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A Künneth Theorem for p-Adic Groups

In memory of my mother Shantha Anantharam.

A. Raghuram

Abstract. Let G_1 and G_2 be p-adic groups. We describe a decomposition of Ext-groups in the category of smooth representations of $G_1 \times G_2$ in terms of Ext-groups for G_1 and G_2. We comment on $\mathrm{Ext}^1_G(\pi, \pi)$ for a supercuspidal representation π of a p-adic group G. We also consider an example of identifying the class, in a suitable Ext^1, of a Jacquet module of certain representations of p-adic GL_{2n}.

1 Introduction and the Main Theorem

Let F be a non-Archimedean local field. Let G stand for the F-points of a connected reductive algebraic group defined over F. We will refer to G as a p-adic group, with the understanding that the base field F is fixed once and for all. We let $\mathfrak{R}(G)$ denote the category of smooth complex representations of G. It is well known [1] that this is an abelian category and has enough projectives and hence, given any two smooth representations π and ρ, we can compute the Ext-groups $\mathrm{Ext}^n_{\mathfrak{R}(G)}(\pi, \rho)$.

In any homological setup, it is a fundamental problem to describe the (co)homology of a product of objects in terms of those of the individual constituents. Given two p-adic groups G_1 and G_2, the Künneth theorem we prove relates extensions for the group $G_1 \times G_2$ to those of G_1 and G_2. Without further ado, we state the main theorem of this article.

Theorem 1.1 *Let G_1 and G_2 be two p-adic groups. Let M_i and N_i be smooth representations of G_i, respectively for $i = 1, 2$. Assume that M_1 and M_2 are representations of finite length. Then*

$$\mathrm{Ext}^n_{\mathfrak{R}(G_1\times G_2)}(M_1 \otimes M_2, N_1 \otimes N_2) = \bigoplus_{a_1+a_2=n} \mathrm{Ext}^{a_1}_{\mathfrak{R}(G_1)}(M_1, N_1) \otimes \mathrm{Ext}^{a_2}_{\mathfrak{R}(G_2)}(M_2, N_2).$$

(The tensors above are over $\mathbb{C}$.)

Some remarks are in order about the hypothesis of this theorem.

Remark 1.2 Specializing M_1 and M_2 to be the trivial representations of G_1 and G_2 in the above theorem yields the result of Borel–Wallach [2, Theorem X.6.1], in the case when both G_1 and G_2 are reductive p-adic groups, describing the continuous cohomology of a tensor product of representations in terms of the individual cohomologies. Our theorem therefore generalizes this theorem of Borel and Wallach. It is

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possible that the proof in [2] can be modified to our setting, however, our approach is quite different and, indeed, our proof gives another proof of their theorem in that special case. Our approach uses the results of Bernstein [1] on the structure of the smooth category which are tailor made for such homological applications.

Remark 1.3 It is not possible to relax the hypothesis that M_1 and M_2 are of finite length. For instance, the theorem is not true if we take G_1 and G_2 to be the trivial group, $N_1 = N_2 = \mathbb{C}$, M_1 and M_2 any two infinite dimensional vector spaces, because then for $n = 0$ we would have $(M_1 \otimes M_2)^* = M_1^* \otimes M_2^*$ which is not true.

The proof of Theorem 1.1 may be explained as follows. Let G be a p-adic group and let M and N be smooth representations of G. To simplify this discussion, assume that M lies in a Bernstein component $\mathfrak{R}^{\mathfrak{s}}(G)$. The first step is to remark that the representation N must lie in the same Bernstein component $\mathfrak{R}^{\mathfrak{s}}(G)$ to have non-trivial extensions. The next step is to remark that extensions in $\mathfrak{R}(G)$ are the same as those within $\mathfrak{R}^{\mathfrak{s}}(G)$. Both these steps follow directly from the Bernstein decomposition of $\mathfrak{R}(G)$. We then use the special idempotents of Bushnell and Kutzko and prove that extensions between M and N in $\mathfrak{R}^{\mathfrak{s}}(G)$ are equal to those between eM and eN in the category of left $e\mathcal{H}(G)e$-modules, where e is a special idempotent and $\mathcal{H}(G) = C_c^\infty(G)$ is the Hecke algebra of G. Applying these remarks to $G = G_1 \times G_2$, the next step is to prove that we may choose e to be of the form $e_1 \otimes e_2$, with e_i special for G_i, and hence $e\mathcal{H}(G_1 \times G_2)e = e_1\mathcal{H}(G_1)e_1 \otimes e_2\mathcal{H}(G_2)e_2$. Finally, since $e_i\mathcal{H}(G_i)e_i$ are complex unital Noetherian algebras, a classical version of the Künneth theorem completes the proof. We refer the reader to Bushnell–Kutzko [3] for details on the Bernstein decomposition. We also follow the notation therein. The proof of Theorem 1.1 is taken up in §2.

