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Conductors and newforms for *U*(1,1)

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Abstract. Let F be a non-Archimedean local field whose residue characteristic is odd. In this paper we develop a theory of newforms for $U(1,1)(F)$, building on previous work on $SL_2(F)$. This theory is analogous to the results of Casselman for $GL_2(F)$ and Jacquet, Piatetski-Shapiro, and Shalika for $GL_n(F)$. To a representation π of $U(1,1)(F)$, we attach an integer $c(\pi)$ called the conductor of π, which depends only on the L-packet Π containing π. A newform is a vector in π which is essentially fixed by a congruence subgroup of level $c(\pi)$. We show that our newforms are always test vectors for some standard Whittaker functionals, and, in doing so, we give various explicit formulae for newforms.

Keywords. Conductor; newforms; representations; $U(1,1)$.

1. Introduction

To introduce the main theme of this paper we recall the following theorem of Casselman [1]. Let F be a non-Archimedean local field whose ring of integers is $\mathcal{O}_F$. Let $\mathcal{P}_F$ be the maximal ideal of $\mathcal{O}_F$. Let ψ_F be a non-trivial additive character of F which is normalized so that the maximal fractional ideal on which it is trivial is $\mathcal{O}_F$.

Theorem 1.0.1 [1]. *Let (π, V) be an irreducible admissible infinite-dimensional representation of $GL_2(F)$. Let ω_π denote the central character of π. Let*

$$\Gamma(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_F) : c \equiv 0 \pmod{\mathcal{P}_F^m} \right\}.$$

Let

$$V_m = \left\{ v \in V : \pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) v = \omega_\pi(d) v, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(m) \right\}.$$

(i) *There exists a non-negative integer m such that $V_m \neq (0)$. If $c(\pi)$ denotes the least non-negative integer m with this property then the epsilon factor $\epsilon(s, \pi, \psi_F)$ of π is up to a constant multiple of the form $q^{-c(\pi)s}$. (Here q is the cardinality of the residue field of F.)*
(ii) *For all $m \geq c(\pi)$ we have* $\dim(V_m) = m - c(\pi) + 1$.

The assertion $\dim(V_{c(\pi)}) = 1$ is sometimes referred to as *multiplicity one theorem for newforms* and the unique vector (up to scalars) in $V_{c(\pi)}$ is called the *newform* for π. This is closely related to the classical Atkin–Lehner theory of newforms for holomorphic cusp forms on the upper half plane [1]. When $c(\pi) = 0$ we have a spherical representation and the newform is nothing but the spherical vector.

Newforms play an important role in the theory of automorphic forms. We cite two examples to illustrate this. First, the zeta integral corresponding to the newform is exactly the local L-factor associated to π (see [4] for instance). In addition, newforms frequently play the role of being 'test vectors' for interesting linear forms associated to π. For example, the newform is a test vector for an appropriate Whittaker linear functional. In showing this, explicit formulae for newforms are quite often needed. For instance, if π is a supercuspidal representation which is realized in its Kirillov model then the newform is the characteristic function of the unit group $\mathcal{O}_F^\times$. This observation is implicit in Casselman [1] and is explicitly stated and proved in Shimizu [18]. Since the Whittaker functional on the Kirillov model is given by evaluating functions at $1 \in F^*$, we get in particular that the functional is non-zero on the newform. In a related vein [15] and [3] show that test vectors for trilinear forms for $GL_2(F)$ are often built from newforms. (See also a recent expository paper of Schmidt [18] where many of these results are documented.)

In addition to Casselman's theory for $GL_2(F)$, newforms have been studied for certain other classes of groups. Jacquet *et al* [4] have developed a theory of newforms for *generic* representations of $GL_n(F)$. In this setting, there is no satisfactory statement analogous to (ii) of the above theorem. However, in his recent thesis, Mann [12] obtained several results on the growth of the dimensions of spaces of fixed vectors and has a conjecture about this in general. For the group $GL_2(D)$, D a p-adic division algebra, Prasad and Raghuram [16] have proved an analogue of Casselman's theorem for irreducible principal series representations and supercuspidal representations coming via compact induction. In an unpublished work, Brooks Roberts has proved part of (i) of the above theorem for representations of $GSp_4(F)$ whose Langlands parameter is induced from a two-dimensional representation of the Weil–Deligne group of F. In a previous paper [11], we develop a theory of conductors and newforms for $SL_2(F)$. In this paper we use the results of [11] to carry out a similar program for the unramified quasi split unitary group $U(1,1)$.

Let $\bar{G} = U(1,1)(F)$. Crucial to our study of newforms are certain filtrations of maximal compact subgroups of $\bar{G}$. Let $\bar{K} = \bar{K}_0$ be the standard hyperspecial maximal compact subgroup of $\bar{G}$. Let $\bar{K}' = \bar{K}_0' = \alpha^{-1}\bar{K}_0\alpha$, where $\alpha = \left(\begin{smallmatrix} \varpi_F & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\bar{K}_0$ and $\bar{K}_0'$ are, up to conjugacy, the two maximal compact subgroups of $\bar{G}$. We define filtrations of these maximal compact subgroups as follows. For m an integer ≥ 1, let

$$\bar{K}_m = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{K}_0 \colon c \equiv 0 \pmod{\mathcal{P}_E^m} \right\} \quad \text{and} \quad \bar{K}_m' = \alpha^{-1}\bar{K}_m\alpha.$$

Let $(\bar{\pi}, V)$ be an irreducible admissible infinite-dimensional representation of $\bar{G}$. Let $\bar{Z}$ denote the center of $\bar{G}$ and let $\omega_{\bar{\pi}}$ be the central character of $\bar{\pi}$. Let $\bar{\eta}$ be any character of $\mathcal{O}_E^\times$ such that $\bar{\eta} = \omega_{\bar{\pi}}$ on the center. Let $c(\bar{\eta})$ denote the conductor of $\bar{\eta}$. For any $m \geq c(\bar{\eta})$, $\bar{\eta}$ gives a character of $\bar{K}_m$ and also $\bar{K}_m'$ given by $\bar{\eta}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \bar{\eta}(d)$. We define for $m \geq c(\bar{\eta})$,

$$\bar{\pi}_{\bar{\eta}}^{\bar{K}_m} := \left\{ v \in V \colon \bar{\pi}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) v = \bar{\eta}(d)v, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{K}_m \right\}.$$

The space $\bar{\pi}_{\bar{\eta}}^{\bar{K}'_m}$ is defined analogously. We define the $\bar{\eta}$-*conductor* $c_{\bar{\eta}}(\bar{\pi})$ of $\bar{\pi}$ as

$$c_{\bar{\eta}}(\bar{\pi}) = \min\{m \geq 0\colon \bar{\pi}_{\bar{\eta}}^{\bar{K}_m} \neq (0) \quad \text{or} \quad \bar{\pi}_{\bar{\eta}}^{\bar{K}'_m} \neq (0)\}. \tag{1.0.2}$$

We define the *conductor* $c(\bar{\pi})$ of $\bar{\pi}$ by

$$c(\bar{\pi}) = \min\{c_{\bar{\eta}}(\bar{\pi}) : \bar{\eta}\}, \tag{1.0.3}$$

where $\bar{\eta}$ runs over characters of $\mathcal{O}_E^\times$ which restrict to the central character $\omega_{\bar{\pi}}$ on $\bar{Z}$.

We deal with the following basic issues in this paper.

(i) Given an irreducible representation $\bar{\pi}$, we determine its conductor $c(\bar{\pi})$. A very easy consequence (almost built into the definition) is that the conductor depends only on the L-packet containing $\bar{\pi}$.

(ii) We identify the conductor with other invariants associated to the representation. For instance, for $SL_2(F)$ we have shown [11] that the conductor of a representation is same as the conductor of a minimal representation of $GL_2(F)$ determining its L-packet. We prove a similar result for $U(1,1)(F)$ in this paper. See §3.4 and §4.4.

(iii) We determine the growth of the space $\dim(V_{\bar{\eta}}^{\bar{K}_m})$ as a function of m. This question is analogous to (ii) of Casselman's theorem quoted above. Computing such dimensions is of importance in 'local level raising' issues. See [12].

(iv) We address the question of whether there is a multiplicity one result for newforms. It turns out that quite often $\dim(V_{\bar{\eta}}^{\bar{K}_{c(\bar{\pi})}}) = 1$, but this fails in general (for principal series representations of a certain kind). In these exceptional cases the dimension of the space of newforms is two, but a canonical quotient of this two-dimensional space has dimension one (see §5).

(v) Are the newforms test vectors for Whittaker functionals? This is important in global issues related to newforms. We are grateful to Benedict Gross for suggesting this question to us. It turns out that our newforms are always test vectors for Whittaker functionals. In the proofs we often need explicit formulae for newforms in various models for the representations. These formulas are interesting for their own sake. For example, if (π, V) is a ramified supercuspidal representation of $U(1,1)(F)$, then the newform can be taken as the characteristic function of $(\mathcal{O}_F^\times)^2$ where V is regarded as a subspace of the Kirillov model of a canonically associated minimal representation of $GL_2(F)$ (cf. [18]).

We set up notation in §2.1 following that used in [11]. We then briefly review the structure of L-packets for SL_2 and $U(1,1)$ in §2.2. As this paper depends crucially on our previous paper [11] on SL_2, we summarize the results of [11] in §3. The heart of this paper is §4. In §4.1 we define the notion of conductor and then make some easy but technically important remarks on spaces of fixed vectors. The next two subsections deal respectively with sub-quotients of principal series representations and supercuspidal representations. In [11], we use Kutzko's construction of supercuspidal representations of $GL_2(F)$ to obtain results for supercuspidals of $SL_2(F)$. In this paper, we use these results, in turn, to obtain information for $U(1,1)(F)$. In general, we will often reduce the proofs of statements concerning $U(1,1)(F)$ to those of the corresponding $SL_2(F)$ statements. In particular, we exploit the fact that $SL_2(F)$ is the derived group of $U(1,1)(F)$ and that $U(1)(F)SL_2(F)$ has index two in $U(1,1)(F)$. In this way we avoid directly dealing with K-types and other intrinsic details for $\mathrm{U}(1,1)(F)$ as much of the work has been done for $SL_2(F)$ in [11]. Finally, in §5, we briefly discuss a multiplicity one result for newforms.

We mention some further directions that arise naturally from this work. To begin with, it would be interesting to see how our theory of newforms and conductors bears upon known results about local factors for $U(1,1)(F)$. In particular, are our conductors the same as (or closely related to) the analytic conductors appearing in epsilon factors? Also, is a zeta-integral corresponding to a newform of a representation equal to a local L-factor for the representation?

2. Preliminaries

2.1 *Notation*

If L is any non-Archimedean local field let $\mathcal{O}_L$ be its ring of integers and let $\mathcal{P}_L$ be the maximal ideal of $\mathcal{O}_L$. Let ϖ_L be a uniformizer for L, i.e., $\mathcal{P}_L = \varpi_L\mathcal{O}_L$. Let $k_L = \mathcal{O}_L/\mathcal{P}_L$ be the residue field of L. Let p be the characteristic of k_L and let the cardinality of k_L be q_L which is a power of p. Let ϵ_L be an element of $\mathcal{O}_L^* - \mathcal{O}_L^{*2}$.

If n is a positive integer, let U_L^n denote the nth filtration subgroup $1 + \mathcal{P}_L^n$ of $\mathcal{O}_L^\times$, and define $U_L^0 = \mathcal{O}_L^\times$. Let $\mathfrak{v}_L$ denote the additive valuation on L^* which takes the value 1 on ϖ_L. We let $|\cdot|_L$ denote the normalized multiplicative valuation given by $|x|_L = q^{-\mathfrak{v}_L(x)}$. If χ is a character of L^* we define the conductor $c(\chi)$ to be the smallest non-negative integer n such that χ is trivial on U_L^n. Let ψ_L be a non-trivial additive character of L which is assumed to be trivial on $\mathcal{O}_L$ and non-trivial on $\mathcal{P}_L^{-1}$. For any $a \in L$ the character given by sending x to $\psi_L(ax)$ will be denoted as $\psi_{L,a}$ or simply by ψ_a. (In all the above notations we may omit the subscript L if there is only one field in the context.)

In the following, F will be a fixed non-Archimedean local field whose residue characteristic is odd and E will be used to denote a quadratic extension of F. We denote by $\omega_{E/F}$ the quadratic character of F^* associated to E/F by local class field theory. Recall that the kernel of $\omega_{E/F}$ is $N_{E/F}(E^*)$, the norms from E^*. We will require the units ϵ_F and ϵ_E to be compatible in the sense that

$$\epsilon_F = N_{E/F}(\epsilon_E).$$

We let $\widetilde{G}$ denote the group $GL_2(F)$. Let $\widetilde{B} = \widetilde{T}N$ be the standard Borel subgroup of upper triangular matrices in $\widetilde{G}$ with Levi subgroup $\widetilde{T}$ and unipotent radical N. Let $\widetilde{Z}$ be the center of $\widetilde{G}$. Let $G = SL_2(F)$. Let $B = TN$ be the standard Borel subgroup of upper triangular matrices in G with Levi subgroup T and unipotent radical N. We set $K = SL_2(\mathcal{O}_F)$ and $\widetilde{K} = GL_2(\mathcal{O}_F)$ and denote by I and $\widetilde{I}$ respectively the standard Iwahori subgroups of G and $\widetilde{G}$.

