COMPARISON RESULTS FOR CERTAIN PERIODS OF CUSP FORMS ON GL_{2n} OVER A TOTALLY REAL NUMBER FIELD

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ABSTRACT. This article grew out of my talk in 'The Legacy of Srinivasa Ramanujan' conference where I spoke about some techniques to prove algebraicity results for the special values of symmetric cube L-functions attached to the Ramanujan Δ-function. If one wishes to compare these different techniques, then one needs to compare various automorphic periods attached to the symmetric cube transfer of Δ. Motivated by this problem, in this article we provide comparison results for *Whittaker-Betti periods, Shalika-Betti periods* and *relative periods* attached to a given cohomological cuspidal automorphic representation of GL_{2n} over a totally real number field.

1. INTRODUCTION

The Ramanujan Δ-function is defined as

$$\Delta(z) \ = \ q \prod_{n=1}^{\infty} (1 - q^n)^{24} \ = \ \sum_{n=1}^{\infty} \tau(n) q^n, \quad q = e^{2\pi i z}, \quad \Im(z) > 0.$$

It is, up to scalar multiples, the unique weight 12 holomorphic cusp form for $\mathrm{SL}_2(\mathbb{Z})$. The Fourier coefficients $\tau(n)$ are multiplicative and hence the L-function of Δ, defined initially as a Dirichlet series, admits an Euler product:

$$L(s, \Delta) \ := \ \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \ = \ \prod_p (1 - \tau(p)p^{-s} + p^{11-2s})^{-1}, \quad \Re(s) \gg 0.$$

The reciprocal of the Euler factor at p may be factored as:

$$1 - \tau(p)p^{-s} + p^{11-2s} \ = \ (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s}).$$

For any $n \geq 1$, the n-th symmetric power L-function attached to Δ is defined as:

$$L(s, \mathrm{Sym}^n, \Delta) \ = \ \prod_p \prod_{j=0}^{n} \left(1 - \alpha_p^j \beta_p^{n-j} p^{-s}\right)^{-1}, \quad \Re(s) \gg 0.$$

The Langlands program predicts that this function admits an analytic continuation to an entire function, and satisfies a functional equation of the expected kind. This is known for $n \leq 4$ by the works of Hecke, Shimura, Gelbart and Jacquet, and Kim and Shahidi; furthermore, partial results are known for $5 \leq n \leq 9$. See [8] and the references therein.

Consider now the symmetric cube L-function, i.e., take $n = 3$. We are interested in algebraicity results for the special values of $L(s, \mathrm{Sym}^3, \Delta)$. The critical points for this L-function are $\{12, 13, 14, \ldots, 22\}$; the center of symmetry for the functional equation here is $s = 17$. There are several approaches to study these special values. Our general approach has been to use known cases of Langlands' principle of functoriality and the theory of cohomology of arithmetic groups to give a cohomological interpretation to certain analytic theories of L-functions. Let's adumbrate this theme: Let $\pi = \pi(\Delta)$ denote the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$, where $\mathbb{A}$ is the adèle ring of $\mathbb{Q}$. Appealing to the Langlands program, for any $n \geq 1$, let $\mathrm{Sym}^n(\pi)$ denote the n-th symmetric transfer which is expected to be a cuspidal automorphic representation of $\mathrm{GL}_{n+1}(\mathbb{A})$. The n-symmetric transfer has been proven for $n \leq 4$. The symmetric cube L-function of Δ appears as a factor in the Rankin-Selberg L-function for $\mathrm{GL}_3 \times \mathrm{GL}_2$, where on GL_3 we take $\mathrm{Sym}^2(\pi)$ and on GL_2 we take π itself; this approach was pursued in my paper [6] to get algebraicity results for $L(s, \mathrm{Sym}^3(\pi))$, which is, up to an explicit shift in the s-variable, the same as $L(s, \mathrm{Sym}^3, \Delta)$. Yet another way to get the special values is to observe that $\mathrm{Sym}^3(\pi)$ has a so-called Shalika model and for such a representation an integral representation of the standard L-function $L(s, \mathrm{Sym}^3(\pi))$ can be interpreted in cohomology (see my paper with Grobner [3]) giving special value results. Typically, one begins by identifying invariants or periods attached to representations such as $\mathrm{Sym}^3(\pi)$, and then one proves via a cohomological interpretation of L-values, that $L(s, \mathrm{Sym}^3(\pi))$ at a critical point $s = m$ is, up to rational numbers and up to archimedean factors (expected to be powers of $(2\pi i)$), given by one of these periods. This begets the problem of comparing the various periods attached to a given representation. *The purpose of this article to provide comparison results between three different families of periods attached to a given representation.*