In §3 we consider a specific example applying the techniques used in §2. Let π be an irreducible supercuspidal representation of $G = \mathrm{GL}_n(F)$. Let $P = (G \times G)N$ be the (n, n) parabolic subgroup of $\mathrm{GL}_{2n}(F)$. We consider the problem of identifying the Jacquet module $\mathrm{Ind}_P^G(\pi \otimes \pi)_N$ in the space $\mathrm{Ext}^1_{G\times G}(\pi \otimes \pi, \pi \otimes \pi)$. See Conjecture 3.3. This calculation, in the context of division algebras, needed for our work [5], was our original motivation to think about the Künneth theorem. Along the way, we also prove a result identifying the dimension of $\mathrm{Ext}^1_G(\pi, \pi)$ for an irreducible supercuspidal representation π of a p-adic group G. See Proposition 3.1.

2 Proof of Theorem 1.1

Lemma 2.1 *Let G be a p-adic group and let* π_1 *and* π_2 *be two smooth representations of G. Assume that there are inertial classes* [3, §2] $\mathfrak{s}_i$ *of G such that* $\pi_i \in \mathfrak{R}^{\mathfrak{s}_i}(G)$ *for* $i = 1, 2$. *If* $\mathfrak{s}_1 \neq \mathfrak{s}_2$, *then* $\mathrm{Ext}^*_{\mathfrak{R}(G)}(\pi_1, \pi_2) = (0)$.

Proof This lemma is well known and follows from the observation that the Bernstein centre of G acts via different scalars on π_1 and π_2 (since they have distinct inertial supports) and then a lemma, classically due to Wigner [2, Theorem I.4.1], shows that π_1 and π_2 have vanishing extensions. ■

It is well known that the Bernstein component $\mathfrak{R}^{\mathfrak{s}}(G)$ is an abelian subcategory with enough projectives and so $\mathrm{Ext}^*_{\mathfrak{R}^{\mathfrak{s}}(G)}(\pi_1, \pi_2)$ makes sense in its own right.

Lemma 2.2 *Let G be a p-adic group and let $\mathfrak{s}$ be an inertial class of G. Let π_1 and π_2 be two smooth representations of G, both in $\mathfrak{R}^{\mathfrak{s}}(G)$. We have*

$$\mathrm{Ext}^n_{\mathfrak{R}(G)}(\pi_1, \pi_2) = \mathrm{Ext}^n_{\mathfrak{R}^{\mathfrak{s}}(G)}(\pi_1, \pi_2).$$

Proof The proof is a direct application of the Bernstein decomposition:

$$\mathfrak{R}(G) = \prod_{\mathfrak{t} \in \mathcal{I}(G)} \mathfrak{R}^{\mathfrak{t}}(G),$$

where $\mathfrak{t}$ runs over the set of all inertial classes of G, which we denote by $\mathcal{I}(G)$. (We differ in notation from Bushnell–Kutzko at this point, because they use $\mathcal{B}(G)$ for $\mathcal{I}(G)$, but we think that $\mathcal{B}(G)$ should be reserved exclusively for the Bruhat–Tits building associated to G.) For the proof, thinking of Ext in terms of Yoneda extensions, it is obvious that an extension of length n of π_2 by π_1 in $\mathfrak{R}^{\mathfrak{s}}(G)$ is also one in $\mathfrak{R}(G)$. Conversely, given an extension of length n in $\mathfrak{R}(G)$, one simply projects down to $\mathfrak{R}^{\mathfrak{s}}(G)$, and it is easy to check that this projection is a Yoneda equivalence (see [4, Theorem III.6.4]). ■