Suppose that E/F is unramified, and let s denote the non-trivial element of $\mathrm{Gal}(E/F)$. We denote by $\bar{G}$ the group $U(1,1)$, i.e., the group of all $g \in GL_2(E)$ such that

$${}^s g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We let $\bar{B}$ be the standard upper triangular Borel subgroup of $\bar{G}$ with diagonal Levi subgroup $\bar{T}$ and unipotent radical N. We note that the elements of $\bar{T}$ are of the form $\begin{pmatrix} t & 0 \\ 0 & {}^s t^{-1} \end{pmatrix}$ for

$t \in E^*$, and those of $\bar{B}$ are of the form $\left(\begin{smallmatrix} t & ta \\ 0 & {}^s t^{-1}\end{smallmatrix}\right)$ with $t \in E^*$ and $a \in F$. We let $\bar{Z}$ be the center of $\bar{G}$. Then $\bar{Z} \cong E^1$, where $E^1 = \ker(N_{E/F})$ is the subgroup of norm one elements of E^*. Denote by $\bar{I}$ the standard Iwahori subgroup of $\bar{G}$ and by $\bar{K}$ the standard hyperspecial maximal compact subgroup of $\bar{G}$.

The following filtrations of maximal compact subgroups of G will be important in our study of newforms. Let $K_{-1} = G$ and $K_0 = K$. Let $K' = K'_0 = \alpha^{-1}K_0\alpha$, where $\alpha = \left(\begin{smallmatrix} \varpi_F & 0 \\ 0 & 1\end{smallmatrix}\right)$. Then K_0 and K'_0 are, up to conjugacy, the two maximal compact subgroups of G. For m an integer ≥ 1,

$$\begin{aligned} K_m &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0 : c \equiv 0 \pmod{\mathcal{P}_F^m} \right\}, \\ K'_m &= \alpha^{-1} K_m \alpha. \end{aligned}$$

We note that for $m \geq 1$ the following inclusions hold up to conjugacy within G:

$$K'_{m+1} \subset K_m \subset K'_{m-1}. \tag{2.1.1}$$

Analogous results hold for the following filtration groups of $\bar{G}$:

$$\begin{aligned} \bar{K}_{-1} &= \bar{G}, \\ \bar{K}_0 &= \bar{K}, \\ \bar{K}_m &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{K}_0 : c \equiv 0 \pmod{\mathcal{P}_E^m} \right\}, \\ \bar{K}'_m &= \alpha^{-1} \bar{K}_m \alpha. \end{aligned}$$

We note that the filtration subgroups for G and $\bar{G}$ are related by

$$\bar{K}_m = K_m \bar{T}_0, \tag{2.1.2}$$

where $\bar{T}_0 = \bar{T} \cap \bar{K}_0$.

In addition to α, we will also make frequent use of the matrices $\beta := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_F\end{smallmatrix}\right)$, $\gamma := \left(\begin{smallmatrix} \epsilon_F & 0 \\ 0 & 1\end{smallmatrix}\right)$ and $\theta := \left(\begin{smallmatrix} \epsilon_E & 0 \\ 0 & {}^s\epsilon_E^{-1}\end{smallmatrix}\right)$.

For any subsets $A, B, C, D \subset F$ we let

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \in A, b \in B, c \in C, d \in D \right\}.$$

We denote $\left[\begin{smallmatrix} 1 & \mathcal{P}^j \\ 0 & 1\end{smallmatrix}\right]$ by $N(\mathcal{P}^j)$ or simply by $N(j)$. We let $\overline{N}$ denote the lower triangular unipotent subgroup of G and a similar meaning is given to $\overline{N}(\mathcal{P}^j)$ and $\overline{N}(j)$.

If $\mathcal{H}$ is a closed subgroup of a locally compact group $\mathcal{G}$ and if σ is an admissible representation of $\mathcal{H}$ then $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\sigma)$ denotes *normalized* induction, and $\mathrm{ind}_{\mathcal{H}}^{\mathcal{G}}(\sigma)$ denotes the subrepresentation of $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\sigma)$ consisting of those functions whose support is compact mod $\mathcal{H}$. The symbol **1** will denote the trivial representation of the group in context.

For any real number ζ we let $\lceil \zeta \rceil$ denote the least integer greater than or equal to ζ and we let $\lfloor \zeta \rfloor = -\lceil -\zeta \rceil$.

2.2 *L-packets for $SL_2(F)$ and $U(1,1)$*

In this section we collect statements about the structure of L-packets for $G = SL_2(F)$ and $\bar{G} = U(1,1)$. All the assertions made here are well-known and can be read off from a combination of Labesse and Langlands [9], Gelbart and Knapp [2] and Rogawski [17].

If $\widetilde{\pi}$ is an irreducible admissible representation of $\widetilde{G}$ then its restriction to G is a multiplicity free finite direct sum of irreducible admissible representations of G which we often write as

$$\mathrm{Res}_{SL_2(F)}\widetilde{\pi} = \pi_1 \oplus \cdots \oplus \pi_r.$$

On the other hand, if π is any irreducible admissible representation of G then there exists an irreducible admissible representation $\widetilde{\pi}$ of $\widetilde{G}$ whose restriction to G contains π.

Note that $\widetilde{G}$ acts on the space of all equivalence classes of irreducible admissible representations of G and an *L-packet* for G is simply an orbit under this action. It turns out that, with the notation as above, the L-packets are precisely the sets $\{\pi_1, \ldots, \pi_r\}$ appearing in the restrictions of irreducible representations $\widetilde{\pi}$ of $\widetilde{G}$.

We now give some general statements concerning the L-packets for $\bar{G} = U(1,1)$. The adjoint group of $U(1,1)$ is PGL_2, and hence $PGL_2(F)$ and $\widetilde{G}$ act via automorphisms on $\bar{G}$, hence act on the set of equivalence classes of irreducible representations of $\bar{G}$. Rogawski ([17], §11.1) defines an L-packet for $\bar{G}$ to be an orbit under this action. If $\bar{\pi}$ is an element of a non-trivial L-packet, then the other element of the L-packet is ${}^{\alpha}\bar{\pi}$.

If $\bar{\Pi}$ is an L-packet for $\bar{G}$, then the set of irreducible components of the restrictions of elements of $\bar{\Pi}$ to G is an L-packet Π for G. The direct sum $\bigoplus_{\pi\in\Pi}\pi$ is therefore the restriction of an irreducible admissible representation $\widetilde{\pi}$ of $\widetilde{G}$. This $\widetilde{\pi}$ is unique up to twisting by a character. In practice, we will choose a convenient $\widetilde{\pi}$. Since $\bigoplus_{\pi\in\Pi}\pi = \mathrm{Res}_G\left(\bigoplus_{\bar{\pi}\in\bar{\Pi}}\bar{\pi}\right)$, we obtain an action of $\widetilde{G}$ on $\bigoplus_{\bar{\pi}\in\bar{\Pi}}\bar{\pi}$ via the representation $\widetilde{\pi}$.

3. Newforms for SL_2

This section collects our results [11] on conductors and newforms for $SL_2(F)$. All these results, along with their complete proofs, can be found in [11].

3.1 *Definitions*

We now give our definition of the conductor of a representation of G. The basic filtration subgroups of G considered in this paper are $K_0 = K = SL_2(\mathcal{O}_F)$ and for $m \geq 1$,

$$K_m = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_F) : c \equiv 0 \pmod{\mathcal{P}_F^m} \right\}.$$

For all $m \geq 0$ we let $K'_m = \alpha^{-1}K_m\alpha$.

Let (π, V) be an irreducible admissible infinite-dimensional representation of G. Let ω_π be the character of $\{\pm 1\}$ such that $\pi\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) = \omega_\pi(-1)1_V$.

We let η be any character of $\mathcal{O}_F^\times$ such that $\eta(-1) = \omega_\pi(-1)$. Let $c(\eta)$ denote the conductor of η. For any $m \geq c(\eta)$, η gives a character of K_m and also K'_m given by

$$\eta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \eta(d).$$

We define

$$\pi_\eta^{K_m} := \left\{ v \in V : \pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) v = \eta(d) v, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_m \right\}.$$

The spaces $\pi_\eta^{K'_m}$ are defined analogously. We note that $\pi_\eta^{K_m} = \pi_\eta^{K'_m} = (0)$ for $m < c(\eta)$.

We define the *η-conductor* $c_\eta(\pi)$ of π as

$$c_\eta(\pi) = \min\{m \geq 0 : \pi_\eta^{K_m} \neq (0) \quad \text{or} \quad \pi_\eta^{K'_m} \neq (0)\}, \tag{3.1.1}$$

We define the *conductor* $c(\pi)$ of π by

$$c(\pi) = \min\{c_\eta(\pi) : \eta\}, \tag{3.1.2}$$

where η runs over characters of $\mathcal{O}_F^\times$ such that $\eta(-1) = \omega_\pi(-1)$. If η is such that $c_\eta(\pi) = c(\pi)$ and $\pi_\eta^{K_{c(\pi)}} \neq (0)$ (resp. $\pi_\eta^{K'_{c(\pi)}} \neq (0)$), then we call $\pi_\eta^{K_{c(\pi)}}$ (resp. $\pi_\eta^{K'_{c(\pi)}}$) a *space of newforms* of π. In this case, we refer to a non-zero element of $\pi_\eta^{K_{c(\pi)}}$ or $\pi_\eta^{K'_{c(\pi)}}$ as a *newform* of π.

3.2 *Principal series representations*

Let χ be a character of F^*. Then χ gives a character of B via the formula $\chi\left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}\right) = \chi(a)$. Let $\pi(\chi)$ stand for the (unitarily) induced representation $\mathrm{Ind}_B^G(\chi)$. It is well-known that $\pi(\chi)$ is reducible if and only if χ is either $|\cdot|_F^{\pm}$ or if χ is a quadratic character.

There is an essential difference between the two kinds of reducibilities. If $\chi = |\cdot|_F^{\pm}$, then $\pi(\chi)$ is the restriction to G of a reducible principal series representation of $\widetilde{G}$. Hence $\pi(\chi)$ will have two representations in its Jordan–Hölder series, namely the trivial representation and the Steinberg representation which we will denote by St_G.

If χ is a quadratic character, then $\pi(\chi)$ is the restriction to G of an irreducible principal series representation of $\widetilde{G}$ and breaks up as a direct sum of two irreducible representations, which constitute an L-packet of G. If $\chi = \omega_{E/F}$ we denote $\pi(\chi)$ by π_E and let $\pi_E \simeq \pi_E^1 \oplus \pi_E^2$. We denote the L-packet by $\xi_E = \{\pi_E^1, \pi_E^2\}$.

As mentioned in the introduction, one of the applications of newforms we have in mind is that they are test vectors for Whittaker functionals. For principal series representations and in fact all their sub-quotients we consider the following ψ-Whittaker functional (see [18]). For any function f in a principal series representation $\pi(\chi)$ we define

$$\Lambda_\psi f := \lim_{r \to \infty} \int_{\mathcal{P}_F^{-r}} f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \overline{\psi}(x)\, \mathrm{d}x, \tag{3.2.1}$$

where the Haar measure is normalized such that $\mathrm{vol}(\mathcal{O}) = 1$.

PROPOSITION 3.2.2 (Unramified principal series representations)

Let χ be an unramified character of F^* and let $\pi(\chi)$ be the corresponding principal series representation of G. We have

$$\dim(\pi(\chi)^{K_m}) = \begin{cases} 1, & \text{if } m = 0, \\ 2m, & \text{if } m \geq 1. \end{cases}$$

COROLLARY 3.2.3 (Test vectors for unramified principal series representations)

For an unramified character χ of F^* such that $\chi \neq |\cdot|_F^{-1}$. Let f_{new} be any non-zero K-fixed vector. Then we have

$$\Lambda_\psi f_{\mathrm{new}} = L(1, \chi)^{-1} \neq 0,$$

where $L(s, \chi)$ is the standard local abelian L-factor associated to χ.

PROPOSITION 3.2.4 (Steinberg representation)

If St_G is the Steinberg representation of G, then the dimension of the space of fixed vectors under K_m is given by

$$\dim(\mathrm{St}_G)^{K_m} = \begin{cases} 0, & \text{if } m = 0, \\ 2m - 1, & \text{if } m \geq 1. \end{cases}$$

COROLLARY 3.2.5 (Test vectors for the Steinberg representation)

Let the Steinberg representation St_G be realized as the unique irreducible subrepresentation of $\pi(|\cdot|)$. The ψ-Whittaker functional Λ_ψ is non-zero on the space of newforms $(\mathrm{St}_G)_{\mathrm{new}} = \mathrm{St}_G^{K_1}$.

PROPOSITION 3.2.6 (Ramified principal series representations)

Let χ be a ramified character of F^*. Let $\pi = \pi(\chi)$ be the corresponding principal series representation of G. Let $c(\chi)$ denote the conductor of χ.

