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ON THE CORRESPONDENCE OF REPRESENTATIONS BETWEEN $GL(n)$ AND DIVISION ALGEBRAS

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Abstract. For a division algebra D over a p-adic field F, we prove that depth is preserved under the correspondence of discrete series representations of $GL_n(F)$ and irreducible representations of D^* by proving that an explicit relation holds between depth and conductor for all such representations. We also show that this relation holds for many (possibly all) discrete series representations of $GL_2(D)$.

1. Introduction

Let F be a non-Archimedean local field. Let G be the group of F-points of a connected reductive algebraic group defined over F. Let (π, V) be an irreducible admissible representation of G on a complex vector space V. To such a π, Moy and Prasad ([12],[13]) have attached a rational number $\rho(\pi)$ called the *depth* of π (see §2.2). The theory of (unrefined) minimal K-types for π depends crucially on $\rho(\pi)$. In addition, the notion of depth is important in harmonic analysis on G.

For the moment let G be the group of units of a central simple algebra over F, i.e., $G = GL_m(D)$ for a division algebra D over F. For any irreducible representation π of G, Godement and Jacquet [6] associate a local L-function and ε-factor to π. From this ε-factor comes an integer invariant $c(\pi)$ called the *conductor* of π. The main result of this note is that a certain relation holds between depth $\rho(\pi)$ and conductor $c(\pi)$ if $G = D^*$ or if $G = GL_n(F)$ and π is essentially square integrable (see Theorem 3.1), namely

$$\rho(\pi) = \max\left\{\frac{c(\pi) - n}{n}, 0\right\}.$$

We also prove this for many representations π of $G = GL_2(D)$ (see §4). We expect this relation to be true for any essentially square integrable representation of any $GL_m(D)$. It is however easy to see via examples that one cannot expect any explicit relation between depth and conductor to hold for *every* representation of G.

Our motivation for proving such a result relating the depth and conductor is as follows. Very generally speaking one may ask: "How functorial is depth?" More specifically, given two such groups G_1 and G_2, supposing there is a map at the level of dual groups ${}^LG_1 \to {}^LG_2$, the Langlands principle of functoriality predicts the existence of a map from the representations of G_1 to those of G_2. The first question

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is how does depth behave under such a map? A deeper (no pun intended) question is how do the K-types behave under such a map? This question is not new and has been investigated under some instances of functoriality; see for example [3], [4].

One such instance is the correspondence of representations between G' and $GL_n(F)$ where G' is any inner form of the latter group. Such a correspondence is known to exist by the work of Deligne, Kazhdan and Vignèras [5] and also Rogawski [16]. (We need to assume here that the characteristic of F is 0.) We shall henceforth abbreviate it as the DKVR correspondence and we now describe it in a little more detail.

Let D be a central division algebra over F of index n, i.e., $\dim_F(D) = n^2$. Let $G' = GL_m(D)$ and $G = GL_{mn}(F)$. The DKVR correspondence asserts that there is a unique bijection $\pi' \leftrightarrow \pi$ between essentially square integrable representations π' of G' and similar representations π of G which is characterized by a certain character identity (see §2.1). Under this correspondence the conductor is preserved. The main theorem of this paper therefore implies that $\rho(\pi) = \rho(\pi')$ if $m = 1$ (see Theorem 3.1), i.e., in the case when $G' = D^*$ and $G = GL_n(F)$. This equality is also shown to hold for a large class of representations if $m = 2$.

In the case of D^*, the relation between depth and conductor is proved using a result of Koch and Zink [10]. For $GL_n(F)$ the relation follows in the supercuspidal case from theorems of Bushnell and Fröhlich [2] and Bushnell [1] which imply that the normalized level $l(\pi)$ of π is equal to the above function of $c(\pi)$. This result was communicated to us by J.-K. Yu in the form

$$\rho(\pi) = \frac{\operatorname{swan}(\phi(\pi))}{n}$$

where $\operatorname{swan}(\phi(\pi))$ is the Swan conductor of the Langlands parameter $\phi(\pi)$ associated to π. If $\operatorname{artin}(\phi(\pi))$ denotes the Artin conductor, then since $\phi(\pi)$ is irreducible (π being supercuspidal) we have $\operatorname{swan}(\phi(\pi)) = \operatorname{artin}(\phi(\pi)) - n$. The desired relation between conductor and depth then follows from the fact that the Langlands correspondence preserves conductor, i.e., $c(\pi) = \operatorname{artin}(\phi(\pi))$. We generalize this relation to the essentially square integrable case. The proofs of these results are given in §3.

