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A. Raghuram

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This paper is in two parts. In the first we work out the asymptotics of functions in the Kirillov model of an irreducible admissible representation of $GL_2(\mathcal{D})$ for a p-adic division algebra $\mathcal{D}$. In the second part we prove a theorem, for $GL_n(\mathcal{H})$ for a quaternionic p-adic division algebra $\mathcal{H}$, of explicitly realizing the contragredient representation and then derive a consequence of this for distinguished representations for $GL_2(\mathcal{H})$.

1. Introduction.

We now describe the contents of this paper in a little more detail. We begin by fixing some notations. Let F be a non-Archimedean local field of arbitrary characteristic and let $\mathcal{D}$ be a central division algebra over F. Let ψ be a nontrivial additive character of F and let Ψ denote the character of $\mathcal{D}$ obtained by composing the reduced trace map from $\mathcal{D}$ to F with ψ.

We now describe the first part of this paper which is Section 2. Let $G = GL_2(\mathcal{D})$ and let P denote the minimal parabolic subgroup consisting of upper triangular matrices in G. Let N denote the unipotent radical of P and Ψ is also thought of as a character of N. If (π, V) is an irreducible admissible infinite dimensional representation of G then Theorem 3.1 of [**8**] gave a realization of the representation space V canonically as a space of functions $K(\pi)$ on $\mathcal{D}^*$ with values in the twisted Jacquet module $\pi_{N,\Psi}$ of π and on this space of functions the action of P can be described very explicitly. Further $K(\pi)$ contains the space $C_c^\infty(\mathcal{D}^*, \pi_{N,\Psi})$ as a subspace of finite codimension and this codimension is 0 if and only if π is supercuspidal.

Here we consider the case when π is not supercuspidal and ask for the asymptotic behaviour of functions in $K(\pi)$. More precisely, we know from [**8**] that functions in $K(\pi)$ vanish outside compact subsets of $\mathcal{D}$ and so the asymptotics are interesting in a neighbourhood of 0 in $\mathcal{D}$. A very special case of this was worked out in Section 4.2 of [**8**]. There π was a spherical principal series representation; then the asymptotic behaviour of the function corresponding to the spherical vector follows from the formula in Theorem 4.2 of [**8**]. In this paper we take the case of any parabolically induced representation and describe the asymptotics near 0 and this is the content of Theorem 2.1 — the main theorem in Section 2.

The analysis given here is a fairly straightforward generalization of a similar analysis for $GL_2(F)$ as presented in Sections 1.9 and 1.10 of Godement's notes [**2**] although there are some technical complications due to vagaries of division algebras as we will point out in appropriate places. We note that the analysis given here is completely independent of [**8**] and in particular gives another way to get hold of the Kirillov model for a principal series representation of G.

While working on [**8**] we were interested in an explicit duality between the Kirillov models of a representation π and its contragredient $\pi^\vee$ which naturally leads us to the question of realizing the contragredient representation. This is answered in Section 3.

We begin by recalling the theorem of Gelfand and Kazhdan (see Theorem 7.1 of [**1**]) which states that if π is an irreducible admissible representation of $GL_n(F)$ then its contragredient $\pi^\vee$ is equivalent to the representation $g \mapsto \pi({}^tg^{-1})$.

We begin Section 3 by observing that the group $GL_n(\mathcal{D})$ admits an outer automorphism if and only if $\mathcal{D} = F$ or $\mathcal{D} = \mathcal{H}$ where $\mathcal{H}$ is the quaternion division algebra over F. Further in the quaternionic case the outer automorphism is easily described using the canonical involution which any quaternion algebra comes equipped with. This automorphism is denoted $g \mapsto {}^Tg^{-1}$. See Proposition 3.1.

It has been observed by Muić and Savin that if the characteristic of F is 0 then for any irreducible admissible representation π of $GL_n(\mathcal{H})$ the contragredient $\pi^\vee$ is equivalent to the representation $g \mapsto \pi({}^Tg^{-1})$. See [**5**]. We are able to prove this in arbitrary characteristic and this is recorded in Theorem 3.1. The proof is modelled on the proof given in [**1**] for the Gelfand-Kazhdan theorem and basically consists in showing that a distribution on G which is invariant under conjugation is invariant under the map $g \mapsto {}^Tg$. This requires a crucial lemma (see Lemma 3.1) which says that g is always conjugate to ${}^Tg$. Muić and Savin prove this for semi-simple g using Galois cohomology but we give a completely elementary proof valid for any g.

The rest of Section 3 is devoted to giving another proof of a recent theorem of D. Prasad [**7**] on distinguished representations, using this realization of the contragredient representation. This theorem is a division algebra analogue of a theorem due to H. Jacquet and S. Rallis [**4**] which states that if $G = GL_{2n}(F)$ and $M = GL_n(F) \times GL_n(F)$ then for any irreducible admissible representation π of G the space $\mathrm{Hom}_M(\pi, \mathbb{C})$ is at most one dimensional. Further if there is such a nontrivial linear functional (in which case π is said to be M-*distinguished*) then π is self-contragredient. There are two inputs to the proof of this result. The first is the above mentioned theorem of Gelfand and Kazhdan on realizing the contragredient representation and the other is a result (the main theorem in [**4**]) on invariant distributions.