(i) *We have $c(\pi) = c(\chi)$ and further $c_\eta(\pi) = c(\pi)$ only for those characters η such that $\eta = \chi^{\pm}$ on the group of units $\mathcal{O}^\times$.*

(ii) *If $\chi^2|_{(\mathcal{O}^\times)^2} \neq \mathbf{1}$ and $\eta = \chi|_{\mathcal{O}^\times}$ then*

$$\dim(\pi(\chi)_\eta^{K_m}) = \begin{cases} 0, & \text{if } m < c(\chi), \\ 1, & \text{if } m = c(\chi), \\ 2(m - c(\chi)) + 1, & \text{if } m > c(\chi). \end{cases}$$

(iii) *If $\chi^2|_{(\mathcal{O}^\times)^2} = \mathbf{1}$ and $\eta = \chi|_{\mathcal{O}^\times}$ then*

$$\dim(\pi(\chi)_\eta^{K_m}) = \begin{cases} 0, & \text{if } m = 0, \\ 2m, & \text{if } m \geq 1 = c(\chi). \end{cases}$$

COROLLARY 3.2.7 (Test vectors for ramified principal series representations)

Let χ be a ramified character of F^*. Let $\pi = \pi(\chi)$ be the corresponding principal series representation of G. Assume that π is irreducible. Let $m = c(\chi) \geq 1$ denote the conductor of χ. The space of newforms $\pi(\chi)_{\mathrm{new}} = \pi(\chi)_\chi^{K_{c(\chi)}}$ is one-dimensional and the Whittaker functional Λ_ψ is non-zero on this space of newforms.

PROPOSITION 3.2.8 (Ramified principal series L-packets)

Let E/F be a quadratic ramified extension. Let $\xi_E = \{\pi_E^1, \pi_E^2\}$ be the corresponding L-packet. Then we have for $\eta = \omega_{E/F}$,

$$\dim((\pi_E^1)_m^\eta) = \dim((\pi_E^2)_m^\eta) = \begin{cases} 0, & \text{if } m = 0, \\ m, & \text{if } m \geq 1. \end{cases}$$

COROLLARY 3.2.9 (Test vectors for ramified principal series L-packets)

Let E/F be a ramified quadratic extension and let $\xi_E = \{\pi_E^1, \pi_E^2\}$ be the corresponding L-packet. Then one and only one of the two representations in the packet is ψ-generic, say, π_E^1. Then π_E^2 is ψ_ϵ-generic. The Whittaker functional Λ_ψ is non-zero on the one dimensional space of newforms $(\pi_E^1)_{\rm new} = (\pi_E^1)_{\omega_{E/F}}^{K_1}$. Any ψ_ϵ-Whittaker functional is non-zero on the one-dimensional space of newforms for π_E^2.

PROPOSITION 3.2.10 (Unramified principal series L-packet)

Let E/F be the quadratic unramified extension. Let $\xi_E = \{\pi_E^1, \pi_E^2\}$ be the corresponding L-packet. Exactly one of the two representations, say π_E^1, has a non-zero vector fixed by K_0. Then the dimensions of the space of fixed vectors under K_m and K'_m for the two representations are as follows:

(i) $\dim((\pi_E^1)^{K_0}) = 1 = \dim((\pi_E^2)^{K'_0})$.

(ii) $\dim((\pi_E^1)^{K'_0}) = 0 = \dim((\pi_E^2)^{K_0})$.

(iii) *For $r \geq 1$,*

$$\dim((\pi_E^1)^{K_r}) = 2\left\lfloor \frac{r}{2} \right\rfloor + 1 = \dim((\pi_E^2)^{K'_r}).$$

(iv) *For $r \geq 1$,*

$$\dim((\pi_E^1)^{K'_r}) = 2\left\lfloor \frac{r-1}{2} \right\rfloor + 1 = \dim((\pi_E^2)^{K_r}).$$

COROLLARY 3.2.11 (Test vectors for unramified principal series L-packet)

Let E/F be the unramified quadratic extension, and let $\xi_E = \{\pi_E^1, \pi_E^2\}$ be the corresponding L-packet. Then one and only one of the two representations in the packet is ψ-generic, namely π_E^1 (using the notation of Proposition 3.2.10). Moreover, a ψ-Whittaker functional is non-zero on the K_0-fixed vector in π_E^1. The representation π_E^2 is not ψ'-generic for any ψ' of conductor $\mathcal{O}_F$. It is ψ_{ϖ_F}-generic and any ψ_{ϖ_F}-Whittaker functional is non-zero on the unique (up to scalars) K'_0-fixed vector in π_E^2.

3.3 *Supercuspidal representations*

We now consider supercuspidal representations of $G = SL_2(F)$. For this we need some preliminaries on how they are constructed. We use Kutzko's construction [5,6] of supercuspidal representations for $\widetilde{G}$ and then Moy and Sally [14] or Kutzko and Sally [8] to obtain information on the supercuspidal representations (L-packets) for G.

We begin by briefly recalling Kutzko's construction of supercuspidal representations of $\widetilde{G}$ via compact induction from very cuspidal representations of maximal open compact-mod-center subgroups.

For $l \geq 1$, let

$$\widetilde{K}(l) = 1 + \mathcal{P}^l M_{2\times 2}(\mathcal{O})$$

be the principal congruence subgroup of $\widetilde{K}$ of level l. Let $\widetilde{K}(0) = \widetilde{K}$. Let $\widetilde{I}$ be the standard Iwahori subgroup consisting of all elements in $\widetilde{K}$ that are upper triangular modulo $\mathcal{P}$. For $l \geq 1$, let

$$\widetilde{I}(l) = \begin{bmatrix} 1+\mathcal{P}^l & \mathcal{P}^l \\ \mathcal{P}^{l+1} & 1+\mathcal{P}^l \end{bmatrix},$$

and let $\widetilde{I}(0) = \widetilde{I}$. We will let $\widetilde{H}$ (resp. $\widetilde{J}$) denote either $Z\widetilde{K}$ (resp. $\widetilde{K}$) or $N_{\widetilde{G}}\widetilde{I}$ (resp. $\widetilde{I}$). Here $N_{\widetilde{G}}\widetilde{I}$ is the normalizer in $\widetilde{G}$ of $\widetilde{I}$. In either case we let $\widetilde{J}(l)$ denote the corresponding filtration subgroup.

DEFINITION 3.3.1 [6,7]

An irreducible (and necessarily finite-dimensional) representation $(\widetilde{\sigma}, W)$ of $\widetilde{H}$ is called a very cuspidal representation of level $l \geq 1$ if

(i) $\widetilde{J}(l)$ is contained in the kernel of $\widetilde{\sigma}$.
(ii) $\widetilde{\sigma}$ does not contain the trivial character of $N(\mathcal{P}^{l-1})$.

We say that an irreducible admissible representation $\widetilde{\pi}$ of $\widetilde{G}$ is minimal if for every character χ of F^* we have $c(\widetilde{\pi}) \leq c(\widetilde{\pi} \otimes \chi)$.

Theorem 3.3.2 [6,7]. *There exists a bijective correspondence given by compact induction $\widetilde{\sigma} \mapsto \mathrm{ind}_{\widetilde{H}}^{\widetilde{G}}(\widetilde{\sigma})$ from very cuspidal representations $\widetilde{\sigma}$ of either maximal open compact-mod-center subgroup $\widetilde{H}$ and irreducible minimal supercuspidal representations of $\widetilde{G}$. Moreover, every irreducible minimal supercuspidal representation of conductor $2l$ (resp. $2l+1$) comes from a very cuspidal representation of $Z\widetilde{K}$ (resp. $N_{\widetilde{G}}\widetilde{I}$) of level l.*

Following Kutzko we use the terminology that a supercuspidal representation of $\widetilde{G}$ is said to be *unramified* if it comes via compact induction from a representation of $Z\widetilde{K}$ and *ramified* if it comes via compact induction from a representation of $N_{\widetilde{G}}\widetilde{I}$. We now take up both types of supercuspidal representations and briefly review how they break up on restriction to G. We refer the reader to [8] and [14] for this.

We begin with the unramified case. Let $\widetilde{\sigma}$ be an irreducible very cuspidal representation of $Z\widetilde{K}$ of level l (≥ 1). Let $\widetilde{\pi}$ be the corresponding supercuspidal representation of $\widetilde{G}$. Let $\sigma = \mathrm{Res}_K(\widetilde{\sigma})$. Then we have

$$\mathrm{Res}_G(\widetilde{\pi}) = \mathrm{ind}_K^G(\sigma) \;\oplus\; {}^{\alpha}(\mathrm{ind}_K^G(\sigma)),$$

where $\alpha = \left(\begin{smallmatrix} \varpi_F & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

If $l \geq 2$, or if $l = 1$ and σ is irreducible, then $\pi = \pi(\sigma) = \mathrm{ind}_K^G(\sigma)$ is irreducible, hence so is $\pi' = {}^{\alpha}\pi$. We thus have an unramified supercuspidal L-packet $\{\pi, \pi'\}$.

If $l = 1$ and σ is reducible, then $\widetilde{\sigma}$ comes from the unique (up to twists) cuspidal representation of $GL_2(\mathbb{F}_q)$ whose restriction to $SL_2(\mathbb{F}_q)$ is reducible and breaks up into the direct sum of the two cuspidal representations of $SL_2(\mathbb{F}_q)$ of dimension $(q-1)/2$. Correspondingly, we have $\sigma = \sigma_1 \oplus \sigma_2$, and if we let $\pi_i = \mathrm{ind}_K^G(\sigma_i)$ and $\pi_i' = {}^{\alpha}(\pi_i)$, then we obtain the unique supercuspidal L-packet $\{\pi_1, \pi_1', \pi_2, \pi_2'\}$ of G containing four elements.

For the ramified case, let $\widetilde{\sigma}$ be a very cuspidal representation of $N_{\widetilde{G}}\widetilde{I}$ of level l (≥ 1) and let $\widetilde{\pi}$ be the corresponding supercuspidal representation of $\widetilde{G}$. Let $\sigma = \mathrm{Res}_I(\widetilde{\sigma})$. Then $\sigma = \sigma_1 \oplus \sigma_2$ for two irreducible representations σ_i ($i = 1, 2$) of I and γ conjugates one to the other, i.e., $\sigma_2 = {}^{\gamma}\sigma_1$. Let $\pi_i = \mathrm{ind}_I^G(\sigma_i)$ and so $\pi_2 = {}^{\gamma}\pi_1$. Then the restriction of $\widetilde{\pi}$ to G breaks up into the direct sum of two irreducible supercuspidal representations as $\mathrm{Res}_G(\widetilde{\pi}) = \pi_1 \oplus \pi_2$. We call $\{\pi_1, \pi_2\}$ a *ramified supercuspidal L-packet* of G.

To summarize, we have three kinds of supercuspidal L-packets for G namely,

(i) unramified supercuspidal L-packets $\{\pi, \pi'\}$;
(ii) the unique (unramified) supercuspidal L-packet $\{\pi_1, \pi_1', \pi_2, \pi_2'\}$ of cardinality four;
(iii) ramified supercuspidal L-packets $\{\pi_1, \pi_2\}$.

PROPOSITION 3.3.3 (Unramified supercuspidal L-packets of cardinality two)

Consider an unramified supercuspidal L-packet $\{\pi, \pi'\}$ determined by a very cuspidal representation $\widetilde{\sigma}$ of level l of $Z\widetilde{K}$ as above. The conductors $c(\pi), c(\pi')$ are both equal to $2l$. The dimensions of the spaces $\pi_\eta^{K_m}$ and $(\pi')_\eta^{K_m}$ are as follows:

(i) *For any η such that $\eta(-1) = \omega_\pi(-1)$ we have*

$$\pi_\eta^{K_{2l-1}} = \pi_\eta^{K'_{2l-1}} = (\pi')_\eta^{K_{2l-1}} = (\pi')_\eta^{K'_{2l-1}} = (0).$$

(ii) *Let $\eta(-1) = \omega_\pi(-1)$ and $c(\eta) \leq l$. If l is odd then for all $m \geq 2l$,*

(a) $\dim(\pi_\eta^{K'_m}) = \dim((\pi')_\eta^{K_m}) = 2\left\lceil \frac{m-2l+1}{2} \right\rceil$,

(b) $\dim(\pi_\eta^{K_m}) = \dim((\pi')_\eta^{K'_m}) = 2\left\lfloor \frac{m-2l+1}{2} \right\rfloor$.

(iii) *Let $\eta(-1) = \omega_\pi(-1)$ and $c(\eta) \leq l$. If l is even then for all $m \geq 2l$,*

(a) $\dim(\pi_\eta^{K_m}) = \dim((\pi')_\eta^{K'_m}) = 2\left\lceil \frac{m-2l+1}{2} \right\rceil$,

(b) $\dim(\pi_\eta^{K'_m}) = \dim((\pi')_\eta^{K_m}) = 2\left\lfloor \frac{m-2l+1}{2} \right\rfloor$.