Now we recall the special idempotents of Bushnell–Kutzko [3, §3]. Given any idempotent $e \in \mathcal{H}(G)$, we let $\mathfrak{R}_e(G)$ denote the subcategory of $\mathfrak{R}(G)$ of all representations which are generated by their e-fixed vectors, *i.e.,* $V \in \mathfrak{R}_e(G)$ if and only if $V = \mathcal{H}(G)eV$. We say e is a *special idempotent* if $\mathfrak{R}_e(G)$ is an abelian subcategory. By [3, Proposition 3.13], given $\mathfrak{s}$, there is an idempotent $e = e_{\mathfrak{s}}$ such that $\mathfrak{R}^{\mathfrak{s}}(G) = \mathfrak{R}_e(G)$. In this case, we say that the *spectrum of e* is $\mathfrak{s}$. Hence, we now have

$$\mathrm{Ext}^n_{\mathfrak{R}(G)}(\pi_1, \pi_2) = \mathrm{Ext}^n_{\mathfrak{R}^{\mathfrak{s}}(G)}(\pi_1, \pi_2) = \mathrm{Ext}^n_{\mathfrak{R}_e(G)}(\pi_1, \pi_2).$$

Consider the functor of e-invariants from $\mathfrak{R}_e(G)$ to the category $e\mathcal{H}e$-Mod of left modules for $e\mathcal{H}e$ given by $(\pi, V) \mapsto (e\pi, eV)$. The idempotent e being special is equivalent to this functor giving a natural equivalence of categories. The functor going in the reverse direction giving this equivalence sends an $e\mathcal{H}e$-module M to $\mathcal{H}e \otimes_{e\mathcal{H}e} M$.

Lemma 2.3 *Let e be a special idempotent of G. For any two representations π_1 and π_2 in $\mathfrak{R}_e(G)$, the functor of e-invariants induces an isomorphism*

$$\mathrm{Ext}^n_{\mathfrak{R}_e(G)}(\pi_1, \pi_2) \simeq \mathrm{Ext}^n_{e\mathcal{H}e\text{-Mod}}(e\pi_1, e\pi_2).$$

Proof It suffices to observe that both the functors $V \mapsto eV$ and $M \mapsto \mathcal{H}e \otimes_{e\mathcal{H}e} M$ are exact functors. To see that the second functor is exact, one only needs to check left exactness, for which it suffices to check that if M is a nonzero $e\mathcal{H}e$ module, then $\widetilde{M} := \mathcal{H}e \otimes_{e\mathcal{H}e} M$ is nonzero. Observe that $e\widetilde{M} = e\mathcal{H}e \otimes_{e\mathcal{H}e} M \simeq M \neq (0)$ and hence $\widetilde{M} \neq (0)$. The lemma follows by applying these functors to Yoneda extensions of length n on either side. We leave the details to the reader. ■

Lemma 2.4 *Let e be a special idempotent for G with spectrum $\mathfrak{s}$. The algebra $e\mathcal{H}e$ is a Noetherian algebra.*

Proof This follows from [1, §3, Corollaire 3.4]. ∎

Consider two p-adic groups G_1 and G_2 and let $\mathfrak{s}_i = [L_i, \sigma_i]_{G_i}$ be an inertial class in G_i. (See [3, §2].) Let $G = G_1 \times G_2$ and let $\mathfrak{s} := \mathfrak{s}_1 \times \mathfrak{s}_2 = [L_1 \times L_2, \sigma_1 \otimes \sigma_2]_{G_1 \times G_2}$ be the corresponding inertial class of G. It is easy to see that every inertial class of G is of this form.

Lemma 2.5 *Let $G = G_1 \times G_2$. Let $\mathfrak{s}_i$ be an inertial class in G_i. Let $\mathfrak{s} = \mathfrak{s}_1 \times \mathfrak{s}_2$. For $i = 1, 2$, consider a special idempotent e_i for G_i with spectrum $\mathfrak{s}_i$. Then $e = e_1 \otimes e_2$ is a special idempotent for G with spectrum $\mathfrak{s}$.*