This result on invariant distributions is true in the context of $G = GL_2(\mathcal{D})$ and has been observed by D. Prasad. It is stated and a proof is sketched in [**6**]. There is a mistake in that sketch and so we present this result with complete details in Proposition 3.2.

Since we have a division algebra version of the first of the two inputs to the theorem of Jacquet and Rallis, namely we have been able to realize the contragredient representation for $GL_n(\mathcal{H})$, it should be possible to prove a corresponding theorem on distinguished representations for $GL_2(\mathcal{H})$ along the lines of [**4**]. We record the statement in Theorem 3.2. We would like to point out that the proof on page 67 of [**4**] goes through mutatis mutandis to our case (with some very minor change in notation), given our Theorem 3.1 and Proposition 3.2. As mentioned above this theorem on distinguished representations for $GL_2(\mathcal{D})$ is by D. Prasad [**7**] but the proof is very different than that of Jacquet-Rallis and comes out of various details in the Kirillov theory developed in [**8**].

2. Asymptotics in the Kirillov model.

We need to introduce some more notations. Let $\mathcal{O}_F$ be the ring of integers of F and let $\mathfrak{P}_F$ be its maximal ideal. Let $\mathcal{O}$ denote the ring of integers of $\mathcal{D}$ and let $\mathfrak{P} = \varpi\mathcal{O}$ be the maximal two-sided ideal of $\mathcal{O}$ with ϖ a uniformizer. Let $\mathfrak{v}$ denote the additive valuation such that $\mathfrak{v}(\varpi) = 1$. Let the multiplicative valuation be given by $|X| = q^{-d\mathfrak{v}(X)}$ where d is the reduced degree of $\mathcal{D}$ and q is the cardinality of residue field of F. We assume that ψ is trivial on $\mathcal{O}_F$ and nontrivial on $\mathfrak{P}_F^{-1} = \varpi_F^{-1}\mathcal{O}_F$. Then Ψ is trivial on $\mathfrak{P}^{1-d} = \varpi^{1-d}\mathcal{O}$ and nontrivial on $\mathfrak{P}^{-d} = \varpi^{-d}\mathcal{O}$. We use the notations dX and $d^\times X$ for the Haar measures on the additive group $\mathcal{D}$ and the multiplicative group $\mathcal{D}^*$ respectively, where $d^\times X$ is chosen to be $|X|^{-1}dX$.

Let (π_1, W_1) and (π_2, W_2) be two smooth irreducible representations of $\mathcal{D}^*$. We let $V(\pi_1, \pi_2)$ denote the representation of G obtained by parabolic induction using π_1 and π_2. To be specific $V(\pi_1, \pi_2)$ consists of locally constant, $W_1 \otimes W_2$ valued functions f on G satisfying

$$f\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} g\right) = |AD^{-1}|^{1/2}(\pi_1(A) \otimes \pi_2(D))f(g)$$

for all $g \in G$ and for all $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in P$. We call this representation $V(\pi_1, \pi_2)$ a *principal series representation* irrespective of whether it is irreducible or not. The aim of this section is to develop a Kirillov model for such a principal series representation and in doing so we get hold of the asymptotics of functions in the Kirillov space. See Theorem 2.1.

Let $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Bruhat decomposition says that G is the disjoint union of P and $PwP = PwN$. Since P is not an open subgroup of G we get that every neighbourhood of 1 in G intersects PwN. Hence, any $f \in V(\pi_1, \pi_2)$, being locally constant, is completely determined by its values on PwN. Now by the defining equivariance on the left with respect to P such a function f is determined by the function $X \longmapsto f(w \left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right))$. As an artifice to have some convenient signs we replace w by $w^{-1} = -w$. We therefore get that the function f is completely determined by the function $f' \in C^\infty(\mathcal{D}, \pi_1 \otimes \pi_2)$, which is the space of locally constant functions on $\mathcal{D}$ taking values in $W_1 \otimes W_2$. This function f' is given by:

$$f'(X) = f\left(w^{-1}\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}\right).$$

Using the matrix identity

$$w^{-1}\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -X^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X^{-1} & 0 \\ 0 & X \end{pmatrix}\begin{pmatrix} 1 & 0 \\ X^{-1} & 1 \end{pmatrix}$$

we get

$$f'(X) = |X|^{-1}(\pi_1(X^{-1}) \otimes \pi_2(X)) f\begin{pmatrix} 1 & 0 \\ X^{-1} & 1 \end{pmatrix}.$$

So f' satisfies the property that $|X|(\pi_1(X) \otimes \pi_2(X^{-1}))f'(X)$ is constant for large $|X|$. With this in view we define the following space of functions which we denote by $\mathcal{F}(\pi_1, \pi_2)$:

$$\Big\{\phi \in C^\infty(\mathcal{D}, \pi_1 \otimes \pi_2) \ : \ |X|(\pi_1(X) \otimes \pi_2(X^{-1}))\phi(X) \\ \text{is constant for } \ |X| \gg 1\Big\}.$$

We omit the proof of the following easy lemma.