We note that $\mathrm{Sym}^3(\pi)$ is a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A})$ of cohomological type, i.e., contributes to the cohomology of a locally symmetric space with coefficients in a local system coming from an algebraic irreducible representation of the algebraic group $\mathrm{GL}_4/\mathbb{Q}$; for more details see [8, Thm. 5.5]. More generally, consider a cohomological cuspidal automorphic representation Π of $\mathrm{GL}_{2n}(\mathbb{A}_F)$, where F is a totally real number field. We consider the following three kinds of periods attached to Π:

(1) *Whittaker-Betti periods,* $p^\epsilon(\Pi_f)$, defined by comparing rational structures on two different models for the finite part Π_f of the representation: one is a Whittaker model $W(\Pi_f)$ and the other is a cohomological model $H^{\mathtt{b}}(\Pi_f \times \epsilon)$; here ϵ is a d-tuple of signs where $d = [F : \mathbb{Q}]$ is the degree of F over $\mathbb{Q}$. The cohomology degree $\mathtt{b} = dn^2$ is the lowest degree in which GL_{2n}/F has nontrivial cuspidal cohomology. Such periods, which can be defined in the context of cusp forms on GL_m over a number field, appear in the special values of Rankin-Selberg L-functions for $\mathrm{GL}_m \times \mathrm{GL}_{m-1}$. See my paper with Shahidi [9] together with my papers [6] and [7].

(2) *Shalika-Betti periods,* $\omega^\epsilon(\Pi_f)$, defined by comparing rational structures on a Shalika model $S(\Pi_f)$, provided Π does indeed admit a Shalika model, and a cohomological model $H^{\mathtt{t}}(\Pi_f \times \epsilon)$. The cohomology degree $\mathtt{t} = d(n^2 + n - 1)$ is the top-most degree in which GL_{2n}/F has nontrivial cuspidal cohomology. Such periods appear in algebraicity results for the critical values of the standard L-function for GL_{2n}/F. See my paper

with Grobner [3]. Let us note that $\mathrm{Sym}^3(\pi(\Delta))$, which is a representation of $\mathrm{GL}_4/\mathbb{Q}$, does indeed admit a Shalika model; see [3, § 8.1].

(3) *Relative periods,* $\Omega_{\mathsf{q}}^\epsilon(\Pi_f)$ defined by comparing rational structures on two cohomological models $H^{\mathsf{q}}(\Pi_f \times \epsilon)$ and $H^{\mathsf{q}}(\Pi_f \times \epsilon')$ attached to Π_f; here $\mathsf{q} = \mathsf{b}$ the bottom degree or $\mathsf{q} = \mathsf{t}$ the top degree for nonvanishing cuspidal cohomology, and if $\epsilon = (\epsilon_v)_{v \in \Sigma_F}$ is a d-tuple of signs indexed by the set Σ_F of all real embeddings of F, then $\epsilon' := -\epsilon = (-\epsilon_v)_{v \in \Sigma_F}$ is the opposite sign. Such periods, especially for $\mathsf{q} = \mathsf{b}$, appear in ratios of successive critical values for Rankin-Selberg L-functions for $\mathrm{GL}_{2n} \times \mathrm{GL}_{2m+1}$. See my announcement with Harder [4] and the forthcoming [5].

In the main theorem of this article (see Thm. 3.1) we prove two reciprocity laws. The first one relating Whittaker-Betti periods with relative periods in degree b says that the complex number

$$\frac{p^\epsilon(\Pi_f)}{p^{-\epsilon}(\Pi_f)} \frac{1}{\Omega_{\mathsf{b}}^\epsilon(\Pi_f)}$$

is equivariant under the action of $\mathrm{Aut}(\mathbb{C})$. Since the representation Π_f is defined over a number field, one may also say that this complex number is algebraic and is equivariant under the absolute Galois group over $\mathbb{Q}$. This implies that $p^\epsilon(\Pi_f)/p^{-\epsilon}(\Pi_f) \approx \Omega_{\mathsf{b}}^\epsilon(\Pi_f)$, where, by $\approx$, we mean equality up an element of a number field $\mathbb{Q}(\Pi, \epsilon)$ which is the compositum of the rationality fields $\mathbb{Q}(\Pi)$ and $\mathbb{Q}(\epsilon)$ of Π and ϵ, respectively. The second reciprocity law looks similar and relates the Shalika-Betti periods with relative periods in degree t. We use these reciprocity laws to show that the relative periods are essentially invariant under twisting by characters; see Cor. 3.2 for a precise statement.