PROPOSITION 3.3.4 (Test vectors for unramified supercuspidal L-packets of cardinality two)

Let $\widetilde{\sigma}$ be a very cuspidal representation of $Z\widetilde{K}$ which determines an unramified supercuspidal L-packet $\{\pi, \pi'\}$ as above. Assume that $\widetilde{\pi} = \mathrm{ind}_{Z\widetilde{K}}^{\widetilde{G}}(\widetilde{\sigma})$ is realized in its Kirillov model with respect to ψ. Define two elements ϕ_1 and ϕ_ϵ in the Kirillov model as follows:

$$\phi_1(x) = \begin{cases} 1, & \text{if } x \in (\mathcal{O}^\times)^2, \\ 0, & \text{if } x \notin (\mathcal{O}^\times)^2, \end{cases}$$

$$\phi_\epsilon(x) = \widetilde{\pi}(\gamma)\phi_1.$$

Let $\eta = \omega_{\widetilde{\pi}}$. We have

(i) $\mathbb{C}\phi_1 \oplus \mathbb{C}\phi_\epsilon = \widetilde{\pi}_\eta^{K_{2l}}$.
(ii) *If l is even, then $\pi_\eta^{K_{2l}} = \widetilde{\pi}_\eta^{K_{2l}}$. In addition, π is ψ-generic and any ψ-Whittaker functional is non-zero on ϕ_1 and vanishes on ϕ_ϵ. Furthermore, π' is not ψ'-generic for*

any character ψ' of conductor $\mathcal{O}$. It is however ψ_ϖ-generic, and any ψ_ϖ-Whittaker functional is non-vanishing on $\widetilde{\pi}(\alpha^{-1})\phi_1$, which is a newform for π'.

(iii) *If l is odd, then* (ii) *holds with π and π' interchanged.*

PROPOSITION 3.3.5 (Unramified supercuspidal L-packet of cardinality four)

Let $\widetilde{\sigma}$ denote a very cuspidal representation of $Z\widetilde{K}$ of level $l = 1$ such that $\mathrm{Res}_K(\widetilde{\sigma}) = \sigma = \sigma_1 \oplus \sigma_2$. Let $\{\pi_1, \pi_1', \pi_2, \pi_2'\}$ be the corresponding L-packet of G. Then $c(\pi_1) = c(\pi_1') = c(\pi_2) = c(\pi_2') = 2$. Moreover,

(i) *Let η be any character such that $\eta(-1) = \omega_\sigma(-1)$. If π denotes any representation in the L-packet, then $\pi_\eta^{K_1} = \pi_\eta^{K_1'} = (0)$.*

(ii) *Let η be any character such that $\eta(-1) = \omega_\sigma(-1)$ and $c(\eta) \leq 1$ then for all $m \geq 2$ we have*

(a) $\dim((\pi_1)_\eta^{K_m'}) = \dim((\pi_2)_\eta^{K_m'}) = \dim((\pi_1')_\eta^{K_m}) = \dim((\pi_2')_\eta^{K_m}) = \left\lceil \frac{m-1}{2} \right\rceil$,

(b) $\dim((\pi_1)_\eta^{K_m}) = \dim((\pi_1')_\eta^{K_m'}) = \dim((\pi_2)_\eta^{K_m}) = \dim((\pi_2')_\eta^{K_m'}) = \left\lfloor \frac{m-1}{2} \right\rfloor$.

PROPOSITION 3.3.6 (Test vectors for unramified supercuspidal L-packets of cardinality four)

With notation as above let $\{\pi_1, \pi_1', \pi_2, \pi_2'\}$ be the unramified supercuspidal L-packet of cardinality four. Let $\overline{\psi}$ be the character of $\mathbb{F}_q$ obtained from ψ by identifying $\mathbb{F}_q$ with $\mathcal{P}^{-1}/\mathcal{O}$. Without loss of generality assume that σ_1 is $\overline{\psi}$-generic. Then

(i) *π_1' is ψ-generic, π_1 is ψ_ϖ-generic, π_2' is ψ_ϵ-generic, and π_2 is $\psi_{\epsilon\varpi}$-generic.*

(ii) *Assume that $\widetilde{\pi}$ is realized in its ψ-Kirillov model. The function ϕ_1 of Proposition* 3.3.4 *is a newform for π_1'. This further implies that $\widetilde{\pi}(\alpha)(\phi_1)$ is a newform for π_1, $\widetilde{\pi}(\gamma)(\phi_1)$ is a newform for π_2' and $\widetilde{\pi}(\alpha\gamma)(\phi_1)$ is a newform for π_2. Finally, each of these newforms is a test vector for an appropriate Whittaker functional coming from* (i).

PROPOSITION 3.3.7 (Ramified supercuspidal L-packets)

Let $\{\pi_1, \pi_2\}$ be a ramified supercuspidal L-packet of level l as above. Then $c(\pi_1) = c(\pi_2) = 2l + 1$. Moreover,

(i) *For any character η of F^* such that $\eta(-1) = \omega_\sigma(-1)$ we have $(\pi_1)_\eta^{K_{2l}} = (\pi_2)_\eta^{K_{2l}} = (\pi_1)_\eta^{K_{2l}'} = (\pi_2)_\eta^{K_{2l}'} = (0)$.*

(ii) *Let $\eta(-1) = \omega_\sigma(-1)$ and $c(\eta) \leq l$. For all $m \geq 2l + 1$ we have $\dim((\pi_1)_\eta^{K_m}) = \dim((\pi_2)_\eta^{K_m}) = \dim((\pi_1)_\eta^{K_m'}) = \dim((\pi_2)_\eta^{K_m'}) = m - 2l$.*

PROPOSITION 3.3.8 (Test vectors for ramified supercuspidal L-packets)

Let $\{\pi_1, \pi_2\}$ be a ramified supercuspidal L-packet coming from a very cuspidal representation $\widetilde{\sigma}$ of $N_{\widetilde{G}}(\widetilde{I})$ of level $l \geq 1$. One and only one of the π_i is ψ-generic, say π_1. Then π_2 is ψ_ϵ-generic. Let $\eta = \omega_\sigma$. If ϕ_1 and ϕ_ϵ have the same meaning as in Proposition 3.3.4 (*assuming that $\widetilde{\pi}$ is realized in its Kirillov model*), *we have*

(i) $(\pi_1)_\eta^{K_{2l+1}} = \mathbb{C}\phi_1$ *and* $(\pi_2)_\eta^{K_{2l+1}} = \mathbb{C}\phi_\epsilon$.
(ii) *Any* ψ*-Whittaker functional is non-zero on* ϕ_1 *and similarly any* ψ_ϵ*-Whittaker functional is non-zero on* ϕ_ϵ.

3.4 *Comparison of conductor with other invariants*

Theorem 3.4.1. *Let* π *be an irreducible admissible representation of* $G = SL_2(F)$. *Let* $\widetilde{\pi}$ *be a representation of* $\widetilde{G} = GL_2(F)$ *whose restriction to* G *contains* π. *Assume that* $\widetilde{\pi}$ *is minimal, i.e.,* $c(\widetilde{\pi} \otimes \chi) \geq c(\widetilde{\pi})$ *for all characters* χ *of* F^*. *Then*

$$c(\pi) = c(\widetilde{\pi}).$$

The next theorem relates the conductor of a representation π of G with the depth (see [13]) $\rho(\pi)$ of π (cf. [10]).

Theorem 3.4.2 (Relation between conductor and depth). *Let* π *be an irreducible representation of* G. *Let* $\rho(\pi)$ *be the depth of* π.

(i) *If* π *is any subquotient of a principal series representation* $\pi(\chi)$, *then*

$$\rho(\pi) = \max\{c(\pi) - 1, 0\}.$$

(ii) *If* π *is an irreducible supercuspidal representation, then*

$$\rho(\pi) = \max\left\{\frac{c(\pi) - 2}{2}, 0\right\}.$$

4. Newforms for U(1,1)

4.1 *Definitions and preliminary remarks*

We now define the basic filtration subgroups of $\bar{G}$ as we did for G in §3. Let $\bar{K}_{-1} = \bar{G}$, $\bar{K}_0 = \bar{K}$, the standard hyperspecial subgroup of $\bar{G}$, and for $m \geq 1$,

$$\bar{K}_m = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{K} : c \equiv 0 \pmod{\mathcal{P}_E^m}\right\}.$$

We let $\bar{K}'_m = \alpha^{-1}\bar{K}_m\alpha$.

Let $(\bar{\pi}, V)$ be an admissible representation of $\bar{G}$ such that $\bar{Z}$ acts by scalars on V. Let $\bar{\eta}$ be a character of $\mathcal{O}_E^\times$ such that $\bar{\eta}|_{E^1} = \omega_{\bar{\pi}}$ (where we have identified $\bar{Z}$ with E^1).

For any such character $\bar{\eta}$ and any subgroup $\bar{H}$ of $\bar{G}$ we define

$$\bar{\pi}_{\bar{\eta}}^{\bar{H}} := \left\{v \in V : \bar{\pi}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) v = \bar{\eta}(d)v, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{H}\right\}.$$

We define the $\bar{\eta}$*-conductor* $c_{\bar{\eta}}(\bar{\pi})$ of $\bar{\pi}$ to be

$$c_{\bar{\eta}}(\bar{\pi}) = \min\{m : \bar{\pi}_{\bar{\eta}}^{\bar{K}_m} \neq (0) \quad \text{or} \quad \bar{\pi}_{\bar{\eta}}^{\bar{K}'_m} \neq (0)\}.$$

We define the *conductor* $c(\bar{\pi})$ of $\bar{\pi}$ as

$$c(\bar{\pi}) = \min\{c_{\bar{\eta}}(\bar{\pi}) : \bar{\eta}|_{E^1} = \omega_{\bar{\pi}}\}. \tag{4.1.1}$$

If $\bar{\eta}$ is such that $c_{\bar{\eta}}(\bar{\pi}) = c(\bar{\pi})$ and $\bar{\pi}_{\bar{\eta}}^{\bar{K}_{c(\bar{\pi})}} \neq (0)$ (resp. $\bar{\pi}_{\bar{\eta}}^{\bar{K}'_{c(\bar{\pi})}} \neq (0)$), then we call $\bar{\pi}_{\bar{\eta}}^{\bar{K}_{c(\bar{\pi})}}$ (resp. $\bar{\pi}_{\bar{\eta}}^{\bar{K}'_{c(\bar{\pi})}}$) a *space of newforms* of $\bar{\pi}$. In this case, we refer to a non-zero element of $\bar{\pi}_{\bar{\eta}}^{\bar{K}_{c(\bar{\pi})}}$ or $\bar{\pi}_{\bar{\eta}}^{\bar{K}'_{c(\bar{\pi})}}$ as a *newform* of $\bar{\pi}$.

In this section, we will compute the dimension of $\bar{\pi}_{\bar{\eta}}^{\bar{K}_m}$ for every irreducible admissible infinite-dimensional representation $\bar{\pi}$ of $\bar{G}$ and every character $\bar{\eta}$ such that $c_{\bar{\eta}}(\bar{\pi}) = c(\bar{\pi})$.

We will often make use of the following fact. Let π be the restriction of $\bar{\pi}$ to G, and let $\eta = \bar{\eta}|_{\mathcal{O}_F^\times}$. By definition, the group K_m acts on $\bar{\pi}_\eta^{K_m}$ via the character η, hence via $\bar{\eta}$. Also, $\bar{Z}$ acts on $\bar{\pi}_\eta^{K_m}$ via the character $\omega_{\bar{\pi}}$, hence via $\bar{\eta}$ since $\bar{\eta}|_{E^1} = \omega_{\bar{\pi}}$. Thus any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \bar{Z}K_m$ acts on $\bar{\pi}_\eta^{K_m}$ by multiplication by $\bar{\eta}(d)$. In the light of (2.1.2),

$$\begin{aligned} \bar{K}_m/\bar{Z}K_m = \bar{T}_0/\bar{Z}T_0 &\simeq \mathcal{O}_E^\times/E^1\mathcal{O}_F^\times \simeq N_{E/F}(\mathcal{O}_E^\times)/N_{E/F}(E^1\mathcal{O}_F^\times) \\ &= \mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2. \end{aligned} \tag{4.1.2}$$

We may therefore take 1 and $\theta = \begin{pmatrix} \epsilon_E & 0 \\ 0 & {}^s\epsilon_E^{-1} \end{pmatrix}$ as coset representatives for $\bar{K}_m/\bar{Z}K_m$. Hence if $v \in \bar{\pi}_\eta^{K_m}$, then $v \in \bar{\pi}_{\bar{\eta}}^{\bar{K}_m}$ if and only if $\bar{\pi}(\theta)v = \bar{\eta}({}^s\epsilon_E^{-1})v$, i.e.,

$$\bar{\pi}_{\bar{\eta}}^{\bar{K}_m} = \left\{ v \in \bar{\pi}_\eta^{K_m} : \bar{\pi}(\theta)v = \bar{\eta}({}^s\epsilon_E^{-1})v \right\}. \tag{4.1.3}$$

4.2 *Principal series representations*

Let $\bar{\chi}$ be a character of E^*. Let $\bar{\pi}(\bar{\chi})$ denote the principal series $\mathrm{Ind}_{\bar{B}}^{\bar{G}}(\bar{\chi})$. According to [17, §11.1], $\bar{\pi}(\bar{\chi})$ is irreducible except in the cases

(i) $\bar{\chi}|_{F^*} = |\cdot|_F^{\pm}$,
(ii) $\bar{\chi}|_{F^*} = \omega_{E/F}$.