Lemma 2.1. *The map $f \longmapsto f'$ gives a bijection from $V(\pi_1, \pi_2)$ onto $\mathcal{F}(\pi_1, \pi_2)$.*

On this space $\mathcal{F}(\pi_1, \pi_2)$ we will define a Fourier transform. Then given a function $\phi \in \mathcal{F}(\pi_1, \pi_2)$ twisting its Fourier transform by a certain representation of $\mathcal{D}^*$ we will get a function in the Kirillov space of $V(\pi_1, \pi_2)$. We would like to point out that the following definition is at the moment a purely formal one and it makes sense only after the convergence of the series is proved which is much of the technical content of this section.

Definition 2.1. Let $\phi \in \mathcal{F}(\pi_1, \pi_2)$. Its Fourier transform $\widehat{\phi}$ which is a function on $\mathcal{D}^*$ is defined by

$$\widehat{\phi}(X) := \sum_{n \in \mathbb{Z}} \int_{\mathfrak{v}(Y)=n} \overline{\Psi(XY)}\phi(Y)\, dY.$$

The set of all the fourier transforms is denoted by

$$\widehat{\mathcal{F}}(\pi_1, \pi_2) := \{\widehat{\phi} : \phi \in \mathcal{F}(\pi_1, \pi_2)\}.$$

Definition 2.2.

$$K(\pi_1,\pi_2) := \{|X|^{1/2}(1\otimes\pi_2(X))\xi(X) : \xi \in \widehat{\mathcal{F}}(\pi_1,\pi_2)\}.$$

This space $K(\pi_1,\pi_2)$ will turn out to be a Kirillov model for the representation $V(\pi_1,\pi_2)$. The nontrivial point will be to show the convergence of the series in Definition 2.1. In the course of proving convergence we will also get asymptotics of functions in $K(\pi_1,\pi_2)$. For any subset Ω of $\mathcal{D}$ and any vector $v \in \pi_1\otimes\pi_2$ we denote χ_Ω to be the characteristic function of Ω and $\chi_\Omega\cdot v$ denotes the function taking the value v on Ω and 0 on $\mathcal{D}-\Omega$. The following lemma is easy and the proof is omitted.

Lemma 2.2. *For $v\in\pi_1\otimes\pi_2$ let ϕ_v be the function in $\mathcal{F}(\pi_1,\pi_2)$ given by $\phi_v(X) = |X|^{-1}(\pi_1(X^{-1})\otimes\pi_2(X))v$ if $|X|\geq 1$ and is zero if $|X|<1$. Let*

$$\mathcal{F}_0(\pi_1,\pi_2) = \{\chi_{\mathcal{O}}\cdot v : v\in\pi_1\otimes\pi_2\}$$

and

$$\mathcal{F}_\infty(\pi_1,\pi_2) = \{\phi_v : v\in\pi_1\otimes\pi_2\}.$$

Then the space $\mathcal{F}(\pi_1,\pi_2)$ can be split up as

$$\mathcal{F}(\pi_1,\pi_2) = C_c^\infty(\mathcal{D}^*,\pi_1\otimes\pi_2)\oplus\mathcal{F}_0(\pi_1,\pi_2)\oplus\mathcal{F}_\infty(\pi_1,\pi_2).$$

Basically the space $\mathcal{F}$ is decomposed into the direct sum of three vector spaces depending on the behaviour at 0 and at ∞. The convergence and the actual value of the Fourier transform on functions in two of these spaces, namely in C_0^∞ and $\mathcal{F}_0$ are easy to describe and this is the content of Lemmas 2.3 and 2.4 respectively. Convergence of the Fourier transform of functions in $\mathcal{F}_\infty$ is much more difficult to prove. We return to this point after disposing off the above mentioned easy cases.

Lemma 2.3. *Let $\phi\in C_c^\infty(\mathcal{D}^*,\pi_1\otimes\pi_2)$. Then the series in Definition 2.1 is actually a finite sum and hence is convergent. The function $\widehat{\phi}$ is a locally constant function on $\mathcal{D}^*$ which vanishes outside compact subsets of $\mathcal{D}$ and is a constant in a neighbourhood of the origin.*

Proof. Let $A\in\mathcal{D}^*$, $n\geq 1$ and $v\in\pi_1\otimes\pi_2$. To this is associated the function $\phi(A,n,v)$ which takes the constant value v on $A(1+\mathfrak{P}^n)$ and is zero outside this set. It is clear that $C_c^\infty(\mathcal{D}^*)$ is spanned by such functions. It is an easy computation which yields that $\widehat{\phi(A,n,v)}(X)$ is $c\overline{\Psi(XA)}v$ if $X\in\mathfrak{P}^{-n-\mathfrak{v}(A)+1-d}$ for some constant c and is zero outside $\mathfrak{P}^{-n-\mathfrak{v}(A)+1-d}$. □

Lemma 2.4. *Let $\phi=\chi_{\mathcal{O}}\cdot v\in\mathcal{F}_0(\pi_1,\pi_2)$. Then $\widehat{\phi}(X) = c\chi_{\mathfrak{P}^{1-d}}(X)v$ for some constant c. Hence the function $\widehat{\phi}$ is a locally constant function on $\mathcal{D}^*$ which vanishes outside a compact subset of $\mathcal{D}$ and is a constant in a neighbourhood of the origin.*

Proof. Obvious. □

Now we go into the proof of convergence of Fourier transform of functions in $\mathcal{F}_\infty$. We begin with a lemma which rephrases this convergence problem into a convergence problem for an operator valued (in fact $\mathrm{End}(\pi_1 \otimes \pi_2)$ valued) series which we denote by $A(X)$. This $A(X)$ is now independent of the function in $\mathcal{F}_\infty$. We would like to point out here that each summand of $A(X)$ is a certain kind of nonabelian Gaussian sum.

