = 1$. In the problem at hand, this consideration requires us to normalize ψ'_0 and ψ'_{N-1} according to $\psi_0 = \psi'_0/c_0$, $\psi_{N-1} = \psi'_{N-1}/c_{N-1}$. For uniformity, we normalize the rest of the eigenvectors as well in the same manner, i.e., as in eq. (A.7).

Appendix B

Expressions for the eigenvectors

The eigenvalues of M are found to be

$$\lambda_0 = 1, \quad \lambda_{N-1} = -1, \quad \lambda_r = 2(pq)^{1/2} \cos\theta_r, \quad (1 \leqslant r \leqslant N - 2), \tag{B.1}$$

where $\theta_r = r\pi/(N-1)$. It is easy to show that

$$\psi'_0 = (1, 1/q, p/q^2, \ldots, p^{N-3}/q^{N-2}, p^{N-2}/q^{N-2})^T \tag{B.2}$$

and that

$$\psi'_{N-1}(m) = (-1)^{m-1}\psi'_0(m). \tag{B.3}$$

Hence $S_{mk} = [\psi'_0(m)]^{1/2}\delta_{mk}$ leads to the left eigenvectors

$$\phi^\dagger_0 = \psi'_0 S^{-2} = (1, 1, \ldots, 1) \tag{B.4}$$

and

$$\phi^\dagger_{N-1} = (1, -1, 1, \ldots, (-1)^{N-1}). \tag{B.5}$$

Further,

$$\phi^\dagger_0 \psi'_0 = c_0 = \frac{2}{q^{N-2}} \left(\frac{p^{N-1} - q^{N-1}}{p - q} \right) \tag{B.6}$$

and also $c_{N-1} = c_0$. Thus the normalized eigenvectors ψ_0 and ψ_{N-1} are given by

$$\psi_0 = \frac{1}{2}\left(\frac{p-q}{p^{N-1} - q^{N-1}} \right)(q^{N-2}, q^{N-3}, pq^{N-4}, \ldots, p^{N-3}, p^{N-2})^T, \tag{B.7}$$

$$\psi_{N-1}(m) = (-1)^{m-1}\psi_0(m). \tag{B.8}$$

For the rest of the eigenvectors $(r = 1, 2, \ldots, N-2)$, we find

$$\left.\begin{array}{l} \psi'_r(1) = 1 \text{ (by construction)}, \\[2mm] \psi'_r(m) = (\sqrt{p/q})^{m-1}(1/p)[\cos(m-1)\theta_r + (p-q)\cot\theta_r \sin(m-1)\theta_r] \\[2mm] \hspace{6cm} (2 \leqslant m \leqslant N-1), \\[2mm] \psi'_r(N) = (-1)^r(\sqrt{p/q})^{N-3}. \end{array}\right\} \tag{B.9}$$

The corresponding left eigenvectors $\phi^\dagger_r(1 \leqslant r \leqslant N-2)$ therefore have elements

$$\phi^\dagger_r(m) = (\sqrt{q/p})^{m-1}[\cos(m-1)\theta_r + (p-q)\cot\theta_r\sin(m-1)\theta_r],$$

$$\hspace{6cm} (1 \leqslant m \leqslant N). \tag{B.10}$$

In particular,

$$\phi^\dagger_r(1) = 1, \phi^\dagger_r(2) = 2(pq)^{1/2}\cos\theta_r, \phi^\dagger_r(N) = (-1)^r(\sqrt{q/p})^{N-1}. \tag{B.11}$$

Hence, for $1 \leqslant r \leqslant N-2$, the normalization constant $c_r = \phi^\dagger_r\psi'_r$ becomes, after simplification,

$$c_r = (N-1)\left(\frac{1 - 4pq\cos^2\theta_r}{2p\sin^2\theta_r} \right). \tag{B.12}$$

The corresponding normalized right eigenvectors are then given by $\psi_r = \psi'_r/c_r$, as indicated in (A.7).

For ready reference, we write down the expressions to which the results above

reduce in the unbiased case $p = q = 1/2$. We obtain

$$c_0 = c_{N-1} = 2(N-1),$$ (B.13)

$$\psi_0 = \frac{1}{(N-1)} \; (1/2, 1, \ldots, 1, 1/2)^T,$$ (B.14)

$$\psi_{N-1}(m) = (-1)^{m-1} \psi_0(m).$$ (B.15)

The left eigenvectors $\phi_0^\dagger$ and $\phi_{N-1}^\dagger$ remain the same as in the biased case, eqs (B.4) and (B.5). For the rest of the eigenvectors $(1 \leqslant r \leqslant N - 2)$, we get

$$c_r = (N-1),$$ (B.16)

$$\psi_r = \frac{1}{(N-1)} \; (1, 2\cos\theta_r, \ldots, 2\cos(N-2)\theta_r, (-1)^r)^T,$$ (B.17)

$$\phi_r^\dagger = (1, \cos\theta_r, \ldots, \cos(N-2)\theta_r, (-1)^r).$$ (B.18)

Using the foregoing, the explicit solutions for $P_n(m|m_0)$ written down in eqs (21) and (25) follow in a straightforward manner.

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