In case (i), let μ be the character of E^1 defined by $\mu(a/{}^sa) = \bar{\chi}|\cdot|_F^{\mp}$. Then $\bar{\pi}(\bar{\chi})$ has two Jordan–Hölder constituents, namely the one-dimensional representation $\xi = \mu \circ \det$ and a square integrable representation $\mathrm{St}(\xi)$. In case (ii), $\bar{\pi}(\bar{\chi})$ is the direct sum of two irreducible representations $\bar{\pi}^1(\bar{\chi})$ and $\bar{\pi}^2(\bar{\chi})$, which together form an L-packet of $\bar{G}$. We distinguish $\bar{\pi}^1(\bar{\chi})$ from $\bar{\pi}^2(\bar{\chi})$ by defining $\bar{\pi}^1(\bar{\chi})$ to be the summand that has a K-spherical vector, hence $\bar{\pi}^i(\bar{\chi})|_G = \pi_E^i$.

Let $\chi = \bar{\chi}|_{F^*}$. Then the restriction of $\bar{\pi}(\bar{\chi})$ to G is isomorphic to $\pi(\chi)$. It is easily seen then that the restriction to G of any irreducible constituent of $\bar{\pi}(\bar{\chi})$ is itself irreducible unless χ is the character corresponding to some ramified quadratic extension E'/F. In this case $\bar{\pi}(\bar{\chi})|_G$ decomposes as the direct sum $\pi_{E'}^1 \oplus \pi_{E'}^2$. We now compute the conductors of the representations in the principal series of $\bar{G}$.

Theorem 4.2.1 (Conductors for principal series representations). *Let $\bar{\chi}$ be a character of E^*. Suppose that $\bar{\pi}$ is an irreducible constituent of the principal series $\bar{\pi}(\bar{\chi})$.*

(i) *If* $\bar\eta$ *is a character of* $\mathcal{O}_E^\times$ *with* $\bar\eta|_{E^1} = \omega_{\bar\pi}$, *then* $c_{\bar\eta}(\bar\pi) = c(\bar\pi)$ *if and only if* $\bar\eta = \bar\chi|_{\mathcal{O}_E^\times}$ *or* ${}^s\bar\chi^{-1}|_{\mathcal{O}_E^\times}$. *Moreover,*

$$c(\bar\pi) = \begin{cases} c(\bar\chi|_{F^*}), & \text{if } \bar\pi \neq \mathrm{St}(\xi), \\ 1, & \text{if } \bar\pi = \mathrm{St}(\xi). \end{cases}$$

(ii) *Suppose* $\bar\eta$ *is as above.*

(a) *If* $\bar\pi = \bar\pi(\bar\chi)$, $\bar\chi$ *is ramified, and* $\bar\chi|_{\mathcal{O}_E^\times} = {}^s\, \bar\chi^{-1}|_{\mathcal{O}_E^\times}$, *then*

$$\dim(\bar\pi_{\bar\eta}^{\bar K_m}) = \begin{cases} 0, & \text{if } m = 0, \\ m+1, & \text{if } m > 0. \end{cases}$$

(b) *For* $\bar\pi = \bar\pi^1(\bar\chi), \bar\pi^2(\bar\chi)$, *we have*

$$\dim\left(\bar\pi^1(\bar\chi)_{\bar\eta}^{\bar K_m}\right) = \dim\left(\bar\pi^2(\bar\chi)_{\bar\eta}^{\bar K'_m}\right) = \left\lceil \frac{m+1}{2} \right\rceil,$$

$$\dim\left(\bar\pi^2(\bar\chi)_{\bar\eta}^{\bar K_m}\right) = \dim\left(\bar\pi^1(\bar\chi)_{\bar\eta}^{\bar K'_m}\right) = \left\lceil \frac{m}{2} \right\rceil.$$

(c) *In all other cases,*

$$\dim \bar\pi_{\bar\eta}^{\bar K_m} = \max\{m - c(\bar\pi) + 1, 0\}.$$

Proof. We may assume without loss of generality that $\bar\chi$ is chosen so that $\bar\pi$ is a subrepresentation of $\bar\pi(\bar\chi)$. Let π be the restriction of $\bar\pi$ to G. Let $\bar\eta$ be any character of $\mathcal{O}_E^\times$ with $\bar\eta|_{E^1} = \omega_{\bar\pi}$. Let $\eta = \bar\eta|_{\mathcal{O}_F^\times}$. Since $\bar\pi_{\bar\eta}^{\bar K_m} \subset \pi_\eta^{K_m}$,

$$c_{\bar\eta}(\bar\pi) \geq c_\eta(\pi) \geq c(\pi).$$

We claim that $c_{\bar\eta}(\bar\pi) = c(\pi)$ precisely for $\bar\eta = \bar\chi$ or ${}^s\bar\chi^{-1}$. The first part of (i) follows immediately from this claim, and the second follows from this together with the conductor calculations in §3.2.

Let $c = c(\pi)$. The only η such that $c_\eta(\pi) = c(\pi)$ are $\bar\chi|_{\mathcal{O}_F^\times}$ and $\bar\chi^{-1}|_{\mathcal{O}_F^\times}$. Hence we cannot have $c_{\bar\eta}(\bar\pi) = c(\pi)$ unless $\bar\eta$ equals $\bar\chi^\pm$ on $\mathcal{O}_F^\times$. We first prove that $\bar\pi_{\bar\eta}^{\bar K_c} \neq (0)$ if and only if $\bar\eta = \bar\chi|_{\mathcal{O}_E^\times}$ or ${}^s\bar\chi^{-1}|_{\mathcal{O}_E^\times}$ in the case where $\bar\eta|_{\mathcal{O}_F^\times} = \eta = \bar\chi|_{\mathcal{O}_F^\times}$ and $\bar\pi \neq \bar\pi^2(\bar\chi)$.

Since $\pi_\eta^{K_c}$ is contained in the restriction of $\bar\pi(\bar\chi)$ to G, which is isomorphic to $\pi(\bar\chi|_{F^*})$, it is an easy consequence of the proofs of the statements in §3.2 (see [11]) that

$$\pi_\eta^{K_c} = \begin{cases} \mathbb{C}\bar f_w, & \text{if } \bar\chi^2|_{\mathcal{O}_F^\times} \neq \mathbf{1}, \\ \mathbb{C}\bar f_w + \mathbb{C}\bar f_1, & \text{if } \bar\chi^2|_{\mathcal{O}_F^\times} = \mathbf{1}, \end{cases}$$

where

$$\bar f_w(g) = \begin{cases} 0, & \text{if } g \notin \bar B w K_c, \\ \bar\chi(t)|t|_E^{1/2}\eta(d) = \bar\chi(td)|t|_E^{1/2}, & \text{if } g = \begin{pmatrix} t & * \\ 0 & {}^s t^{-1} \end{pmatrix} w \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \end{cases}$$

$$\bar f_1(g) = \begin{cases} 0, & \text{if } g \notin \bar B K_c, \\ \bar\chi(t)|t|_E^{1/2}\eta(d) = \bar\chi(td)|t|_E^{1/2}, & \text{if } g = \begin{pmatrix} t & * \\ 0 & {}^s t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{cases}$$

We now determine when $\bar{f}_w, \bar{f}_1$ lie in $\bar{\pi}_{\bar{\eta}}^{\bar{K}_c}$. In the light of (4.1.3), this reduces to verifying whether $\bar{\pi}(\theta)$ acts as the scalar $\bar{\eta}({}^s\epsilon_E^{-1})$ on these vectors. It is easily checked that

$$\begin{aligned}\bar{\pi}(\theta)\bar{f}_w &= \bar{\chi}({}^s\epsilon_E^{-1})\bar{f}_w,\\ \bar{\pi}(\theta)\bar{f}_1 &= \bar{\chi}(\epsilon_E)\bar{f}_1.\end{aligned}$$

Hence $\bar{f}_w \in \bar{\pi}_{\bar{\eta}}^{\bar{K}_c}$ if and only if $\bar{\eta}({}^s\epsilon_E) = \bar{\chi}({}^s\epsilon_E)$. This is equivalent to $\bar{\eta} = \bar{\chi}|_{\mathcal{O}_E^\times}$ since $\bar{\eta}$ and $\bar{\chi}$ already agree on $\mathcal{O}_F^\times$ and E^1 (by assumption) and since ${}^s\epsilon_E$ is a representative for the non-trivial coset in $\mathcal{O}_E^\times/E^1\mathcal{O}_F^\times$. Similarly, if $\bar{\chi}^2|_{\mathcal{O}_F^\times} = \mathbf{1}$, then $\bar{f}_1 \in \bar{\pi}_{\bar{\eta}}^{\bar{K}_c}$ if and only if $\bar{\eta}(\epsilon_E) = {}^s\bar{\chi}^{-1}(\epsilon_E)$, which is equivalent to $\bar{\eta} = {}^s\bar{\chi}^{-1}|_{\mathcal{O}_E^\times}$ since $\bar{\eta}$ and ${}^s\bar{\chi}^{-1}$ already agree on $\mathcal{O}_F^\times$ and E^1 and since the non-trivial coset in $\mathcal{O}_E^\times/E^1\mathcal{O}_F^\times$ is represented by ϵ_E. Summarizing, we have that when $\bar{\pi} \neq \bar{\pi}^2(\bar{\chi})$ and $\eta = \bar{\chi}|_{\mathcal{O}_F^\times}$, $\bar{\pi}_{\bar{\eta}}^{\bar{K}_c} \neq (0)$ if and only if $\bar{\eta} = \bar{\chi}|_{\mathcal{O}_E^\times}$ or ${}^s\bar{\chi}^{-1}|_{\mathcal{O}_E^\times}$.

On the other hand, if $\eta = \bar{\chi}^{-1}|_{\mathcal{O}_F^\times}$, note that we may exchange $\bar{\chi}$ and ${}^s\bar{\chi}^{-1}$ in the above proof since $\bar{\pi}(\bar{\chi})$ and $\bar{\pi}({}^s\bar{\chi}^{-1})$ have the same constituents. (Of course, exchanging $\bar{\chi}$ and ${}^s\bar{\chi}^{-1}$ may make our assumption that $\bar{\pi}$ is a subrepresentation of $\bar{\pi}(\bar{\chi})$ false. The only case in which this matters, however, is when $\bar{\pi} = \mathrm{St}(\xi)$, and in this case we are already done since $\bar{\chi} = \bar{\chi}^{-1}$ on $\mathcal{O}_F^\times$.) Then carrying out the proof *mutatis mutandis*, we obtain again that $\bar{\pi}_{\bar{\eta}}^{\bar{K}_c} \neq (0)$ if and only if $\bar{\eta} = \bar{\chi}$ or ${}^s\bar{\chi}^{-1}$. This establishes our claim if $\bar{\pi}$ is in a singleton L-packet since for all $m \geq 0$,

$$\dim(\bar{\pi}_{\bar{\eta}}^{\bar{K}'_m}) = \dim({}^\alpha\bar{\pi}_{\bar{\eta}}^{\bar{K}_m}) = \dim(\bar{\pi}_{\bar{\eta}}^{\bar{K}_m}).$$

Finally, suppose that $\bar{\pi} = \bar{\pi}^1(\bar{\chi})$. By the above, $\big(\bar{\pi}^1(\bar{\eta})\big)_{\bar{\eta}}^{\bar{K}_0} \neq (0)$ if and only if $\bar{\eta} = \bar{\chi}$ or ${}^s\bar{\chi}^{-1}$. Also, if $\bar{\eta}$ is any character of $\mathcal{O}_E^\times$, then since $\big(\bar{\pi}^1(\bar{\chi})\big)_{\bar{\eta}}^{\bar{K}'_0} = \big({}^\alpha\bar{\pi}^1(\bar{\chi})\big)_{\bar{\eta}}^{\bar{K}_0}$ and ${}^\alpha\bar{\pi}^1(\bar{\chi}) \cong \bar{\pi}^2(\bar{\chi})$, we have that

$$\dim(\bar{\pi}^1(\bar{\chi}))_{\bar{\eta}}^{\bar{K}'_0} = \dim({}^\alpha\bar{\pi}^1(\bar{\chi}))_{\bar{\eta}}^{\bar{K}_0} = \dim(\bar{\pi}^2(\bar{\chi}))_{\bar{\eta}}^{\bar{K}_0}.$$

But $\big(\bar{\pi}^2(\bar{\chi})\big)_{\bar{\eta}}^{\bar{K}_0} \subset \big(\bar{\pi}^2(\bar{\chi})\big)_{\eta}^{K_0} = (0)$ by Proposition 3.2.10 since $\mathrm{Res}_G\,\bar{\pi}^2(\bar{\chi}) \cong \pi_E^2$. Thus $\dim\big(\bar{\pi}^1(\bar{\chi})\big)_{\bar{\eta}}^{\bar{K}'_0} = 0$ so again $c_{\bar{\eta}}(\bar{\pi}) = 0 = c(\pi)$ precisely for $\bar{\eta} = \bar{\chi}$ or ${}^s\bar{\chi}^{-1}$. Finally, conjugating by α as above, one easily obtains the claim in the case $\bar{\pi} = \bar{\pi}^2(\bar{\chi})$.

We now compute the dimensions of $\bar{\pi}_{\bar{\eta}}^{\bar{K}_m}$ to prove (ii). Since $\bar{\pi}(\bar{\chi})$ and $\bar{\pi}({}^s\bar{\chi}^{-1})$ have the same irreducible constituents, we may assume that $\bar{\eta} = \bar{\chi}|_{\mathcal{O}_E^\times}$. (As above, the representations $\mathrm{St}(\xi)$ present no problem here since in this case $\bar{\chi} = {}^s\bar{\chi}^{-1}$ on $\mathcal{O}_E^\times$.)