In §4, the result is proved for $G' = GL_2(D)$ using a computation involving the Bruhat-Tits building and results of a previous work of the second author and Dipendra Prasad [14] where many (possibly all) supercuspidal representations of G' were constructed. We prove that depth is preserved when π' is one of these supercuspidals or if π' is an essentially square integrable non-supercuspidal representation of G'.

It should be noted that the results of Bushnell and Fröhlich ([1] and [2]) can be shown to imply that $l(\pi) = \max\{(c(\pi) - n)/n, 0\}$ for supercuspidal π for many groups of the form $GL_m(D)$, including all those in the preceding paragraph. We would like to point out that our arguments for the case of D^* and in the last section dealing with $GL_2(D)$ are completely independent of [1] and [2].

2. Preliminaries

2.1. **DKVR correspondence.** Let F be a non-Archimedean local field of characteristic 0. Let the ring of integers of F be $\mathcal{O}_F$. Let $\mathfrak{P}_F$ be the unique maximal ideal in $\mathcal{O}_F$. Let q be the cardinality of the residue field k_F of F. Let D be a central division algebra over F of index n, i.e., $\dim_F(D) = n^2$. Let $G' = GL_m(D)$ and let $G = GL_{mn}(F)$.

An irreducible representation of G or G' is said to be essentially square integrable if one (and hence every) matrix coefficient is up to a twist square integrable modulo the center.

The *DKVR correspondence* asserts that there is a unique bijection $\pi' \leftrightarrow \pi$ between essentially square integrable representations π' of G' and essentially square integrable representations π of G such that for all regular elliptic γ,

$$(-1)^m \chi_{\pi'}(\gamma) = (-1)^{mn} \chi_\pi(\gamma)$$

where $\chi_{\pi'}$ and χ_π are the characters of π' and π respectively. This bijection satisfies a number of properties (see pp. 34-35 of [5]). For us the most important one is the equality of epsilon factors

$$(-1)^m \varepsilon(s, \pi', \psi) = (-1)^{mn} \varepsilon(s, \pi, \psi)$$

where ψ is a non-trivial additive character of F normalized such that the largest fractional ideal of F on which ψ is trivial is $\mathcal{O}_F$. These epsilon factors have the form $\varepsilon(s, \Pi, \psi) = Aq^{-c(\Pi)s}$ for a non-negative integer $c(\Pi)$ which is called the *conductor* of Π. In particular we get

$$c(\pi') = c(\pi).$$

Our proof that depths match now rests on the observation that for both π and π' the depth bears the same formal relation to the conductor.

2.2. **Depth.** In this section we review the definition of depth of a representation. Just for this section, let G be the F-points of a connected reductive algebraic group defined over F. Let $\mathcal{B} = \mathcal{B}(G)$ be the Bruhat-Tits building attached to G. For any $x \in \mathcal{B}$ let G_x be the parahoric subgroup of G associated to x. Moy and Prasad have defined a decreasing filtration $\{G_{x,r}\}$ of G_x indexed by the non-negative real number r. (See §3 of [13].) Let $G_{x,r^+} := \bigcup_{s>r} G_{x,s}$. The *depth* $\rho(\pi)$ of an irreducible admissible representation (π, V) of G is the smallest non-negative number such that the space $V^{G_{x,r^+}}$ is non-trivial for some $x \in \mathcal{B}$ (see [12]).

We will need the following result of Moy and Prasad which says that depth is invariant under parabolic induction (see Theorem 5.2 in [13]).

Proposition 2.1. *Let P be a parabolic subgroup of G with Levi decomposition $P = MN$. Let τ be an irreducible admissible representation of M. Let π be an irreducible constituent of the parabolic induction $\mathrm{Ind}_P^G(\tau)$ of τ to a representation of G. Then*

$$\rho(\pi) = \rho(\tau).$$

For irreducible supercuspidal representations of $GL_n(F)$ we need the following result of Bushnell and Fröhlich (see Theorem 3.3.8 of [2], Theorem 3 and especially paragraph (5.1) of [1]). We note that the terminology of [1] is different and the result was communicated to us in the form below by J.-K. Yu, using the Swan Conductor of the Langlands parameter associated to π. We prefer to write it as stated below since it is in this form that the formula admits generalizations to other representations and related groups.