In § 2 we briefly recall some basics about cohomological representations and the definitions of these families of periods. The main results are stated in § 3 and their proofs are given in § 4. The proof of Thm. 3.1 is modeled along the lines of the proof of the period relations as in my paper with Shahidi [9]. One considers a certain diagram of isomorphisms of $\mathrm{GL}_{2n}(\mathbb{A}_{F,f})$-modules and the requisite reciprocity law captures the failure of commutativity this diagram up to a corresponding diagram of rational structures; see the diagrams in (4.1) and (4.5).

2. Notation and preliminaries

I will be very brief here, basically to set up the notations and define the periods. Any serious reader of this article is recommended to have by his/her side the following: my paper with Freydoon Shahidi [9] for the Whittaker-Betti periods; my paper with Harald Grobner [3] together with my paper with Wee Teck Gan [2] for the Shalika-Betti periods; and my paper with Günter Harder [4] for the relative periods.

2.1. **Cohomological representations.** (Main Reference: [8, § 3.3].) Let F be a totally real field of degree $d = [F : \mathbb{Q}]$. We identify the set S_∞ of infinite places of F with the set $\Sigma_F = \mathrm{Hom}(F, \mathbb{C}) = \mathrm{Hom}(F, \mathbb{R})$ of all real embeddings of F. Let T be the diagonal torus in the standard Borel subgroup B of upper triangular matrices in GL_{2n}/F. Let $\mu = (\mu_v)_{v \in \Sigma_F}$, a character of T, be a dominant integral weight. Let (ρ_μ, M_μ) be the algebraic finite-dimensional irreducible representation of $\mathrm{GL}_{2n}(F \otimes \mathbb{C}) = \prod_{v \in S_\infty} \mathrm{GL}_{2n}(\mathbb{C})$ with highest weight μ.

A cuspidal automorphic representation Π is said to be cohomological if there is a dominant integral weight μ such that the relative Lie algebra cohomology group

$$H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; \Pi_\infty \otimes M_\mu) \neq 0,$$

where, $\mathfrak{g}_\infty$ is the complexified Lie algebra of $G_\infty = \mathrm{GL}_{2n}(F \otimes \mathbb{R}) = \prod_{v \in S_\infty} \mathrm{GL}_{2n}(\mathbb{R})$; $K_\infty = \prod_{v \in S_\infty}(\mathrm{O}(2n)Z_{2n}(\mathbb{R}))$ is the maximal compact subgroup times the center of G_∞ and K_∞° is the connected component of the identity in K_∞; hence $K_\infty^\circ = \prod_{v \in S_\infty} \mathrm{SO}(2n)Z_{2n}(\mathbb{R})^\circ$. For a cuspidal Π as above, we will write $\Pi \in \mathrm{Cusp}(\mathrm{GL}_{2n}/F, \mu)$ to denote that it is cohomological with respect to μ-coefficients.

Given $\Pi \in \mathrm{Cusp}(\mathrm{GL}_{2n}/F, \mu)$, it is known that

$$H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; \Pi_\infty \otimes M_\mu) \neq 0 \iff dn^2 \leq \bullet \leq d(n^2 + n - 1).$$

Let $\mathtt{b} = dn^2$ denote the bottom-degree, and $\mathtt{t} = d(n^2 + n - 1)$ the top-degree for non-vanishing of the above cohomology group. Let $\mathtt{q} = \mathtt{b}$ or $\mathtt{q} = \mathtt{t}$ be either of these two extreme degrees. The group of connected components $\pi_0(K_\infty)$ acts on these relative Lie algebra cohomology groups. As a $\pi_0(K_\infty) \cong \prod_{v \in S_\infty} \mathrm{O}(2n)/\mathrm{SO}(2n)$-module we have:

$$H^\mathtt{q}(\mathfrak{g}_\infty, K_\infty^\circ; \Pi_\infty \otimes M_\mu) = \bigoplus_{\epsilon \in \widehat{\pi_0(K_\infty)}} \epsilon.$$