Lemma 2.5. *Let*

$$A(X) = \sum_{m \leq \mathfrak{v}(X)} \int_{\mathfrak{v}(T)=m} \overline{\Psi(T)}(\pi_1(T^{-1}) \otimes \pi_2(T))\ d^\times T.$$

Let $\phi = \phi_v \in \mathcal{F}_\infty(\pi_1, \pi_2)$. Then the series defining $\widehat{\phi_v}(X)$ converges if and only if the series defining $A(X)$ converges and in this case we have

$$\widehat{\phi_v}(X) = (1 \otimes \pi_2(X^{-1})) \cdot A(X) \cdot (\pi_1(X) \otimes 1)v.$$

Proof. Note that

$$\widehat{\phi_v}(X) = \sum_{m \leq 0} \int_{\mathfrak{v}(Y)=m} |Y|^{-1} \overline{\Psi(XY)}(\pi_1(Y^{-1}) \otimes \pi_2(Y))v\ dY.$$

In the above integral, notice that $|Y|^{-1}\ dY = d^\times Y$ and by putting $XY = T$ we get

$$\begin{aligned} \widehat{\phi_v}(X) &= \sum_{m \leq \mathfrak{v}(X)} \int_{\mathfrak{v}(T)=m} \overline{\Psi(T)}(\pi_1(T^{-1}X) \otimes \pi_2(X^{-1}T))v\ d^\times T \\ &= (1 \otimes \pi_2(X^{-1})) \cdot A(X) \cdot (\pi_1(X) \otimes 1)v. \end{aligned}$$

□

Now the main point is the convergence (and then getting the asymptotics) of the 'function' $A(X)$. This is the content of Lemma 2.6 and Corollary 2.1.

Lemma 2.6. *The defining series for $A(X)$ (see Lemma 2.5) is a finite series and hence is convergent. Further, it vanishes outside a compact subset of $\mathcal{D}$.*

Proof. Recall that for an irreducible representation σ of $\mathcal{D}^*$, the *level* of σ is the least nonnegative integer m such that σ is trivial on $U(m) = 1 + \mathfrak{P}^m$. (By convention, $U(0) = U = \mathcal{O}^\times$ the group of units in $\mathcal{O}$.) Let ℓ_i be the level of π_i and let $\ell = 1 + \max\{\ell_1, \ell_2\}$.

In the defining formula for $A(X)$ (see Lemma 2.5) use the substitution $T = \varpi^m u$ to get

$$A(X) = \sum_{m \leq \mathfrak{v}(X)} \int_U \overline{\psi(\varpi^m u)}(\pi_1(u^{-1}\varpi^{-m}) \otimes \pi_2(\varpi^m u))d^\times u$$

which can be rewritten as:

$$\sum_{m\leq \mathfrak{v}(X)} (1\otimes \pi_2(\varpi^m)) \left(\int_U \overline{\psi(\varpi^m u)}(\pi_1(u^{-1})\otimes \pi_2(u))d^\times u\right)(\pi_1(\varpi^{-m})\otimes 1).$$

For brevity we denote the inner integral in the above formula by I_m. We claim that I_m vanishes for all $m < -\ell + 1 - d$. Clearly this claim will prove the lemma. Note that

$$\begin{aligned} I_m &= \sum_{a\in U/U(\ell)} \int_{b\in U(\ell)} \overline{\Psi(\varpi^m ab)}(\pi_1(b^{-1}a^{-1})\otimes \pi_2(ab))\ d^\times b \\ &= \sum_{a\in U/U(\ell)} \left(\int_{b\in U(\ell)} \overline{\Psi(\varpi^m ab)}\ d^\times b\right)(\pi_1(a^{-1})\otimes \pi_2(a)). \end{aligned}$$

The inner integral in the right-hand side vanishes for $m < -\ell + 1 - d$. This can be seen by going to $\mathfrak{P}^\ell$ via the substitution $b = 1 + \beta$ and noting that $\beta \mapsto \overline{\Psi(\varpi^m a\beta)}$ is a nontrivial character (since $m < -\ell + 1 - d$) on the compact group $\mathfrak{P}^\ell$.