Proof It is easily checked that $\mathfrak{R}_e(G) = \mathfrak{R}^{\mathfrak{s}}(G)$ by checking that both subcategories have the same set of irreducible representations and then appealing to [3, Proposition 3.5]. Since $\mathfrak{R}^{\mathfrak{s}}(G)$ is an abelian subcategory, so is $\mathfrak{R}_e(G)$, and hence e is special. To see that both subcategories have the same set of irreducibles, observe that an irreducible representation $\pi = \pi_1 \otimes \pi_2 \in \mathfrak{R}^{\mathfrak{s}}(G)$ if and only if each $\pi_i \in \mathfrak{R}^{\mathfrak{s}_i}(G_i)$. But e_i is special with spectrum $\mathfrak{s}_i$, hence $\mathfrak{R}^{\mathfrak{s}_i}(G_i) = \mathfrak{R}_{e_i}(G_i)$. Observing that each $\pi_i \in \mathfrak{R}_{e_i}(G_i)$ if and only if $\pi \in \mathfrak{R}_e(G)$, finishes the proof.[1] ∎

The last lemma we need is a classical Künneth theorem in the context of complex unital Noetherian algebras. We state this as the following.

Lemma 2.6 *Let Λ_1 and Λ_2 be unital Noetherian $\mathbb{C}$-algebras. For $i = 1, 2$, let M_i be a left Λ_i-module of finite length and let N_i be any left Λ_i-module. We have*

$$\mathrm{Ext}^n_{\Lambda_1 \otimes \Lambda_2}(M_1 \otimes M_2, N_1 \otimes N_2) = \bigoplus_{a_1 + a_2 = n} \mathrm{Ext}^{a_1}_{\Lambda_1}(M_1, N_1) \otimes \mathrm{Ext}^{a_2}_{\Lambda_2}(M_2, N_2).$$

Proof A module for a Noetherian algebra of finite length admits a projective resolution by free modules of finite rank. Take such resolutions for M_1 and M_2; then the tensor product complex is such a resolution for $M_1 \otimes M_2$. Now apply [4, Theorem VIII.1.2] to prove the lemma. ∎

Proof of Theorem 1.1 Using the Bernstein decomposition (stated in the proof of Lemma 2.2) one can write

$$M_i = \sum_{\mathfrak{s}_i \in \mathfrak{I}(G_i)} M_i^{\mathfrak{s}_i}.$$

Since M_i is of finite length, there are only finitely many summands and each is of finite length. Note that a direct sum commutes with tensor products and that a finite direct sum also commutes with Hom and hence Ext^*. We may assume therefore that each M_i is a finite length module supported on a single inertial class, and say, $M_i \in \mathfrak{R}^{\mathfrak{s}_i}(G_i)$

[1] I thank Phil Kutzko for suggesting this proof.

for $i = 1, 2$. Let $\mathfrak{s} = \mathfrak{s}_1 \times \mathfrak{s}_2$ be the corresponding inertial class in G. Using Lemma 2.1 we may replace each N_i by $N_i^{\mathfrak{s}_i}$ and assume that each N_i is supported on $\mathfrak{s}_i$. Let e_i be a special idempotent for G_i with spectrum $\mathfrak{s}_i$ and let $e = e_1 \otimes e_2$, which is special with spectrum $\mathfrak{s}$ by Lemma 2.5. Applying Lemmas 2.2 and 2.3 we get

$$\mathrm{Ext}^n_{\mathfrak{R}(G_1\times G_2)}(M_1 \otimes M_2, N_1 \otimes N_2) \simeq \mathrm{Ext}^n_{e\mathcal{H}(G_1\times G_2)e\text{-Mod}}(e(M_1 \otimes M_2), e(N_1 \otimes N_2)).$$

Denoting $\Lambda_i = e_i\mathcal{H}_i e_i$ we get

$$\mathrm{Ext}^n_{e\mathcal{H}(G_1\times G_2)e\text{-Mod}}(e(M_1 \otimes M_2), e(N_1 \otimes N_2)) \simeq \mathrm{Ext}^n_{\Lambda_1\otimes\Lambda_2}(e_1M_1 \otimes e_2M_2, e_1N_1 \otimes e_2N_2).$$

Note that Λ_i is a Noetherian algebra by Lemma 2.4 and that each e_iM_i is a finite length module for Λ_i. Applying Lemma 2.6, we get that the latter is isomorphic to

$$\bigoplus_{a_1+a_2=n} \mathrm{Ext}^{a_1}_{\Lambda_1}(e_1M_1, e_1N_1) \otimes \mathrm{Ext}^{a_2}_{\Lambda_2}(e_2M_2, e_2N_2).$$