If $\bar{\pi} \neq \bar{\pi}^2(\bar{\chi})$ the proof of (i) shows that $\bar{\pi}_{\bar{\eta}}^{\bar{K}_c} = \pi_\eta^{K_c}$. Thus $\dim\bar{\pi}_{\bar{\eta}}^{\bar{K}_c}$ is 1 if $\bar{\chi}|_{\mathcal{O}_E^\times} \neq {}^s\bar{\chi}^{-1}|_{\mathcal{O}_E^\times}$ and 2 if $\bar{\chi}|_{\mathcal{O}_E^\times} = {}^s\bar{\chi}^{-1}|_{\mathcal{O}_E^\times}$. The proof also shows that

$$\begin{aligned}\dim(\bar{\pi}^1(\bar{\chi}))_{\bar{\eta}}^{\bar{K}_0} &= \dim(\bar{\pi}^2(\bar{\chi}))_{\bar{\eta}}^{\bar{K}'_0} = 1,\\ \dim(\bar{\pi}^2(\bar{\chi}))_{\bar{\eta}}^{\bar{K}_0} &= \dim(\bar{\pi}^1(\bar{\chi}))_{\bar{\eta}}^{\bar{K}'_0} = 0.\end{aligned}$$

This shows that the formulae for the dimensions are valid when $m = c$.

Suppose that $m > c$. As with Theorem 5.3 of [16], it follows from Lemma 3.2.1 and the following proofs in §3.2 of [11] that $\pi_\eta^{K_m}$ is the direct sum of $\pi_\eta^{K_c}$ together with certain two-dimensional spaces $\pi_{\eta,i}^{K_m}$ of the form $\mathbb{C}\bar{f}_{i,1} + \mathbb{C}\bar{f}_{i,\epsilon}$ $(1 \le i \le m-c)$, where

$$\bar{f}_{i,1}(g) = \begin{cases} 0, & \text{if } g \notin \bar{B}\begin{pmatrix} 1 & 0 \\ \varpi^m & 1 \end{pmatrix} K_m, \\ \bar{\chi}(t)|t|_E^{1/2}\eta(d) = \bar{\chi}(td)|t|_E^{1/2}, & \text{if } g = \begin{pmatrix} t & * \\ 0 & {}^s t^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \end{cases}$$

$$\bar{f}_{i,\epsilon}(g) = \begin{cases} 0, & \text{if } g \notin \bar{B}\begin{pmatrix} 1 & 0 \\ \varpi^m \epsilon_F & 1 \end{pmatrix} K_m, \\ \bar{\chi}(t)|t|_E^{1/2}\eta(d) = \bar{\chi}(td)|t|_E^{1/2}, & \text{if } g = \begin{pmatrix} t & * \\ 0 & {}^s t^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i \epsilon_F & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{cases}$$

We will now verify that whenever $\pi_{\eta,i}^{K_m} \subset \pi_\eta^{K_m}$,

(1) $\pi_{\eta,i}^{K_m}$ is $\bar{K}_m$-stable, and
(2) the subspace of $\pi_{\eta,i}^{K_m}$ on which $\bar{K}_m$ acts via the character $\bar{\eta}$ is one-dimensional.

If this holds, then

$$\dim \bar{\pi}_{\bar{\eta}}^{\bar{K}_m} - \dim \bar{\pi}_{\bar{\eta}}^{\bar{K}_c} = \frac{1}{2}(\dim \pi_\eta^{K_m} - \dim \pi_\eta^{K_c}),$$

and the formulae for the dimension of $\bar{\pi}_{\bar{\eta}}^{\bar{K}_m}$ follow easily from this equation and the dimension results of §3.2. The dimension of $\bar{\pi}_{\bar{\eta}}^{\bar{K}'_m}$ is computed analogously.

We now show (1) and (2). By (4.1.3), this reduces to showing that $\pi_{\eta,i}^{K_m}$ is θ-stable, and that the subspace of $\pi_{\eta,i}^{K_m}$ on which θ acts as the scalar $\bar{\eta}({}^s\epsilon_E^{-1})$ is one-dimensional. Let δ be either 1 or ϵ. Then

$$(\bar{\pi}(\theta)\bar{f}_{i,\delta})(g) = \bar{f}_{i,\delta}(g\theta) = \bar{f}_{i,\delta}(\theta\ (\theta^{-1}g\theta)) = \bar{\chi}(\epsilon_E)\bar{f}_{i,\delta}(\theta^{-1}g\theta).$$

If $\delta = 1$, this is non-zero if and only if $\theta^{-1}g\theta \in \bar{B}\left(\begin{smallmatrix} 1 & 0 \\ \varpi^i & 1 \end{smallmatrix}\right) K_m$, i.e., if and only if $g \in \bar{B}\left(\begin{smallmatrix} 1 & 0 \\ \varpi^i\epsilon_F & 1 \end{smallmatrix}\right) K_m$. This together with the fact that $\bar{\pi}(\theta)\bar{f}_{i,1} \in \pi_\eta^{K_m}$ implies that $\bar{\pi}(\theta)\bar{f}_{i,1}$ is a multiple of $\bar{f}_{i,\epsilon}$. The exact multiple is determined by evaluating

$$\begin{aligned}(\bar{\pi}(\theta)\bar{f}_{i,1})\left(\begin{pmatrix} 1 & 0 \\ \varpi^i\epsilon_F & 1 \end{pmatrix}\right) &= \bar{f}_{i,1}\left(\begin{pmatrix} 1 & 0 \\ \varpi^i\epsilon_F & 1 \end{pmatrix}\theta\right) \\ &= \bar{f}_{i,1}\left(\begin{pmatrix} {}^s\epsilon_E^{-1} & 0 \\ 0 & \epsilon_E \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}\begin{pmatrix} \epsilon_F & 0 \\ 0 & \epsilon_F^{-1} \end{pmatrix}\right) \\ &= \bar{\chi}({}^s\epsilon_E^{-1}\epsilon_F^{-1}).\end{aligned}$$

Thus

$$\bar{\pi}(\theta)\bar{f}_{i,1} = \bar{\chi}({}^s\epsilon_E^{-1}\epsilon_F^{-1})\bar{f}_{i,\epsilon}.$$

Similarly,

$$\bar{\pi}(\theta)\bar{f}_{i,\epsilon} = \bar{\chi}(\epsilon_E)\bar{f}_{i,1}.$$

As claimed, θ stabilizes $\bar{\pi}_{\eta,i}^{K_m}$. Moreover, the characteristic polynomial of θ on this two-dimensional space is

$$X^2 - \bar{\chi}({}^s\epsilon_E^{-1}\epsilon_F^{-1})\bar{\chi}(\epsilon_E) = X^2 - \bar{\chi}({}^s\epsilon_E^{-1})^2.$$

The eigenvalues of θ on $\bar{\pi}_{\eta,i}^{K_m}$ are therefore $\pm\bar{\chi}({}^s\epsilon_E^{-1}) = \pm\bar{\eta}({}^s\epsilon_E^{-1})$. It follows that the subspace of $\pi_{\eta,i}^{K_m}$ on which θ acts as the scalar $\bar{\eta}({}^s\epsilon_E^{-1})$ is one-dimensional. $\square$

Now suppose that $\bar{\pi}$ is an irreducible representation of conductor c in the principal series of $\bar{G}$ and that $\bar{\eta}$ is such that $c_{\bar{\eta}}(\bar{\pi}) = c$. We consider the effect of the Whittaker functional Λ_ψ given by (3.2.1) on $\bar{\pi}_{\bar{\eta}}^{\bar{K}_c}$, $\bar{\pi}_{\bar{\eta}}^{\bar{K}_c'}$.

PROPOSITION 4.2.2 (Test vectors for principal series representations)

Suppose that $\bar{\pi}$ is an irreducible representation in the principal series of $\bar{G}$. Let $\bar{\eta}$ be a character of $\mathcal{O}_E^\times$ with $\bar{\eta}|_{E^1} = \omega_{\bar{\pi}}$ such that $c_{\bar{\eta}}(\bar{\pi}) = c(\bar{\pi})$. Let $\psi = \psi_F$.

(i) *If $\bar{\pi} = \bar{\pi}(\bar{\chi})$, $\bar{\chi}$ is ramified, and $\bar{\chi} = {}^s\bar{\chi}^{-1}|_{\mathcal{O}_E^\times}$, then $\bar{\pi}$ is ψ-generic. Moreover, the space of vectors $\bar{\pi}_{\bar{\eta}}^{\bar{K}_1}$ on which Λ_ψ vanishes is one-dimensional.*

(ii) *If $\bar{\pi} = \bar{\pi}^1(\bar{\chi})$, then $\bar{\pi}$ is ψ-generic and Λ_ψ is non-zero on the one-dimensional space of newforms $\bar{\pi}_{\bar{\eta}}^{\bar{K}_0}$.*

(iii) *If $\bar{\pi} = \bar{\pi}^2(\bar{\chi})$, then $\bar{\pi}$ is not ψ-generic, but it is ψ_ϖ-generic. Moreover, Λ_{ψ_ϖ} is non-zero on the one-dimensional space of newforms $\bar{\pi}_{\bar{\eta}}^{\bar{K}_0'}$.*

(iv) *In all other cases, $\bar{\pi}$ is ψ-generic. In addition, if $c = c(\bar{\pi})$, then $\Lambda_\psi(v) \neq 0$ for any newform v in $\bar{\pi}_{\bar{\eta}}^{\bar{K}_c}$.*

Proof. Let π be the restriction of $\bar{\pi}$ to G. Note that since G and $\bar{G}$ have Borel subgroups with the same unipotent radical (namely, N), the restriction of Λ_ψ to any ψ-generic component of π is a non-zero ψ-Whittaker functional on that component, while its restriction to any non-ψ-generic component is 0.

Let $c = c(\bar{\pi})$. Assume we are in case (ii), (iii), or (iv). Let $\bar{L}$ be either $\bar{K}_c$ or $\bar{K}_c'$, according to the case, and let $L = \bar{L} \cap G$, i.e., L is either K_c or K_c'. Assume that v is a non-zero vector in $\bar{\pi}_{\bar{\eta}}^{\bar{L}}$. By Theorem 4.2.1, the restriction of $\bar{\pi}$ to G is irreducible of conductor c, and $\bar{\pi}_{\bar{\eta}}^{\bar{L}} = \pi_\eta^L$ is one-dimensional. The statements in each of these cases now follow easily from the analogous results about π in §3.2.

Suppose now that $\bar{\pi} = \bar{\pi}(\bar{\chi})$ with $\bar{\chi}$ ramified and $\bar{\chi} = {}^s\bar{\chi}^{-1}$. Then π has conductor $c = 1$ and $\bar{\pi}_{\bar{\eta}}^{\bar{K}_1} = \pi_\eta^{K_1}$ has dimension 2.

If π is irreducible, then π is ψ-generic according to Corollary 3.2.7. Also, according to the proof of Theorem 4.2.1 (and using its notation), $\bar{\pi}_{\bar{\eta}}^{K_1} = \mathbb{C}f_1 \oplus \mathbb{C}f_w$. It follows from Corollary 3.2.7 that $\Lambda_\psi(f_1) \neq 0$. Since the image of Λ_ψ has dimension 1, Λ_ψ must vanish on a one-dimensional subspace of $\bar{\pi}_{\bar{\eta}}^{\bar{K}_1}$.

If π is reducible, then as discussed in §3.2, π decomposes as the direct sum of two representations π_1 and π_2. Moreover, only one of these representations, say π_1, is ψ-generic by Corollary 3.2.9. Then Λ_ψ vanishes on $(\pi_2)_\eta^{K_1} \subset \pi_\eta^{K_1} = \bar{\pi}_{\bar{\eta}}^{\bar{K}_1}$. Moreover, by Corollary 3.2.9, $\Lambda_\psi(v) \neq 0$ for all non-zero $v \in (\pi_1)_\eta^{K_1} \subset \pi_\eta^{K_1} = \bar{\pi}_{\bar{\eta}}^{\bar{K}_1}$. Hence, as in the preceding paragraph, the subspace of $\bar{\pi}_{\bar{\eta}}^{\bar{K}_1}$ on which Λ_ψ vanishes is one-dimensional. □

4.3 *Supercuspidal representations*

We now consider the supercuspidal representations of $\bar{G}$. Let $\bar{\pi}$ be such a representation. It is easily deduced from analogous results on $\widetilde{G}$ and G that $\bar{\pi}$ is compactly induced from an irreducible representation of $\bar{K}$, $\bar{K}'$, or $\bar{I}$. We will call $\bar{\pi}$ an *unramified* (*resp. ramified*) *supercuspidal representation* of $\bar{G}$ if its restriction to G contains an unramified (resp. ramified) supercuspidal representation of G.