Let V_Π denote the unique subspace of the space of cusp forms on $\mathrm{GL}_{2n}(\mathbb{A}_F)$ realizing the representation Π. Then in degree $\mathtt{q} = \mathtt{b}$ or $\mathtt{q} = \mathtt{t}$ we have

$$H^\mathtt{q}(\mathfrak{g}_\infty, K_\infty^\circ; V_\Pi \otimes M_\mu) = \bigoplus_{\epsilon \in \widehat{\pi_0(K_\infty)}} \epsilon \otimes \Pi_f$$

as a $\pi_0(K_\infty) \times \mathrm{GL}_{2n}(\mathbb{A}_{F,f})$-module; where $\mathbb{A}_{F,f}$ is the ring of finite adèles of F. Therefore for each $\epsilon \in \widehat{\pi_0(K_\infty)}$ and an extreme degree $\mathtt{q}$, we get a *cohomological model* for the finite part of the representation which we denote:

$$H^\mathtt{q}(\Pi_f \times \epsilon).$$

There is an action of $\mathrm{Aut}(\mathbb{C})$ on all these cohomological realizations. First of all, $\mathrm{Aut}(\mathbb{C})$ acts on Σ_F by composition, or one can say $\mathrm{Aut}(\mathbb{C})$ acts on S_∞ by permutations induced by composition. Next, given a cuspidal representation Π, define an abstract representation $^\sigma\Pi = {}^\sigma\Pi_\infty \otimes {}^\sigma\Pi_f$ as in [3, § 2.5]. It is deep theorem of Clozel [1, Thm. 3.13] that if $\Pi \in \mathrm{Cusp}(\mathrm{GL}_{2n}/F, \mu)$ then $^\sigma\Pi \in \mathrm{Cusp}(\mathrm{GL}_{2n}/F, {}^\sigma\mu)$, and we have a σ-semi-linear, $\mathrm{GL}_{2n}(\mathbb{A}_{F,f})$-equivariant isomorphism:

$$\tilde\sigma : H^\mathtt{q}(\Pi_f \times \epsilon) \longrightarrow H^\mathtt{q}({}^\sigma\Pi_f \times {}^\sigma\epsilon), \quad \mathtt{q} \in \{\mathtt{b}, \mathtt{t}\}.$$

Define the rationality field $\mathbb{Q}(\Pi_f)$ as the subfield of $\mathbb{C}$ fixed by $\{\sigma \in \mathrm{Aut}(\mathbb{C}) : {}^\sigma\Pi_f = \Pi_f\}$. Similarly, define $\mathbb{Q}(\mu)$ as the subfield of $\mathbb{C}$ fixed by $\{\sigma \in \mathrm{Aut}(\mathbb{C}) : {}^\sigma\mu = \mu\}$. By strong multiplicity one for GL_n, we have $\mathbb{Q}(\mu) \subset \mathbb{Q}(\Pi_f)$. Now rename $\mathbb{Q}(\Pi_f)$ as $\mathbb{Q}(\Pi)$. For any ϵ as above, define $\mathbb{Q}(\epsilon)$ in a similar way; and $\mathbb{Q}(\Pi, \epsilon)$ is the compositum of $\mathbb{Q}(\Pi)$ and $\mathbb{Q}(\epsilon)$. We also know from Clozel, *loc. cit.,* that the cohomological models $H^\mathtt{q}(\Pi_f \times \epsilon)$ have a $\mathbb{Q}(\Pi, \epsilon)$ structure.

2.2. **Whittaker-Betti periods.** (Main Reference: [9].) Fix a non-trivial additive character $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$ as, for example, in Tate's thesis. A cuspidal automorphic representation (Π, V_Π) of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ has a global ψ-Whittaker model, and hence a local ψ_v-Whittaker model for all places v of F. This gives a Whittaker model $W(\Pi_f) := \otimes_{v\notin S_\infty} W(\Pi_v)$ for Π_f. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, as described in [9, §3.2], we get a σ-semi-linear, $\mathrm{GL}_{2n}(\mathbb{A}_{F,f})$-equivariant isomorphism:

$$\tilde{\sigma} : W(\Pi_f) \longrightarrow W(^\sigma\Pi_f).$$

In particular, this says that $W(\Pi_f)$ has a $\mathbb{Q}(\Pi_f)$-structure which we denote $W(\Pi_f)_0$; one can see that a rational structure is generated by a new-vector; see [9, Lem. 3.2].