Proposition 2.2. *Let π be an irreducible supercuspidal representation of $GL_n(F)$. The depth $\rho(\pi)$ of π is related to the conductor of π by the formula*

$$\rho(\pi) = \frac{c(\pi) - n}{n}.$$

Remark 2.3. The purpose of this note is to show that this formula continues to hold with a minor modification for any essentially square integrable representation of either $GL_n(F)$ or D^* (and also for many such representations of $GL_2(D)$).

3. Main theorem

Theorem 3.1. *Let π be either an irreducible representation of D^* or an essentially square integrable representation of $GL_n(F)$. The conductor $c(\pi)$ of π and the depth $\rho(\pi)$ of π are related by*

$$\rho(\pi) = \max\left\{\frac{c(\pi) - n}{n}, 0\right\}.$$

Proof. We begin with the division algebra. Let $\mathfrak{P}$ denote the maximal ideal in the maximal compact subring $\mathcal{O}$ of D. Let π be an irreducible representation of D^*. We recall the definition of *level* of π. For any non-negative integer k, let $U_k = 1 + \mathfrak{P}^k$, with the convention that $U_0 = \mathcal{O}^\times$, which is the group of units in $\mathcal{O}$. The level of π denoted $\ell(\pi)$ is the least non-negative integer m such that π is trivial on U_m. A standard computation involving local factors (see [10]) relates the conductor to the level by the formula

$$c(\pi) = \ell(\pi) + n - 1.$$

It is an easy exercise to see that that if x is any point in the building of D^*, then for any non-negative real number r we have

$$D^*_{x,r} = 1 + \mathfrak{P}^{\lceil nr \rceil}.$$

From this and the definition of depth, it is clear that $\rho(\pi) = \max\{(\ell(\pi) - 1)/n, 0\}$ from which it follows that

$$\rho(\pi) = \max\left\{\frac{c(\pi) - n}{n}, 0\right\}.$$

Now let π be an essentially square integrable representation of $GL_n(F)$. According to the classification of such representations by Bernstein and Zelevinskii (see pp. 369–370 of [11]), π is equivalent to the Langlands quotient $Q(\tau)$ of the parabolic induction $\mathrm{Ind}_P^G(\tau)$ where P is a standard parabolic subgroup of $GL_n(F)$ whose Levi subgroup is of the form $GL_a(F) \times \cdots \times GL_a(F)$ and σ is a representation of P of the form $\tau \otimes \tau|\det| \otimes \cdots \otimes \tau|\det|^{b-1}$ where τ is a supercuspidal representation of $GL_a(F)$. (Note that $ab = n$.) By Proposition 2.1 we get

$$\rho(\pi) = \rho(\tau \otimes \cdots \otimes \tau(b - 1)).$$

It is easily seen that $\rho(\tau \otimes \cdots \otimes \tau(b - 1)) = \rho(\tau)$. Hence we have

$$\rho(\pi) = \rho(\tau).$$

We now consider two cases. First suppose that $a = 1$. Then τ is just a character of F^*. We can view F^* as a special case of a division algebra of index 1 and so by the preceding discussion we have

$$\rho(\tau) = \max\left\{\frac{c(\tau) - a}{a}, 0\right\}.$$

Next we consider the case where $a > 1$. Now τ is an irreducible supercuspidal representation of $GL_a(F)$. Then by Proposition 2.2 we have

$$\rho(\pi) = \frac{c(\tau) - a}{a} = \max\left\{\frac{c(\tau) - a}{a}, 0\right\}.$$

(For the last equality we have used that the conductor of a supercuspidal representation τ of $GL_a(F)$ is at least a.)