Ramified case. Suppose first that $\bar{\pi}$ is ramified. Let π be the restriction of $\bar{\pi}$ to G. Let π_1 be an irreducible component of the restriction of $\bar{\pi}$ to G. Then π_1 is a ramified supercuspidal representation of G. We extend π_1 to a representation of $\bar{Z}G$ via the central character $\omega_{\bar{\pi}}$, also denoted by π_1. Then $\bar{\pi}$ is contained in $\mathrm{ind}_{\bar{Z}G}^{\bar{G}}\pi_1$, and the restriction of $\mathrm{ind}_{\bar{Z}G}^{\bar{G}}\pi_1$ to $\bar{Z}G$ is $\pi_1 \oplus {}^\theta\pi_1$. But conjugation by θ and γ have the same effect on G so, by the discussion in the beginning of §3.3, π_1 and $\pi_2 = {}^\theta\pi_1$ comprise an L-packet for G. Since $\pi_1 \ncong {}^\theta\pi_1$, $\mathrm{ind}_{\bar{Z}G}^{\bar{G}}\pi_1$ is irreducible and hence equal to $\bar{\pi}$. Thus $\mathrm{Res}_G\,\bar{\pi} = \pi_1 \oplus \pi_2$, where $\pi_2 \cong {}^\theta\pi_1$.

From Theorem 3.4.2, we see that the conductor of both π_1 and π_2 is $2\rho + 2$, where ρ is the depth of both π_1 and π_2. We note that the depth of a twist of $\bar{\pi}$ is no less than ρ. To see this, let x be a point in the Bruhat–Tits building of $\bar{G}$ (which is the same as that of G) and let r be a non-negative real number. Then any vector in the twist of $\bar{\pi}$ that is fixed by $\bar{G}_{x,r+}$ is fixed by $G_{x,r+}$ since $G_{x,r+} \subset \bar{G}_{x,r+}$ (see [13]). It follows that the depth of the twist of $\bar{\pi}$ is no less than the depth of its restriction to G. But this restriction is π, which has depth equal to ρ.

On the other hand, we may select a character χ of $\bar{G}$ such that $\chi^{-2} = \omega_{\bar{\pi}}$ on $E^1 \cap (1+\mathcal{P}_E)$ (viewed as a subgroup of $\bar{Z}$). If $\bar{\pi}' = \bar{\pi} \otimes \chi$, then $\omega_{\bar{\pi}\otimes\chi}$ is trivial on $E^1 \cap (1+\mathcal{P}_E)$, and it is easily seen that $\rho(\bar{\pi}') = \rho$. Define $\rho_0(\bar{\pi}) = \min\{\rho(\bar{\pi} \otimes \chi)\}$ as χ ranges over all characters of $\bar{G}$. Then we have $\rho_0(\bar{\pi}) = \rho$.

Theorem 4.3.1 (Ramified supercuspidal representation). *Let $(\bar{\pi}, V)$ be a ramified supercuspidal representation of $\bar{G}$. Let $\bar{\eta}$ be any character of $\mathcal{O}_E^\times$ with $\bar{\eta}|_{E^1} = \omega_{\bar{\pi}}$ and $c(\bar{\eta}|_{\mathcal{O}_F^\times}) \le \rho_0(\bar{\pi}) + 1/2$. Then we have $c(\bar{\pi}) = c_{\bar{\eta}}(\bar{\pi}) = 2\rho_0(\bar{\pi}) + 2$ and*

$$\dim(\bar{\pi}_{\bar{\eta}}^{\bar{K}_m}) = \max\{m - c(\bar{\pi}) + 1, 0\}.$$

Proof. Let π be the restriction of $\bar{\pi}$ to G. Set $c = 2\rho_0(\bar{\pi}) + 2$ and $\eta = \bar{\eta}|_{\mathcal{O}_F^\times}$. As discussed above, the restriction of $\bar{\pi}$ to G is the direct sum of two ramified supercuspidal representations π_1, π_2, each of conductor c. By Proposition 3.3.7, $\dim(\pi_1)_\eta^{K_m} = \dim(\pi_2)_\eta^{K_m}$ is non-zero if and only if $m \ge c$. Hence if $m < c$, $\dim \bar{\pi}_{\bar{\eta}}^{\bar{K}_m} = 0$ since

$$\bar{\pi}_{\bar{\eta}}^{\bar{K}_m} \subset \pi_\eta^{K_m} = (\pi_1)_\eta^{K_m} \oplus (\pi_2)_\eta^{K_m} = (0).$$

Suppose $m \geq c$. As in §4.2, we compute $\dim \bar{\pi}_{\bar{\eta}}^{\bar{K}_m}$ using the fact (4.1.3) that $\bar{\pi}_{\bar{\eta}}^{\bar{K}_m}$ is the subspace of $\pi_\eta^{K_m}$ on which $\bar{\pi}(\theta)$ acts as the scalar $\bar{\eta}({}^s\epsilon_E^{-1})$.

Since $\pi_2 = {}^\theta\pi_1$ and the conjugation action of θ and γ are the same on G, π_1 and π_2 form an L-packet according to §3.3. Thus $\pi = \pi_1 \oplus \pi_2$ is the restriction to G of a minimal ramified supercuspidal representation $\widetilde{\pi}$ of $\widetilde{G}$. In particular, we have an action of $\widetilde{G}$ on V. Let W be the one-dimensional space $(\pi_1)_\eta^{K_c}$. Then according to the proof [11] of Proposition 3.3.7

$$(\pi_1)_\eta^{K_m} = \bigoplus_{i=0}^{m-c} \widetilde{\pi}(\beta)^i W.$$

Now $\bar{\pi}(\theta)$ intertwines ${}^\theta\pi_1$ and π_2 and takes $(\pi_1)_\eta^{K_m} = \left({}^\theta\pi_1\right)_\eta^{K_m}$ to $(\pi_2)_\eta^{K_m}$. Therefore,

$$(\pi_2)_\eta^{K_m} = \bigoplus_{i=0}^{m-c} \bar{\pi}(\theta)\widetilde{\pi}(\beta)^i W.$$

Let $W(i) = \widetilde{\pi}(\beta)^i W \oplus \bar{\pi}(\theta)\widetilde{\pi}(\beta)^i W$ for $i = 0, \ldots, m-c$.

Note that

$$\begin{aligned}\bar{\pi}(\theta)^2 = \bar{\pi}(\theta^2) &= \bar{\pi}\left(\begin{pmatrix} \epsilon_E{}^s\epsilon_E^{-1} & 0 \\ 0 & \epsilon_E{}^s\epsilon_E^{-1}\end{pmatrix}\begin{pmatrix} \epsilon_F & 0 \\ 0 & \epsilon_F^{-1}\end{pmatrix}\right) \\ &= \omega_{\bar{\pi}}(\epsilon_E/{}^s\epsilon_E)\bar{\pi}\left(\begin{pmatrix} \epsilon_F & 0 \\ 0 & \epsilon_F^{-1}\end{pmatrix}\right).\end{aligned}$$

Thus $\bar{\pi}(\theta)^2$ acts via the scalar $\omega_{\bar{\pi}}(\epsilon_E/{}^s\epsilon_E)\eta(\epsilon_F^{-1}) = \bar{\eta}({}^s\epsilon_E^{-1})^2$ on $\bar{\pi}_\eta^{K_m}$. It follows that $\bar{\pi}(\theta)$ exchanges the one-dimensional spaces $\widetilde{\pi}(\beta)^i W, \bar{\pi}(\theta)\widetilde{\pi}(\beta)^i W$ since

$$\begin{aligned}&\bar{\pi}(\theta)(\widetilde{\pi}(\beta)^i W) = \bar{\pi}(\theta)\widetilde{\pi}(\beta)^i W, \\ &\bar{\pi}(\theta)(\bar{\pi}(\theta)\widetilde{\pi}(\beta)^i W) = \bar{\pi}(\theta)^2(\widetilde{\pi}(\beta)^i W) = \bar{\eta}({}^s\epsilon_E^{-1})^2\widetilde{\pi}(\beta)^i W = \widetilde{\pi}(\beta)^i W.\end{aligned}$$

In particular, each $W(i)$ is stabilized by $\bar{\pi}(\theta)$. Moreover, since $\bar{\pi}(\theta)^2$ acts via the scalar $\bar{\eta}({}^s\epsilon_E^{-1})^2$ on $W(i)$, the eigenspaces of $\bar{\pi}(\theta)$ on $W(i)$ corresponding to the eigenvalues $\pm\bar{\eta}({}^s\epsilon_E^{-1})$ must each be one-dimensional. Hence the subspace of

$$\pi_\eta^{K_m} = (\pi_1)_\eta^{K_m} \oplus (\pi_2)_\eta^{K_m} = \bigoplus_{i=0}^{m-c} W(i)$$

on which $\bar{\pi}(\theta)$ acts via the scalar $\bar{\eta}({}^s\epsilon_E^{-1})$ has dimension $m-c+1$, as required. □

Unramified case. Suppose that $(\bar{\pi}, V)$ is an unramified supercuspidal representation induced from a representation $\bar{\sigma}$ of $\bar{K}$. It is easily seen that the restriction π of $\bar{\pi}$ to G is either

(i) an irreducible unramified supercuspidal representation of G induced from K if the restriction of $\bar{\sigma}$ to K is irreducible, or

(ii) the direct sum of two irreducible unramified supercuspidal representations of G induced from K if the restriction of $\bar{\sigma}$ to K is isomorphic to $\sigma_1 \oplus \sigma_2$, where σ_1 and σ_2 come from the two cuspidal representations of $SL_2(\mathbb{F}_q)$ of dimension $(q-1)/2$ (as in §3.3).

In case (ii), we note that if π decomposes into the direct sum of (π_1, V_1) and (π_2, V_2), then $V_2 = \widetilde{\pi}(\gamma)V_1$.

As discussed in the ramified case, if $\rho_0(\bar{\pi}) = \min\{\rho(\bar{\pi} \otimes \chi)\}$ as χ ranges over all characters of $\bar{G}$, then the conductors of the components of π are $2\rho_0(\bar{\pi}) + 2$.

Theorem 4.3.2 (Unramified supercuspidal representation). *Let $(\bar{\pi}, V)$ be an unramified supercuspidal representation of $\bar{G}$ that is induced from $\bar{K}$, and let $\bar{\pi}' = {}^{\alpha}\bar{\pi}$. Let $\bar{\eta}$ be any character of $\mathcal{O}_E^{\times}$ with $\bar{\eta}|_{E^1} = \omega_{\bar{\pi}}$ and $c(\bar{\eta}|_{\mathcal{O}_F^{\times}}) \leq \rho_0(\bar{\pi}) + 1$. Then $c(\bar{\pi}) = c_{\bar{\eta}}(\bar{\pi}) = 2\rho_0(\bar{\pi}) + 2$.*

(i) *If ρ_0 is odd, then*

$$\dim(\bar{\pi}_{\bar{\eta}}^{\bar{K}_m}) = \max\left\{\left\lceil \frac{m - c(\bar{\pi}) + 1}{2} \right\rceil, 0\right\} = \dim((\bar{\pi}')_{\bar{\eta}}^{\bar{K}'_m}),$$

$$\dim(\bar{\pi}_{\bar{\eta}}^{\bar{K}'_m}) = \max\left\{\left\lceil \frac{m - c(\bar{\pi}) - 1}{2} \right\rceil, 0\right\} = \dim((\bar{\pi}')_{\bar{\eta}}^{\bar{K}_m}).$$

(ii) *If ρ_0 is even, then*

$$\dim((\bar{\pi}')_{\bar{\eta}}^{\bar{K}_m}) = \max\left\{\left\lceil \frac{m - c(\bar{\pi}) + 1}{2} \right\rceil, 0\right\} = \dim(\bar{\pi}_{\bar{\eta}}^{\bar{K}'_m}),$$

$$\dim((\bar{\pi}')_{\bar{\eta}}^{\bar{K}'_m}) = \max\left\{\left\lceil \frac{m - c(\bar{\pi}) - 1}{2} \right\rceil, 0\right\} = \dim(\bar{\pi}_{\bar{\eta}}^{\bar{K}_m}).$$

Proof. We give a proof only for Case (ii) ($\rho_0(\bar{\pi})$ even) as the proof for Case (i) is easily obtained therefrom by interchanging the representations $\bar{\pi}$ and $\bar{\pi}'$. Moreover, we prove only the first equality of each line as the second follows by conjugating by α.

Let (π', V) be the restriction of $(\bar{\pi}', V)$ to G. Set $c = 2\rho_0(\bar{\pi}') + 2$ and $\eta = \bar{\eta}|_{\mathcal{O}_F^{\times}}$. Now π' is a direct summand of the restriction to G of a minimal unramified supercuspidal representation $(\widetilde{\pi}, \widetilde{V})$ of $\widetilde{G}$. Since $\widetilde{\pi}$ is unramified, it follows from §3.3 that ${}^{\gamma}\pi'$ is isomorphic to π' and hence that $\widetilde{\pi}(\gamma)$ maps V onto V. (Here we view V as a subrepresentation of $\widetilde{V}$.)

As discussed above, π' is either an irreducible unramified supercuspidal representation of conductor c or the direct sum of two such representations (π'_1, V_1) and (π'_2, V_2), where $V_2 = \widetilde{\pi}(\gamma)V_1$. By Propositions 3.3.3 and 3.3.5, the level l of the inducing data for these representations is $c/2 = \rho_0(\bar{\pi}') + 1$. As in the ramified case, we have $\dim\left(\bar{\pi}'\right)_{\bar{\eta}}^{\bar{K}_m} = 0$ if $m < c$.