There is a comparison isomorphism between the Whittaker model $W(\Pi_f)$ and a cohomological model $H^{\flat}(\Pi_f \times \epsilon)$, for any $\epsilon \in \widehat{\pi_0(K_\infty)}$. This isomorphism depends on a choice of a generator $[\Pi_\infty]^\epsilon$ of the one-dimensional subspace of $H^{\flat}(\mathfrak{g}_\infty, K_\infty^\circ; \Pi_\infty \otimes M_\mu)$ realizing ϵ. These generators are chosen in a compatible fashion, i.e., $[\Pi_\infty]^\epsilon$ and $[^\sigma\Pi_\infty]^{\sigma\epsilon}$ are related by the permutation action of σ on S_∞. We take these generators to be fixed for the rest of the paper. The comparison isomorphism is denoted by:

$$\mathcal{F}^\epsilon(\Pi_f) : W(\Pi_f) \longrightarrow H^{\flat}(\Pi_f \times \epsilon).$$

The rational structure on $W(\Pi_f)$ has no relation to the rational structure on $H^{\flat}(\Pi_f \times \epsilon)$. However, both sides are representation spaces for an irreducible action of $\mathrm{GL}_{2n}(\mathbb{A}_{F,f})$. There is a uniqueness principle at play in such a situation (see Waldspurger [10, Lem. I.1]) which defines certain periods: there exists $p^\epsilon(\Pi_f) \in \mathbb{C}^\times$ such that

$$(2.1) \qquad \tilde{\sigma} \circ \left(\frac{1}{p^\epsilon(\Pi_f)} \mathcal{F}^\epsilon(\Pi_f) \right) \; = \; \frac{1}{p^{\sigma\epsilon}(^\sigma\Pi_f)} \mathcal{F}^{\sigma\epsilon}(^\sigma\Pi_f) \circ \tilde{\sigma};$$

i.e., the normalized map $\frac{1}{p^\epsilon(\Pi_f)}\mathcal{F}^\epsilon(\Pi_f)$ preserves rational structures. See [9, Def./Prop. 3.3].

2.3. **Shalika-Betti periods.** (Main reference: [3].) Suppose now that the representation $\Pi \in \mathrm{Cusp}(\mathrm{GL}_{2n}/F, \mu)$ admits a Shalika model; see [3, §3]. Any discussion of a Shaika model for Π entails a choice of an additive character ψ as with Whittaker models, and a choice of a Hecke character $\eta : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ such that $\eta^n = \omega_\Pi$ the central character of Π. We may take η to be an algebraic Hecke character and then we fix our choice of η; we also have its rationality field $\mathbb{Q}(\eta)$. To emphasize the dependence on these characters, we say that Π has an (η, ψ)-Shalika model. (See [3, Rem. 3.8.2] on multiplicity one for Shalika models.) For $\sigma \in \mathrm{Aut}(\mathbb{C})$, the representation $^\sigma\Pi$ has a $(^\sigma\eta, \psi)$-Shalika model by the arithmeticity property for Shalika periods; see [2]. As described in [3, §3.7] we have a σ-semi-linear, $\mathrm{GL}_{2n}(\mathbb{A}_{F,f})$-equivariant isomorphism between the (η, ψ)-Shalika model of Π_f, denoted for brevity as $S(\Pi_f)$, and the $(^\sigma\eta, \psi)$-Shalika model $S(^\sigma\Pi_f)$ of $^\sigma\Pi_f$:

$$\tilde{\sigma} : S(\Pi_f) \longrightarrow S(^\sigma\Pi_f).$$

In particular, this says that $S(\Pi_f)$ has a $\mathbb{Q}(\Pi_f, \eta)$-structure which we denote $S(\Pi_f)_0$; furthermore, one can see that $S(\Pi_f)_0$ is generated by a new-vector; see [3, Lem. 3.8.1]. For $\epsilon \in \widehat{\pi_0(K_\infty)}$ we fix a generator (which are all compatible as before) for the the one-dimensional subspace of $H^{\mathfrak{t}}(\mathfrak{g}_\infty, K_\infty^\circ; \Pi_\infty \otimes M_\mu)$ realizing ϵ. We have a comparison isomorphism (see [3, §4.2]):

$$\Theta^\epsilon(\Pi_f) : S(\Pi_f) \longrightarrow H^{\mathfrak{t}}(\Pi_f \times \epsilon).$$