We now further analyze the conductor of $\pi = Q(\tau)$. For this we need the following formula describing the epsilon factor of such a π in terms of the local factors of τ and its contragredient $\tau^\vee$ (see Equation (5), §2.6 of [7]):

$$\varepsilon(s, \pi, \psi) = \prod_{i=0}^{b-1} \varepsilon(s + i, \tau, \psi) \prod_{i=0}^{b-2} \frac{L(-s - i, \tau^\vee)}{L(s + i, \tau)}. \tag{3.2}$$

We consider two cases again. First suppose that $a = 1$ and τ is unramified. This implies that π is an unramified twist of the Steinberg representation St_n of $GL_n(F)$. Hence the conductor of π is the conductor of St_n (by the main theorem of [9]). An easy exercise using the formula above gives $c(\mathrm{St}_n) = n - 1$. Also since π is a constituent of an unramified principal series representation we get, using [13], that $\rho(\pi) = 0$. Hence

$$\rho(\pi) = 0 = \max\left\{\frac{c(\pi) - n}{n}, 0\right\}.$$

We now consider the case where $a > 1$ or where $a = 1$ and τ is ramified. It is well known that in this case we have $L(s, \tau) = L(s, \tau^\vee) = 1$. Using the above formula for the epsilon factor for π we get $c(\pi) = c(\tau)b$. Hence again we have

$$\rho(\pi) = \max\left\{\frac{c(\tau) - a}{a}, 0\right\} = \max\left\{\frac{c(\pi) - n}{n}, 0\right\}.$$

□

Corollary 3.3. *Under the correspondence between irreducible representations π' of D^* and essentially square integrable representations π of $GL_n(F)$, depths are preserved, i.e.,*

$$\rho(\pi') = \rho(\pi).$$

Proof. The corollary follows from Theorem 3.1 since $c(\pi) = c(\pi')$. □

4. The case of $GL_2(D)$

For this section let $G' = GL_2(D)$. We begin by briefly recalling a construction of supercuspidal representations of G' by D. Prasad and the second author. (See §5 in [14] and also §3.3 of [15].) We will need to revisit the filtrations introduced there and recognize them as Moy-Prasad filtrations.

Let $K(m) = 1 + \mathfrak{P}^m M_2(\mathcal{O})$ for $m \geq 1$. Let $K(0) = GL_2(\mathcal{O})$ and let H_1 be the normalizer in G' of $K(0)$. It is an easy exercise using §3 of [13] to see that there is a point x in the Bruhat-Tits building $\mathcal{B} = \mathcal{B}(GL_2(D))$ such that for any non-negative real number r we have

$$GL_2(D)_{x,r} = K(\lceil nr \rceil).$$

Now let $I(0) = \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathfrak{P} & \mathcal{O}^\times \end{bmatrix}$ denote the subgroup of $GL_2(\mathcal{O})$ which is upper triangular modulo $\mathfrak{P}$. (The notation is self explanatory.) Let $I(2m) = \begin{bmatrix} 1+\mathfrak{P}^m & \mathfrak{P}^m \\ \mathfrak{P}^{m+1} & 1+\mathfrak{P}^m \end{bmatrix}$ and

$I(2m+1) = \begin{bmatrix} 1+\mathfrak{P}^{m+1} & \mathfrak{P}^m \\ \mathfrak{P}^{m+1} & 1+\mathfrak{P}^{m+1} \end{bmatrix}$ for any $m \geq 1$. Let H_2 be the normalizer in G' of $I(0)$. Using §3 of [13] it can be seen that there is a point $y \in \mathcal{B}$ such that for any non-negative real number we have

$$GL_2(D)_{y,r} = I(\lceil 2nr \rceil).$$

(Recall from Proposition 1.4 of [14] that H_1 and H_2 are up to conjugacy the two maximal open compact-mod-center subgroups of G'. We would also like to point out that in the notation of [14] we have $H_1(m) = K(m)$ and $H_2(m) = I(2m)$.) Finally it can be seen that for any $z \in \mathcal{B}$ and any non-negative real number t, the group $G'_{z,t}$ can be conjugated to either $G'_{x,r}$ or $G'_{y,r}$ for some r. In other words it suffices to consider just these two filtrations corresponding to the points x and y.