So the summation in the formula for $A(X)$ runs between $-\ell + 1 - d$ and $\mathfrak{v}(X)$ which implies that if $X \notin \mathfrak{P}^{-\ell+1-d}$ then $A(X) = 0$, i.e., $A(X)$ vanishes outside a compact subset of $\mathcal{D}$. □

Corollary 2.1. *For any function $\phi_v \in \mathcal{F}_\infty(\pi_1, \pi_2)$ the series defining $\widehat{\phi_v}$ is finite and hence converges. Further, $\widehat{\phi_v}$ is a locally constant function on $\mathcal{D}^*$ which vanishes outside a compact subset of $\mathcal{D}$.*

Proof. Use Lemmas 2.5 and 2.6. □

Remark 2.1. We would like to point out that the integral I_m in Lemma 2.6 is a certain kind of nonabelian Gaussian sum. The fact that it vanishes for all m less than some number is a partial analogue of Equation 22 in [**2**]. We can prove (although this is not required for the sequel) that if π_1 and π_2 have distinct levels then the integral I_m vanishes for all $m \neq -\max\{\ell_1, \ell_2\} + 1 - d$.

We now state and prove the main theorem in this section which gives a Kirillov model for representations $V(\pi_1, \pi_2)$ and also gives asymptotics for the functions in the corresponding Kirillov space $K(\pi_1, \pi_2)$. Note that the asymptotics given below is a direct generalization of the table on page 1.36 of [**2**]. It might seem that our theorem below, on specializing to the case $\mathcal{D} = F$ is weaker than that given in [**2**]. This is because some of the features of the formulae there are absorbed into our function $A(X)$. Indeed, for $\mathcal{D} = F$ one can say more precisely what $A(X)$ looks like using information on nonvanishing of (abelian) Gaussian sums and our theorem can be rephrased exactly as the above mentioned table in [**2**].

Theorem 2.1. *Let π_1 and π_2 be two smooth irreducible representations of $\mathcal{D}^*$. For each $f \in V(\pi_1, \pi_2)$ let $\xi_f \in C^\infty(\mathcal{D}^*, \pi_1 \otimes \pi_2)$ be given by*

$$\xi_f(X) = |X|^{1/2}(1 \otimes \pi_2(X))\widehat{f'}(X).$$

Then:

1. *For all $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in P$ and for all $X \in \mathcal{D}^*$ we have*
$$\xi_{\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) f}(X) = \Psi(D^{-1}XB)(\pi_1(D) \otimes \pi_2(D))\xi_f(D^{-1}XA).$$
2. *There exists a function $X \mapsto A(X)$ in $C^\infty(\mathcal{D}^*, \mathrm{End}(\pi_1 \otimes \pi_2))$ such that given any $f \in V(\pi_1, \pi_2)$ there exists vectors α and β (depending on f) in $W_1 \otimes W_2$ such that in some neighbourhood of* 0 *we have*
$$\xi_f(X) = |X|^{1/2}(1 \otimes \pi_2(X))\alpha + |X|^{1/2}A(X)(\pi_1(X) \otimes 1)\beta.$$

Proof. The proof of (1) is an easy computation and we give a sketch of it below. Using the definition we get $\xi_{\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) f}(X)$ is equal to

$$|X|^{1/2}(1 \otimes \pi_2(X)) \sum_{n \in \mathbb{Z}} \int_{\mathfrak{v}(Y)=n} \overline{\Psi(XY)} \cdot \left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} f\right)\left(w^{-1}\begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}\right) dY.$$

Simplifying the above integral and making the substitution $Z = A^{-1}(B + YD)$ we get

$$\Psi(D^{-1}XB)|D^{-1}XA|^{1/2}(\pi_1(D) \otimes \pi_2(XA)) \cdot \sum_{n \in \mathbb{Z}} \int_{\mathfrak{v}(Z)=n} \overline{\Psi(D^{-1}XAZ)} f'(Z)\, dZ$$

and this expression simplifies to the right-hand side of the equation in (1).

For the proof of (2), given $f \in V(\pi_1, \pi_2)$ if f' which is in $\mathcal{F}(\pi_1, \pi_2)$ is actually in $C_c^\infty(\mathcal{D}^*, \pi_1 \otimes \pi_2) \oplus \mathcal{F}_0(\pi_1, \pi_2)$ then Lemmas 2.3, 2.4 show that in a neighbourhood of 0 we have $\xi_f(X) = |X|^{1/2}(1 \otimes \pi_2(X))\alpha$ for some α depending on f. Similarly, if $f' \in \mathcal{F}_\infty(\pi_1, \pi_2)$ then Lemmas 2.5, 2.6 and Corollary 2.1 show that in a neighbourhood of 0 we have $\xi_f(X) = |X|^{1/2}A(X)(\pi_1(X) \otimes 1)\beta$ for some β depending on f. The general case follows using Lemma 2.2. □

Remark 2.2. Theorem 2.1 of [**8**] says that $\pi_1 \otimes \pi_2$ is the twisted Jacquet module of the principal series representation $V(\pi_1, \pi_2)$. Hence statement (1) in the theorem above gives the action of P on the 'Kirillov space' $K(\pi_1, \pi_2)$ exactly as in Lemma 3.1 of [**8**].

Remark 2.3. We conclude this section by pointing out that the above analysis is closely related to reduciblity of such principal series representations. The relation being that reduciblity boils down to understanding the kernel of the map $f \mapsto \xi_f$ which then would require a finer analysis (than given here) of the map $X \mapsto A(X)$. Indeed, for $GL_2(F)$, understanding reduciblity was the reason why this analysis was given in the first place. However, for $GL_2(\mathcal{D})$, reduciblity of principal series representations has been answered by Tadic [**9**].