Applying Lemma 2.3 and then Lemma 2.2 for each G_i we get that the above is isomorphic to

$$\bigoplus_{a_1+a_2=n} \mathrm{Ext}^{a_1}_{\mathfrak{R}(G_1)}(M_1, N_1) \otimes \mathrm{Ext}^{a_2}_{\mathfrak{R}(G_2)}(M_2, N_2).$$

This concludes the proof of Theorem 1.1. ■

3 An Example

The motivation for the Künneth theorem came from our work [5] analyzing the restriction of representations of $\mathrm{GL}_2(\mathcal{D})$ to the diagonal subgroup $\mathcal{D}^* \times \mathcal{D}^*$. For the purposes of [5], we can also argue in a different manner for the required Ext calculations, since this diagonal subgroup is compact modulo its centre. However, if one tries to pursue a similar strategy as in [5] to analyze representations of $\mathrm{GL}_4(F)$ restricted to $\mathrm{GL}_2(F)\times\mathrm{GL}_2(F)$, then the above Künneth theorem would be an essential ingredient. In this section we sketch some such calculations using the techniques of this paper. We begin with the following proposition, which is stated in a much more general setting.

Proposition 3.1 *Let F be a non-Archimedean local field and let G be the F-points of a connected reductive group defined over F. Let G^0 be the subgroup of G generated by all compact subgroups, equivalently, the intersection of the kernels of all unramified characters of G. Let r be the F-rank of the maximal central torus of G, which is also the rank of the free abelian group G/G^0. Let π be an irreducible supercuspidal representation of G. Assume that the restriction of π to G^0 is multiplicity free. Then*

$$\dim(\mathrm{Ext}^1_{\mathfrak{R}(G)}(\pi, \pi)) = r.$$

Proof Let $\mathfrak{s}$ be the inertial class of π. From Lemma 2.2, we have $\mathrm{Ext}^1_{\mathfrak{R}(G)}(\pi,\pi) = \mathrm{Ext}^1_{\mathfrak{R}^{\mathfrak{s}}(G)}(\pi,\pi)$. Instead of using special idempotents, now we use another identification of the component $\mathfrak{R}^{\mathfrak{s}}(G)$ as the category of modules over a suitable algebra. Toward this we collect some well-known facts.

Let π_0 be an irreducible subrepresentation of G^0 in the restriction of π to G^0. Let $\Pi = \mathrm{ind}^G_{G^0}(\pi_0)$ be the compact induction of π_0 to G. Let $A_{\mathfrak{s}} = \mathrm{End}_G(\Pi)$. As the notation suggests, $A_{\mathfrak{s}}$ depends only on the inertial class of π. Then Π is a progenerator for $\mathfrak{R}^{\mathfrak{s}}(G)$. Hence the category $\mathfrak{R}^{\mathfrak{s}}(G)$ is naturally equivalent to $A_{\mathfrak{s}}$-Mod, the category of modules over $A_{\mathfrak{s}}$. Further, since the restriction of π to G^0 is multiplicity free, one has $A_{\mathfrak{s}} \simeq \mathbb{C}[X_1^{\pm},\dots,X_r^{\pm}]$, the $\mathbb{C}$-algebra of Laurent polynomials in r variables.[2]

Since π is irreducible, it corresponds to a simple $A_{\mathfrak{s}}$ module, say χ_π, which is necessarily one-dimensional. The proposition follows using the well-known fact:

$$\dim(\mathrm{Ext}^1_{\mathbb{C}[X_1^{\pm},\dots,X_r^{\pm}]}(\chi_\pi,\chi_\pi)) = r. \qquad \blacksquare$$

The assumption that π restricted to G^0 is multiplicity-free is satisfied in many (possibly all) cases. It is true for $G = \mathrm{GL}_n(F)$ and more generally if the F-rank of the maximal split central torus is 1. It is also true if G is quasi-split and π is a generic, (*i.e.,* admitting a Whittaker model) supercuspidal representation.