Let $H = H_1$ or H_2. Let $m \geq 1$ and let (σ, W) be a very cuspidal representation of H of level m. (See Definition 5.1 of [14].) Let $\pi = \mathrm{ind}_H^{G'}(\sigma)$ be the compact induction of σ to a representation of G'. Proposition 5.1 of [14] says that such a π is an irreducible supercuspidal representation of G'. Also Proposition 5.2 of [14] computes the conductor of π and this is given by

$$c(\pi) = 2m + i - 1 + 2(n-1)$$

where i is such that $H = H_i$.

Now comparing the definition of an unrefined minimal K-type of Moy and Prasad and the definition of a very cuspidal representation (at this point it is better to use the second author's thesis [15] rather than [14]) leads us to the following easy observations whose proof we omit.

(1) For $m \geq 1$, if σ is a very cuspidal representation of H_1 of level m, then σ contains an unrefined minimal K-type of depth $(m-1)/n = (2m-2)/2n$.
(2) For $m \geq 1$, if σ is a very cuspidal representation of H_2 of level m, then σ contains an unrefined minimal K-type of depth $(2m-1)/2n$.

In particular if $H = H_i$ and $\pi = \mathrm{ind}_H^{G'}(\sigma)$ we get

$$\rho(\pi) = \frac{2m+i-3}{2n} = \frac{c(\pi)-2n}{2n}.$$

In other words the required relation between depth and conductor holds for every supercuspidal representation constructed in [14].

We now consider essentially square integrable representations of G' which are not supercuspidal. (See Theorem B.2.b of [5].) They are obtained as follows. Let τ be an irreducible representation of D^*. Then there is a unique unramified character ν such that $\tau \otimes \tau\nu$ as a representation of $D^* \times D^*$ gives a reducible representation when parabolically induced to G'. This induced representation has a unique irreducible quotient, which we denote $Q(\tau)$. This $Q(\tau)$ is essentially square integrable and every essentially square integrable representation of G' which is not supercuspidal is equivalent to a $Q(\tau)$ for a uniquely determined τ.

By Proposition 2.1 and the division algebra part of the proof of our Theorem 3.1 we get

$$\rho(Q(\tau)) = \rho(\tau) = \max\left\{\frac{c(\tau)-n}{n}, 0\right\} = \max\left\{\frac{2c(\tau)-2n}{2n}, 0\right\}.$$

As in §3 we can similarly compute the conductor of $Q(\tau)$ and get $c(Q(\tau)) = 2c(\tau)$ if $c(\tau) \geq n$ or that $c(Q(\tau)) < 2n$. (This can be seen using Equation (3.2) and

Theorem B.2.b of [5].) This latter case happens when $c(\tau) < n$ and hence $\ell(\tau) = 0$, i.e., τ is an unramified character of D^* which would give that $\rho(Q(\tau)) = 0$. Hence

$$\rho(Q(\tau)) = \max\left\{\frac{c(Q(\tau)) - 2n}{2n}, 0\right\}.$$

To summarize, we have proved the following proposition in this section.

Proposition 4.1. *Let π' be either a supercuspidal representation constructed in* [14] *or an essentially square integrable non-supercuspidal representation of $GL_2(D)$. Then*

$$\rho(\pi') = \max\left\{\frac{c(\pi') - 2n}{2n}, 0\right\}.$$

Corollary 4.2. *Let $G' = GL_2(D)$ and let $G = GL_{2n}(F)$. Under the DKVR correspondence between essentially square integrable representations of G' and those of G we have*

$$\rho(\pi') = \rho(\pi)$$

if π' is either a supercuspidal representation constructed in [14] *or is an essentially square integrable non-supercuspidal representation of G'.*

We end this article by stating the following conjecture.

Conjecture 4.3. *Let A be a central simple algebra over F such that $\dim_F(A) = n^2$. Let $G = A^\times$. Let π be an essentially square integrable representation of G. Then the relation between the depth $\rho(\pi)$ and the conductor $c(\pi)$ of π is given by*

$$\rho(\pi) = \max\left\{\frac{c(\pi) - n}{n}, 0\right\}.$$

As in the proof of Theorem 3.1 using Equation (3.2) and Theorem B.2.b of [5] it suffices to prove this conjecture for supercuspidal representations of G. It is tempting to speculate that a similar equality holds for any reductive G for which there is a theory of epsilon factors (which would give us the conductor $c(\pi)$) with n replaced by the absolute rank of G.

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