3. The contragredient representation.

In this section we consider the question of realizing the contragredient representation analogous to the Gelfand-Kazhdan theorem for $GL_n(F)$. We begin with a proposition which says that this is possible for $GL_n(\mathcal{D})$ if and only if $\mathcal{D}$ is the quaternion division algebra over F and then use this to prove the main theorem of this section.

Proposition 3.1. *Let $G = GL_n(\mathcal{D})$ where $\mathcal{D}$ is a division algebra of index d over F. Assume that $d > 1$.*

1. *If $d > 2$ then any automorphism of G is inner.*
2. *If $d = 2$, i.e., if $\mathcal{D} = \mathcal{H}$ the quaternionic division algebra, then there is an outer automorphism given by $g \mapsto {}^T g^{-1}$ where ${}^T g(i,j) = \overline{g(j,i)}$ and $\overline{x} = T_{\mathcal{H}/F}(x) - x$ is the canonical involution on $\mathcal{H}$.*

Proof. Let M denote $M(n,\mathcal{D})$. Fix an algebraic closure $\overline{F}$ of F. Let $\overline{M} = M \otimes_F \overline{F} \simeq M(nd, \overline{F})$ and let $\overline{G} = \overline{M}^\times = GL(nd, \overline{F})$. If f is any map from G to G or M to M let $\overline{f}$ denote the map obtained on going to the closure.

Let f be an automorphism of G and suppose $\overline{f}$ is an inner automorphism of $\overline{G}$ then $\overline{f}$ extends to an algebra theoretic automorphism of $\overline{M}$ and a fortiori f extends to an algebra automorphism of M. The Skolem-Noether theorem says that f is inner.

Now suppose $\overline{f}$ is not inner (then necessarily f is not inner) then since $\overline{G}$ has, up to inner automorphisms, only one outer automorphism we have that there exists an $\alpha \in \overline{G}$ such that for all $\beta \in \overline{G}$, $\overline{f}(\beta) = \alpha({}^t\beta^{-1})\alpha^{-1}$. Consider now the map f_1 from G to G given by $f_1(g) = f(g^{-1})$. It is an anti-automorphism of G and going to the closure gives

$$\overline{f_1}(g \otimes 1) = f_1(g) \otimes 1 = f(g^{-1}) \otimes 1 = \overline{f}(g^{-1} \otimes 1) = \alpha({}^t(g \otimes 1))\alpha^{-1}.$$

Therefore $\overline{f_1}$ extends to an algebra theoretic anti-automorphism of $\overline{M}$ namely $\beta \mapsto \alpha({}^t\beta)\alpha^{-1}$. As above, a fortiori, f_1 extends to an algebra theoretic anti-automorphism of M. This implies that $M \simeq M^{op} \simeq M(n, D^{op})$ which implies that $\mathcal{D} \simeq \mathcal{D}^{op}$. (Here $\mathcal{D}^{op}$ stands for the opposite division algebra of $\mathcal{D}$.) Hence $\mathcal{D} = \mathcal{H}$ (since we assumed that $d > 1$) where $\mathcal{H}$ is (the) quaternionic division algebra since there is a unique nontrivial element of order two in the Brauer group of a local field. Finally, it is easy to see that for $G = GL_n(\mathcal{H})$ the map $g \mapsto {}^T g^{-1}$ is not inner. □

Theorem 3.1. *Let $\mathcal{H}$ be the quaternionic division algebra over F. Let $x \mapsto \overline{x} = T_{\mathcal{H}/F}(x) - x$ be the canonical involution on $\mathcal{H}$. Let $G = GL_n(\mathcal{H})$. Let w be the element in G given by $w(i,j) = (-1)^i \delta_{i,n-j+1}$. Let ${}^T g = {}^t\overline{g}$ for g in G, i.e., ${}^T g(i,j) = \overline{g(j,i)}$. Let $\sigma : G \to G$ be given by $\sigma(g) = w \cdot {}^T g^{-1} \cdot w^{-1}$. Let π be an irreducible admissible representation of G. Let ${}^\sigma\pi$ be the representation defined as ${}^\sigma\pi(g) = \pi(\sigma(g))$. Then ${}^\sigma\pi$ is equivalent to $\pi^\vee$ the contragredient representation of π.*

Remark 3.1. Note that for the purpose of realizing the contragredient representation we can work as well with $g \mapsto {}^T g^{-1}$. We specifically consider the automorphism σ because conjugation by w is to ensure, as can be easily seen, that σ preserves the standard minimal parabolic P consisting of upper triangular matrices and its unipotent radical N. We let Ψ also denote the character of N given by sending $u \in N$ to $\Psi(u_{1,2} + \cdots + u_{n-1,n})$. Then σ takes Ψ to $\overline{\Psi}$. Let M be the Levi subgroup of P consisting of diagonal matrices. So M is a product of n copies of $\mathcal{H}^*$. Let $\Delta\mathcal{H}^*$ denote the subgroup $\mathcal{H}^*$ sitting diagonally in M. Then $\Delta\mathcal{H}^*$ leaves the character Ψ invariant.