For the rest of this section let $G = \mathrm{GL}_n(F)$ and let π be an irreducible supercuspidal representation of G. We know from the above proposition that

$$\mathrm{Ext}^1_{\mathfrak{R}(G\times G)}(\pi\otimes\pi,\pi\otimes\pi)$$

is a two dimensional space. Consider the parabolically induced representation Π of $\mathrm{GL}_{2n}(F)$ obtained by inducing $\pi\otimes\pi$ from the (n,n) parabolic subgroup P. Let $P = (G\times G)N$ be the Levi decomposition of P. The normalized Jacquet module of Π with respect to P sits in an exact sequence of $G\times G$ modules as

$$0 \to \pi\otimes\pi \to \Pi_N \to \pi\otimes\pi \to 0.$$

This sequence does not split. *We are interested in explicitly identifying the Jacquet module Π_N in the two dimensional space* $\mathrm{Ext}^1_{\mathfrak{R}(G\times G)}(\pi\otimes\pi,\pi\otimes\pi)$. Toward this, we first fix a basis for this Ext^1 space. This is given by the following proposition, which is stated in a more general setting.

Proposition 3.2 *For $1 \le i \le r$, let π_i be an irreducible supercuspidal representation of $\mathrm{GL}_{n_i}(F)$. Let $\pi = \pi_1\otimes\cdots\otimes\pi_r$ be the corresponding representation of $\mathcal{G} = \mathrm{GL}_{n_1}(F)\times\cdots\times\mathrm{GL}_{n_r}(F)$. Let W be the representation space of π. Let $\mathfrak{v}\colon F^* \to \mathbb{Z}$ be a valuation on F. Then $\mathrm{Ext}^1_{\mathfrak{R}(\mathcal{G})}(\pi,\pi)$ is an r-dimensional vector space which may be realized as the set of all short exact sequences*

$$0 \to \pi \to \pi \otimes \begin{bmatrix} 1_W & f_{(a_1,\dots,a_r)} \\ 0 & 1_W \end{bmatrix} \to \pi \to 0,$$

where $f_{(a_1,\dots,a_r)}\colon \mathcal{G} \to \mathrm{End}(W)$ is any function of the form $f_{(a_1,\dots,a_r)}(x_1,\dots,x_r) = (a_1\mathfrak{v}(\det(x_1)) + \cdots + a_r\mathfrak{v}(\det(x_r)))\mathbf{1}_W$ for an r-tuple $(a_1,\dots,a_r)$ of complex numbers.

[2]Unpublished notes of a course by Alan Roche on the Bernstein decomposition, given at the Fields Institute, University of Ottawa Workshop in May 2004.

Proof That this Ext^1 is r-dimensional follows directly from Proposition 3.1. Thinking of Ext in terms of Yoneda extensions, it is easy to see that each r-tuple $(a_1, \ldots, a_r)$ gives a short exact sequence, and distinct tuples give distinct Yoneda extensions, *i.e.*, are Yoneda inequivalent. ■

The above proposition gives an isomorphism $\mathrm{Ext}^1_{\mathfrak{R}(G\times G)}(\pi \otimes \pi, \pi \otimes \pi) \simeq \mathbb{C}^2$. Observe that the standard basis for $\mathbb{C}^2$ corresponds to the decomposition coming from the Künneth theorem.

Conjecture 3.3 *With the notations as above, the Jacquet module Π_N corresponds to the element $(1, -1) \in \mathbb{C}^2 \simeq \mathrm{Ext}^1_{\mathfrak{R}(G\times G)}(\pi \otimes \pi, \pi \otimes \pi)$.*

For $n = 1$ this conjecture can be proved using Kirillov theory for $\mathrm{GL}_2(F)$.[3] We do not know of a proof for $n > 1$. The statement also makes sense if π is an irreducible representation of $\mathcal{D}^*$, where $\mathcal{D}$ is a division algebra over F. In this case, too, we do not know of a proof and this is one of the reasons that in the main theorem of [5] we need to avoid a representation like Π for $\mathrm{GL}_2(\mathcal{D})$. We end this paper on the speculative note that a possible strategy for proving the conjecture is to transfer the entire issue to the level Hecke algebras appealing to the commutative diagrams in [3, Corollary 8.4].

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Department of Mathematics
Oklahoma State University
Stillwater, OK 74078
U.S.A.
e-mail: araghur@math.okstate.edu

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