Suppose $m \geq c$. By (4.1.3), to find $\dim\left(\bar{\pi}'\right)_{\bar{\eta}}^{\bar{K}_m}$, we compute the dimension of the subspace of $(\pi')_{\eta}^{K_m}$ on which $\bar{\pi}'(\theta)$ acts as the scalar $\bar{\eta}({}^{s}\epsilon_E^{-1})$. Let $W = (\pi')_{\eta}^{K_c}$. Since $l = \rho_0(\bar{\pi}') + 1$ is odd, it follows from Propositions 3.3.3 and 3.3.5 and their proofs [11] that $\dim(W) = 2$ and

$$(\pi')_{\eta}^{K_m} = \bigoplus_{i=0}^{\lfloor (m-c)/2 \rfloor} \pi\left(\begin{pmatrix} \varpi_F & 0 \\ 0 & \varpi_F^{-1} \end{pmatrix}\right)^{i} W.$$

In fact, from the proof of Proposition 3.3.4 in [11], it follows that for a certain vector $\phi \in W$, $W = \mathbb{C}\phi \oplus \mathbb{C}\widetilde{\pi}(\gamma)\phi$. If π' is irreducible, then since conjugation by γ and θ have the same effect on G, $\widetilde{\pi}(\gamma)$ and $\bar{\pi}'(\theta)$ are both elements of the one-dimensional space $\mathrm{Hom}({}^{\theta}\pi', \pi')$. They are therefore equal up to scalars so $W = \mathbb{C}\phi \oplus \mathbb{C}\bar{\pi}'(\theta)\phi$. If π' is reducible, then we may further assume that $\phi \in W \cap V_1$ by Proposition 3.3.6. In this case, $\widetilde{\pi}(\gamma)$ and $\bar{\pi}'(\theta)$ are both elements of the one-dimensional space $\mathrm{Hom}({}^{\theta}\pi'_1, \pi'_2)$ so $W = \mathbb{C}\phi \oplus \mathbb{C}\bar{\pi}'(\theta)\phi$ as above.

As in the ramified case, $\bar{\pi}'(\theta)^2$ acts via the scalar $\bar{\eta}({}^{s}\epsilon_E^{-1})^2$ on $\left(\bar{\pi}'\right)_{\eta}^{K_m}$. It follows that $\bar{\pi}'(\theta)$ exchanges the one-dimensional spaces $\mathbb{C}\phi$, $\bar{\pi}'(\theta)\mathbb{C}\phi$ since

$$\begin{aligned}&\bar{\pi}'(\theta)(\mathbb{C}\phi) = \mathbb{C}\bar{\pi}'(\theta)\phi,\\ &\bar{\pi}'(\theta)\left(\mathbb{C}\bar{\pi}'(\theta)\phi\right) = \bar{\pi}'(\theta)^2\left(\mathbb{C}\phi\right) = \bar{\eta}({}^{s}\epsilon_E^{-1})^2\mathbb{C}\phi = \mathbb{C}\phi.\end{aligned}$$

In particular, $\bar{\pi}'(\theta)$ stabilizes W. Again as in the ramified case, these facts imply that the eigenspaces of $\bar{\pi}'(\theta)$ on W corresponding to the eigenvalues $\pm\bar{\eta}({}^{s}\epsilon_E^{-1})$ must each be one-dimensional. The same is clearly true of $\pi\left(\begin{pmatrix}\varpi_F & 0\\ 0 & \varpi_F^{-1}\end{pmatrix}\right)^{i} W$ for $i = 0, \ldots, \lfloor(m-c)/2\rfloor$. It follows that the subspace of $(\pi')_{\eta}^{K_m}$ on which $\bar{\pi}'(\theta)$ acts via the scalar $\bar{\eta}({}^{s}\epsilon_E^{-1})$ has dimension $\lceil(m-c+1)/2\rceil$ as required.

The computation of $\dim\left(\bar{\pi}'\right)_{\bar{\eta}}^{\bar{K}'_m}$ is entirely analogous. $\square$

Now suppose that $\bar{\pi}$ is a supercuspidal representation of $\bar{G}$ of conductor c. We consider the effect of a Whittaker functional Λ_{ψ} on $\bar{\pi}_{\bar{\eta}}^{\bar{K}_c}, \bar{\pi}_{\bar{\eta}}^{\bar{K}'_c}$. For this we need to choose the character $\bar{\eta}$ somewhat carefully. Let $\bar{\Pi}$ be the L-packet of $\bar{G}$ containing $\bar{\pi}$. Then the restriction to G of the direct sum of representations in $\bar{\Pi}$ is also the restriction to G of a minimal supercuspidal representation $\widetilde{\pi}$ coming via Kutzko's construction. We require $\bar{\eta}|_{\mathcal{O}_F^{\times}} = \omega_{\widetilde{\pi}}$.

PROPOSITION 4.3.3 (Test vectors for supercuspidal representations)

Suppose that $\bar{\pi}$ is an irreducible supercuspidal representation of $\bar{G}$ of conductor c. Let $\bar{\eta}$ be a character of $\mathcal{O}_E^{\times}$ with $\bar{\eta}|_{E^1} = \omega_{\bar{\pi}}$ and $\bar{\eta}|_{\mathcal{O}_F^{\times}} = \omega_{\widetilde{\pi}}$ (see above). Let $\psi = \psi_F$.

(i) *If $\bar{\pi}$ is ramified, then $\bar{\pi}$ is ψ-generic. Moreover, $\Lambda_{\psi}(v) \neq 0$ for all non-zero v in $\bar{\pi}_{\bar{\eta}}^{\bar{K}_c}$ or $\bar{\pi}_{\bar{\eta}}^{\bar{K}'_c}$.*

(ii) *If $\bar{\pi}$ is unramified and induced from $\bar{K}$, let $\bar{\pi}' = {}^{\alpha}\bar{\pi}$.*

  (a) *If $\rho_0(\bar{\pi}) = \rho_0(\bar{\pi}')$ is odd, then $\bar{\pi}$ is ψ-generic and $\bar{\pi}'$ is ψ_{ϖ}-generic. Moreover, $\Lambda_{\psi}(v) \neq 0$ for all non-zero $v \in \bar{\pi}_{\bar{\eta}}^{\bar{K}_c}$ and $\Lambda_{\psi_{\varpi}}(v) \neq 0$ for all non-zero $v \in \left(\bar{\pi}'\right)_{\bar{\eta}}^{\bar{K}'_c}$.*

  (b) *If $\rho_0(\bar{\pi}) = \rho_0(\bar{\pi}')$ is even, then $\bar{\pi}'$ is ψ-generic and $\bar{\pi}$ is ψ_{ϖ}-generic. Moreover, $\Lambda_{\psi}(v) \neq 0$ for all non-zero $v \in \left(\bar{\pi}'\right)_{\bar{\eta}}^{\bar{K}_c}$ and $\Lambda_{\psi_{\varpi}}(v) \neq 0$ for all non-zero $v \in \bar{\pi}_{\bar{\eta}}^{\bar{K}'_c}$.*

Proof. Let π be the restriction of $\bar{\pi}$ to G. As in Proposition 4.2.2, we note that the restriction of Λ_{ψ} to any ψ-generic component of π is a ψ-Whittaker functional on that component, while its restriction to any non-ψ-generic component is 0.

Suppose first that $\bar{\pi}$ is ramified (case (i)). Then π decomposes as the direct sum $\pi_1 \oplus \pi_2$ of irreducible ramified supercuspidal representations of conductor c. By Proposition 3.3.8, only one summand, say π_1, is ψ-generic and we have that Λ_ψ is non-zero on $(\pi_1)_\eta^{K_c}$. Now $\bar{\pi}_{\bar{\eta}}^{\bar{K}'_c}$ is the space of vectors in $\pi_\eta^{K_c} = (\pi_1)_\eta^{K_c} \oplus (\pi_2)_\eta^{K_c}$ on which $\bar{\pi}(\theta)$ acts as the scalar $\bar{\eta}({}^s\epsilon_E^{-1})$. As observed in the proof of Theorem 4.3.1, $\bar{\pi}(\theta)$ exchanges $(\pi_1)_\eta^{K_c}$ and $(\pi_2)_\eta^{K_c}$. Therefore, $\bar{\pi}_{\bar{\eta}}^{\bar{K}_c}$ cannot lie in either $(\pi_1)_\eta^{K_c}$ or $(\pi_2)_\eta^{K_c}$. In particular, if $v \in \bar{\pi}_{\bar{\eta}}^{\bar{K}_c}$ is written as $v_1 + v_2$ with $v_i \in (\pi_i)_\eta^{K_c}$, then $v_1, v_2 \neq 0$. Since π_1 is ψ-generic and π_2 is not, we get

$$\Lambda_\psi(v) = \Lambda_\psi(v_1) \neq 0.$$

We now give a proof in case (ii). We only prove (a) as the proof of (b) is obtained by interchanging $\bar{\pi}$ and $\bar{\pi}'$.

Suppose that $\bar{\pi}$ is unramified and induced from $\bar{K}$ and that $\rho_0(\bar{\pi})$ is odd. Then π is also unramified, induced from K, and has conductor c. As noted in the proof of Theorem 4.3.2, $\pi_\eta^{K_c} = \mathbb{C}\phi_1 \oplus \mathbb{C}\bar{\pi}(\theta)\phi_1$. Since the level of the inducing data of π is $c/2$, which is even, π is ψ-generic by Proposition 3.3.4. Moreover, $\Lambda_\psi(\phi_1) \neq 0$, while $\Lambda_\psi(\bar{\pi}(\theta)\phi_1) = \Lambda_\psi(\phi_\epsilon) = 0$. By the proof of Theorem 4.3.1, $\bar{\pi}(\theta)$ exchanges $\mathbb{C}\phi_1$ and $\mathbb{C}\bar{\pi}(\theta)\phi_1$. Therefore, just as in the ramified case, if $v = a\phi_1 + b\bar{\pi}(\theta)\phi_1$, then $a, b \neq 0$. It follows that

$$\Lambda_\psi(v) = a\Lambda_\psi(\phi_1) \neq 0.$$

The proof of the non-vanishing of Λ_{ψ_ϖ} is entirely analogous. □

Remark 4.3.4. We have only considered the unitary group $U(1,1)$ for an unramified extension E/F. The entire series of results go through with some minor modifications if instead we considered ramified extensions.

4.4 *Comparison of conductor with other invariants*

Theorem 4.4.1 (Relation of conductor with other invariants for $\bar{G}$). *Let $\bar{\pi}$ be an irreducible admissible supercuspidal representation of $\bar{G}$. The relation between its conductor $c(\bar{\pi})$ and its minimal depth $\rho_0(\bar{\pi})$ is given by*

$$\rho_0(\bar{\pi}) = \frac{c(\bar{\pi}) - 2}{2}.$$

If π is an irreducible subrepresentation of the restriction of $\bar{\pi}$ to G then

$$c(\bar{\pi}) = c(\pi).$$

Proof. This follows from Theorems 4.3.1 and 4.3.2. □

5. Towards multiplicity one for newforms

Given an irreducible representation $\bar{\pi}$ of $\bar{G}$ and a character $\bar{\eta}$ of $\mathcal{O}_E^\times$ such that $c_{\bar{\eta}}(\bar{\pi}) = c(\bar{\pi})$, one can ask if we have $\dim(V_{\bar{\eta}}^{\bar{K}_{c(\bar{\pi})}}) = 1$. The answer is that this is often the case but is not true in general. Indeed, we have $\dim(V_{\bar{\eta}}^{\bar{K}_{c(\bar{\pi})}}) = 1$ unless $\bar{\pi}$ is the principal series representation $\bar{\pi}(\bar{\chi})$, where $\bar{\chi}$ is ramified and $\bar{\chi}|_{\mathcal{O}_E^\times} = {}^s\bar{\chi}^{-1}|_{\mathcal{O}_E^\times}$. For these exceptional representations, the dimension of the space of newforms is two.

Nevertheless, in all cases we have proved that an appropriate Whittaker functional is non-vanishing on some newform. This can be used to formulate a kind of a multiplicity one result if we consider the quotient of the space of newforms by the kernel of this Whittaker functional. More precisely, if Ψ is a non-trivial additive character of F of conductor either $\mathcal{O}_F$ or $\mathcal{P}_F^{-1}$ such that $\bar{\pi}$ is Ψ-generic, and Λ_Ψ is a Ψ-Whittaker functional for $\bar{\pi}$, then we have

$$\dim\left(\frac{V_{\bar{\eta}}^{\bar{K}_{c(\bar{\pi})}}}{V_{\bar{\eta}}^{\bar{K}_{c(\bar{\pi})}} \cap \mathrm{kernel}(\Lambda_\Psi)}\right) = 1.$$

Another possibility is to consider some canonical non-degenerate bilinear form on the space $V^{\bar{K}_{c(\bar{\pi})}}$ and consider the orthogonal complement of the subspace $V_{\bar{\eta}}^{\bar{K}_{c(\bar{\pi})}} \cap \mathrm{kernel}(\Lambda_\Psi)$ as a candidate for a one-dimensional space of newforms. Then the multiplicity one result is formulated as

$$\dim\left(V_{\bar{\eta}}^{\bar{K}_{c(\bar{\pi})}} \cap \mathrm{kernel}(\Lambda_\Psi)^\perp\right) = 1.$$

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