The rational structure on $S(\Pi_f)$ has no relation to the rational structure on $H^{\mathtt{t}}(\Pi_f \times \epsilon)$, and this situation defines $\omega^{\epsilon}(\Pi_f) \in \mathbb{C}^{\times}$ such that

$$(2.2) \qquad \tilde{\sigma} \circ \left(\frac{1}{\omega^{\epsilon}(\Pi_f)} \Theta^{\epsilon}(\Pi_f) \right) \;=\; \frac{1}{\omega^{\sigma_{\epsilon}}({}^{\sigma}\Pi_f)} \Theta^{\sigma_{\epsilon}}({}^{\sigma}\Pi_f) \circ \tilde{\sigma};$$

i.e., the normalized map $\frac{1}{\omega^{\epsilon}(\Pi_f)}\Theta^{\epsilon}(\Pi_f)$ preserves rational structres. See [3, Def./Prop. 4.2.1].

2.4. **Relative periods.** (Main Reference: [4, § 2].) These periods are intrinsic to theory of cohomology of arithmetic groups and their definition is independent of any transcendentally defined models like the Whittaker or the Shalika models considered above. Given ϵ and its opposite sign ϵ' we have an isomorphism of $\mathrm{GL}_{2n}(\mathbb{A}_{F,f})$-modules:

$$(2.3) \qquad \mathcal{T}_{\mathtt{q}}^{\epsilon}(\Pi_f) : H^{\mathtt{q}}(\Pi_f \times \epsilon) \longrightarrow H^{\mathtt{q}}(\Pi_f \times \epsilon'), \quad \mathtt{q} \in \{\mathtt{b}, \mathtt{t}\}.$$

(See [4, (2)].) Later, somewhat artificially, we will arrange for this isomorphism to be related to an appropriate (Whittaker or Shalika) model; such an arrangement is what will ultimately make the proof work. There is no reason for this isomorphism to preserve rational structures on either side, and this defines the relative period $\Omega_{\mathtt{q}}^{\epsilon}(\Pi_f) \in \mathbb{C}^{\times}$ as that complex number needed to modify the isomorphism to make it $\mathrm{Aut}(\mathbb{C})$-equivariant:

$$(2.4) \qquad \tilde{\sigma} \circ \left(\Omega_{\mathtt{q}}^{\epsilon}(\Pi_f)\, \mathcal{T}_{\mathtt{q}}^{\epsilon}(\Pi_f) \right) \;=\; \Omega_{\mathtt{q}}^{\sigma_{\epsilon}}({}^{\sigma}\Pi_f)\, \mathcal{T}_{\mathtt{q}}^{\sigma_{\epsilon}}({}^{\sigma}\Pi_f) \circ \tilde{\sigma}.$$

3. Statement of the main results on period relations

Theorem 3.1. *Let Π be a cohomological cuspidal automorphic representation of GL_{2n} over a totally real number field F.*

(1) *The Whittaker-Betti periods $p^{\epsilon}(\Pi_f)$ are related to the relative periods $\Omega_{\mathtt{b}}^{\epsilon}(\Pi_f)$ in degree $\mathtt{b}$ as follows:*

$$\sigma\left(\frac{p^{\epsilon}(\Pi_f)}{p^{-\epsilon}(\Pi_f)}\, \frac{1}{\Omega_{\mathtt{b}}^{\epsilon}(\Pi_f)} \right) \;=\; \frac{p^{\sigma_{\epsilon}}({}^{\sigma}\Pi_f)}{p^{-\sigma_{\epsilon}}({}^{\sigma}\Pi_f)}\, \frac{1}{\Omega_{\mathtt{b}}^{\sigma_{\epsilon}}({}^{\sigma}\Pi_f)}, \quad \forall \sigma \in \mathrm{Aut}(\mathbb{C}).$$

In particular, we have

$$\frac{p^{\epsilon}(\Pi_f)}{p^{-\epsilon}(\Pi_f)} \;\approx\; \Omega_{\mathtt{b}}^{\epsilon}(\Pi_f),$$

where, by $\approx$, we mean equality up an element of the number field $\mathbb{Q}(\Pi, \epsilon)$.