Recall that the *twisted Jacquet module* of π, denoted $\pi_{N,\Psi}$, is the maximal quotient of π on which N acts via Ψ and is given by $\pi_{N,\Psi} = \pi/\pi(N,\Psi)$ where $\pi(N,\Psi)$ is the span of all vectors of the form $\pi(n)v - \Psi(n)v$ with $v \in V$ and $n \in N$. The theorem gives an identification of $\pi(N,\Psi)$ with $\pi^\vee(N,\overline{\Psi})$ as subspaces of π. Hence the twisted Jacquet modules $\pi_{N,\Psi}$ and $(\pi^\vee)_{N,\overline{\Psi}}$ are canonically isomorphic as vector spaces.

Further as $\Delta\mathcal{H}^*$ modules one differs from the other by the canonical involution on $\mathcal{H}$. So by the theorem for $n = 1$ we get one is the dual of the other, i.e., $(\pi_{N,\Psi})^\vee \simeq (\pi^\vee)_{N,\overline{\Psi}}$ as $\Delta\mathcal{H}^*$ modules. (Although the theorem is only for irreducible representations, it applies to any representation of $\mathcal{H}^*$ because $\mathcal{H}^*$ is compact mod center and so any representation with a central character is completely reducible.)

This fundamental fact that the twisted Jacquet functor 'commutes' with the functor of taking contragredients (which is obvious for quasi-split groups by Shalika's multiplicity one for Whittaker models) was observed for $GL_2(\mathcal{D})$ in [**8**] and the above remark proves this for $GL_n(\mathcal{H})$. To the author's knowledge it is still not known for $GL_n(\mathcal{D})$ in general. Finally, the fact that the contragredient and the twisted Jacquet functor commute, can be used to give an explicit duality between π and $\pi^\vee$ in terms of their Kirillov models exactly as in Corollary 3.1 of [**8**].

Lemma 3.1. *With the hypothesis as in Theorem* 3.1*, for all $g \in G$, g and ${}^T g$ are conjugate in G.*

Proof. Let $\overline{F}$ be an algebraic closure of F. It is easy to see that g and ${}^T g$ are conjugate after going to $\overline{F}$. This is because in $M_n(\overline{F})$ any anti-involution is conjugate to transpose and for any matrix A in $M_n(\overline{F})$, A and its usual transpose are conjugate (consider the Jordan canonical form). Now the point is to do this rationally. The argument given below is not very different from a Galois cohomology argument.

Consider the space $\Omega = \{u \in M_n(\mathcal{H}) : {}^T g \cdot u = u \cdot g\}$. Then Ω is a vector space over F and can be thought of as an affine subvariety of $\mathbb{A}_F^{4n^2}$. We would like to say that Ω contains invertible elements of $M_n(\mathcal{H})$. Recall that $u \in M_n(\mathcal{H})$ is invertible if and only if the reduced norm (from $M_n(\mathcal{H})$ to F) of u is nonzero.

Let $f : \Omega \to F$ denote the restriction to Ω of the reduced norm map from $M_n(\mathcal{H})$ to F. Then f is a polynomial map on Ω with F coefficients. Since F is an infinite field it suffices to show that f is not identically zero on Ω.

The fact that g and ${}^T g$ are conjugate over the closure means that the space $\Omega_{\overline{F}} = \Omega \otimes_F \overline{F}$ contains invertible elements. Hence there are elements in $\Omega_{\overline{F}}$ on which f does not vanish which implies that it can not be identically zero on Ω. □

Proof of Theorem 3.1. Once we have Lemma 3.1 the proof of the theorem is a completely routine argument following Bernshtein-Zelevinskii [**1**]. We merely sketch the easy parts of the proof.

Clearly ${}^\sigma\pi$ and $\pi^\vee$ are both irreducible and admissible representations of G. Hence ${}^\sigma\pi \simeq \pi^\vee$ if and only if $\Theta_{{}^\sigma\pi} = \Theta_{\pi^\vee}$, i.e., when their characters are equal as distributions on G. For any f in $C_c^\infty(G)$ and for any x in G let f^*, f^T and f^x be defined as $f^*(g) = f(g^{-1})$, $f^T(g) = f({}^T g)$ and $f^x(g) = f(xgx^{-1})$ respectively. It is easy to see that $\pi^\vee(f)$ and $\pi(f^*)$ are adjoints of each other with respect to the canonical duality $\pi^\vee \times \pi \to \mathbb{C}$ and hence have same trace, i.e., $\Theta_{\pi^\vee}(f) = \Theta_\pi(f^*)$. Also $\Theta_{{}^\sigma\pi}(f) = \Theta_\pi(f^\sigma) = \Theta_\pi(((f^T)^*)^w)$. Since G is unimodular Θ_π is conjugation invariant hence $\Theta_{{}^\sigma\pi}(f) = \Theta_\pi((f^T)^*)$ which is by the earlier remark $\Theta_{\pi^\vee}(f^T)$. All this gives us that, for any $f \in C_c^\infty(G)$, $\Theta_{{}^\sigma\pi}(f) = \Theta_{\pi^\vee}(f)$ if and only if $\Theta_{\pi^\vee}(f) = \Theta_{\pi^\vee}(f^T)$. Therefore it is enough to prove that a conjugation invariant distribution on G is invariant under $g \mapsto {}^T g$. Now such a statement on distributions follows from Lemma 3.1 and Theorems 6.13 and 6.15 of [**1**]. □

The rest of this section is devoted, as mentioned in the introduction, to point out another proof of a theorem due to D. Prasad on distinguished representations of $GL_2(\mathcal{H})$. This requires a result on invariant distributions observed by D. Prasad in [**6**]. A proof has been sketched there which is correct in principle although it has a little snag. We begin by giving complete details of the proof of this result.