(2) *Suppose that Π admits an (η, ψ)-Shalika model. Then the Shalika-Betti periods $\omega^{\epsilon}(\Pi_f)$ are related to the relative periods $\Omega_{\mathtt{t}}^{\epsilon}(\Pi_f)$ in degree $\mathtt{t}$ as follows:*

$$\sigma\left(\frac{\omega^{\epsilon}(\Pi_f)}{\omega^{-\epsilon}(\Pi_f)}\, \frac{1}{\Omega_{\mathtt{t}}^{\epsilon}(\Pi_f)} \right) \;=\; \frac{\omega^{\sigma_{\epsilon}}({}^{\sigma}\Pi_f)}{\omega^{-\sigma_{\epsilon}}({}^{\sigma}\Pi_f)}\, \frac{1}{\Omega_{\mathtt{t}}^{\sigma_{\epsilon}}({}^{\sigma}\Pi_f)}, \quad \forall \sigma \in \mathrm{Aut}(\mathbb{C}).$$

In particular, we have

$$\frac{\omega^{\epsilon}(\Pi_f)}{\omega^{-\epsilon}(\Pi_f)} \;\approx\; \Omega_{\mathtt{t}}^{\epsilon}(\Pi_f),$$

where, by $\approx$, we mean equality up an element of the number field $\mathbb{Q}(\Pi, \epsilon, \eta)$.

Corollary 3.2 (Behaviour of relative periods under twisting by characters). *Let Π be a cohomological cuspidal automorphic representation of GL_{2n} over a totally real number field F and let χ be an algebraic Hecke character of F. Then:*

(1) *For $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have*

$$\sigma\left(\frac{\Omega_{\mathsf{b}}^{\epsilon}(\Pi_f \otimes \chi_f)}{\Omega_{\mathsf{b}}^{\epsilon\epsilon_\chi}(\Pi_f)}\right) \;=\; \frac{\Omega_{\mathsf{b}}^{\sigma_\epsilon}({}^\sigma\Pi_f \otimes {}^\sigma\chi_f)}{\Omega_{\mathsf{b}}^{\sigma_\epsilon \sigma_{\epsilon_\chi}}({}^\sigma\Pi_f)}.$$

In particular, we have

$$\Omega_{\mathsf{b}}^{\epsilon}(\Pi_f \otimes \chi_f) \;\approx\; \Omega_{\mathsf{b}}^{\epsilon\epsilon_\chi}(\Pi_f),$$

where, by $\approx$, we mean equality up to an element of $\mathbb{Q}(\Pi, \epsilon, \chi)$.

(2) *For $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have*

$$\sigma\left(\frac{\Omega_{\mathsf{t}}^{\epsilon}(\Pi_f \otimes \chi_f)}{\Omega_{\mathsf{t}}^{\epsilon\epsilon_\chi}(\Pi_f)}\right) \;=\; \frac{\Omega_{\mathsf{t}}^{\sigma_\epsilon}({}^\sigma\Pi_f \otimes {}^\sigma\chi_f)}{\Omega_{\mathsf{t}}^{\sigma_\epsilon \sigma_{\epsilon_\chi}}({}^\sigma\Pi_f)}.$$
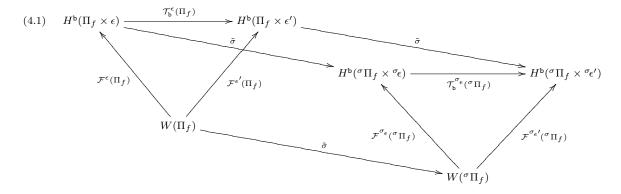
(We do not need to assume that Π has a Shalika model.) In particular, we have

$$\Omega_{\mathsf{t}}^{\epsilon}(\Pi_f \otimes \chi_f) \;\approx\; \Omega_{\mathsf{t}}^{\epsilon\epsilon_\chi}(\Pi_f),$$

where, by $\approx$, we mean equality up to an element of $\mathbb{Q}(\Pi, \epsilon, \chi)$.

4. Proofs

Lets begin with the proof of (1) of Theorem 3.1 which is a comparison between the Whittaker-Betti periods $p^\epsilon(\Pi_f)$ and the relative periods $\Omega_{\mathsf{b}}^\epsilon(\Pi_f)$ in degree b. The proof is roughly along the lines of the proof of Theorem 4.1 in my paper with Shahidi [9]. Consider the following diagram of isomorphisms of $\mathrm{GL}_{2n}(\mathbb{A}_{F,f})$-modules. *Warning:* This diagram is not commutative! Indeed the period relation captures the failure of commutativity up to a corresponding diagram of rational structures.

In this diagram, consider the composition $\tilde{\sigma} \circ \mathcal{T}_{\mathsf{b}}^\epsilon(\Pi_f) \circ \mathcal{F}^\epsilon(\Pi_f)$. We are going to evaluate this in two different ways.