Proposition 3.2. *Let $G = GL_2(\mathcal{D})$ and let $M = \mathcal{D}^* \times \mathcal{D}^*$ be the diagonal subgroup of G. Then any bi-M-invariant distribution on G is also invariant under the involution $g \mapsto \tau(g) = g^{-1}$.*

Proof. For any locally compact totally disconnected topological space X we let $C_c^\infty(X)$ denote the space of locally constant compactly supported $\mathbb{C}$-valued functions on X. We let $D(X) = \mathrm{Hom}_{\mathbb{C}}(C_c^\infty(X), \mathbb{C})$ denote the space of distributions on X.

Let P denote the minimal parabolic subgroup of upper triangular matrices and let w denote the Weyl group element $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Further let C denote the closed subset of G given by:

$$C = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : ad = 0 \right\} = Pw \cup wP.$$

(This C was incorrectly chosen in [**6**].)

It is trivial to see that C and so also $G - C$ are both union of (M, M) double cosets. Further, some easy matrix manipulations show that every (M, M) double coset in $G - C$ is also stable under τ.

Let $T \in D(G)$ be a bi-M-invariant distribution which is also skew τ invariant. We need to show that such a T is 0. Consider the exact sequence

$$0 \to D(C) \to D(G) \to D(G - C) \to 0.$$

The above remarks on (M, M) double cosets in $G - C$ show that the image of T in $D(G-C)$ is 0. Hence T comes from $D(C)$. Since τ does not stabilize the double cosets in C one has to argue in a different way.

Note that $C = Pw \cup wP = MNw \cup MwN$. So $M \backslash C$ can be identified with $Nw \cup wN$. Let Y denote $\mathcal{D} \times 0 \cup 0 \times \mathcal{D} \subset \mathcal{D} \times \mathcal{D}$. Then $M \backslash C$ may be identified with Y in the obvious way. The M action (on the right) on C and so also the action of τ may be transferred to Y which gives the formulae:

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} (a, 0) = (yax^{-1}, 0), \qquad \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} (0, b) = (0, xby^{-1})$$

and

$$\tau(a, 0) = (0, -a), \qquad \tau(0, b) = (-b, 0).$$

Now it remains to show that, with these actions, any M invariant distribution T on Y which is also skew τ invariant is 0. Let $U = \mathcal{D}^* \times 0 \cup 0 \times \mathcal{D}^*$ which is an open subset of Y. Consider the exact sequence:

$$0 \to D(0 \times 0) \to D(Y) \to D(U) \to 0.$$

The main point is to show that the image of T in $D(U)$ is 0 since then T would be up to a constant, evaluating functions at 0×0. But then skew τ invariance will give that this constant is 0 and we would be done.

Using the action of M on Y and the definition of a Haar measure on $\mathcal{D}^*$, denoted $d^\times y$, we get that there two constants a and b such that

$$\langle T, g \rangle = a \int_{\mathcal{D}^*} g(x, 0) d^\times x + b \int_{\mathcal{D}^*} g(0, y) d^\times y$$

for all $g \in C_c^\infty(U) \subset C_c^\infty(Y)$. It is easy to see by evaluating T on some suitably chosen functions g gives that both a and b are 0 and hence $T = 0$. $\square$

As pointed out in the introduction, this above result on invariant distributions, along with Theorem 3.1 on realizing the contragredient representation, gives a proof of the following theorem on distinguished representations, exactly along the lines of Jacquet and Rallis [**4**]. This theorem is true in the context of $GL_2(\mathcal{D})$ and this has been proved by D. Prasad [**7**] using quite a few technical details coming from Kirillov theory as developed in [**8**].

Theorem 3.2. *Let $G = GL_2(\mathcal{H})$ and let $M = \mathcal{H}^* \times \mathcal{H}^*$ be the diagonal subgroup of G. Let π be an irreducible admissible representation of G. Then:*

1. $\dim_{\mathbb{C}} \mathrm{Hom}_M(\pi, \mathbb{C}) \leq 1$.
2. *If* $\dim_{\mathbb{C}} \mathrm{Hom}_M(\pi, \mathbb{C}) = 1$, *i.e., if π is M distinguished then π is equivalent to its contragredient representation $\pi^\vee$.*

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
100 ST. GEORGE STREET
TORONTO, ON M5S 3G3
CANADA
E-mail address: raghuram@math.toronto.edu

SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
DR. HOMI BHABHA ROAD
COLABA, MUMBAI - 400005
INDIA