On the one hand, using (2.4) for the relative periods, and then using (2.1) for the Whittaker-Betti periods, we have

$$
\tilde{\sigma} \circ \mathcal{T}_{\mathsf{b}}^{\epsilon}(\Pi_f) \circ \mathcal{F}^{\epsilon}(\Pi_f) \;\; = \;\; \frac{\Omega_{\mathsf{b}}^{\sigma_\epsilon}({}^\sigma \Pi_f)}{\sigma(\Omega_{\mathsf{b}}^{\epsilon}(\Pi_f))} \, \mathcal{T}_{\mathsf{b}}^{\sigma_\epsilon}({}^\sigma \Pi_f) \circ \tilde{\sigma} \circ \mathcal{F}^{\epsilon}(\Pi_f)
$$

(4.2)

$$
= \;\; \frac{\Omega_{\mathsf{b}}^{\sigma_\epsilon}({}^\sigma \Pi_f)}{\sigma(\Omega_{\mathsf{b}}^{\epsilon}(\Pi_f))} \, \frac{\sigma(p^{\epsilon}(\Pi_f))}{p^{\sigma_\epsilon}({}^\sigma \Pi_f)} \, \mathcal{T}_{\mathsf{b}}^{\sigma_\epsilon}({}^\sigma \Pi_f) \circ \mathcal{F}^{\sigma}\epsilon({}^\sigma \Pi_f) \circ \tilde{\sigma}.
$$

Now we come to an important point that the isomorphism $\mathcal{T}_{\mathsf{b}}^{\epsilon}(\Pi_f)$ in (2.3), needed to define the relative periods in bottom degree, may be taken to be so that

(4.3)
$$
\mathcal{T}_{\mathsf{b}}^{\epsilon}(\Pi_f) \circ \mathcal{F}^{\epsilon}(\Pi_f) = \mathcal{F}^{\epsilon'}(\Pi_f).
$$

This follows from [4, (2)]. (In the notations of [4], the Hecke summand π_f appearing rationally in inner cohomology may be identified with a rational structure $W(\Pi_f)_0$ on the Whittaker model; after making this identification over $\mathbb{Q}(\Pi_f)$, (4.3) follows from [4, (2)].) In other words, if the diagram (4.1) is thought of as a triangular prism, then the triangles at both the ends commute.

So, on the other hand, using (4.3) and (2.1), we have

$$
\tilde{\sigma} \circ \mathcal{T}_{\mathsf{b}}^{\epsilon}(\Pi_f) \circ \mathcal{F}^{\epsilon}(\Pi_f) \;\; = \;\; \tilde{\sigma} \circ \mathcal{F}^{\epsilon'}(\Pi_f)
$$

(4.4)

$$
= \;\; \frac{\sigma(p^{\epsilon'}(\Pi_f))}{p^{\sigma_{\epsilon'}}({}^\sigma \Pi_f)} \, \mathcal{F}^{\sigma_{\epsilon'}}({}^\sigma \Pi_f) \circ \tilde{\sigma}
$$

$$
= \;\; \frac{\sigma(p^{\epsilon'}(\Pi_f))}{p^{\sigma_{\epsilon'}}({}^\sigma \Pi_f)} \, \mathcal{T}_{\mathsf{b}}^{\sigma_\epsilon}({}^\sigma \Pi_f) \circ \mathcal{F}^{\sigma}\epsilon({}^\sigma \Pi_f) \circ \tilde{\sigma}.
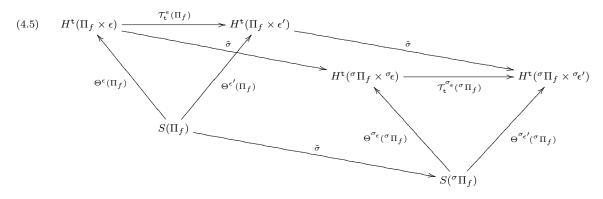$$

Comparing the right hand sides of (4.2) and (4.4) we get the following relation:

$$
\frac{\Omega_{\mathsf{b}}^{\sigma_\epsilon}({}^\sigma \Pi_f)}{\sigma(\Omega_{\mathsf{b}}^{\epsilon}(\Pi_f))} \, \frac{\sigma(p^{\epsilon}(\Pi_f))}{p^{\sigma_\epsilon}({}^\sigma \Pi_f)} \;\; = \;\; \frac{\sigma(p^{\epsilon'}(\Pi_f))}{p^{\sigma_{\epsilon'}}({}^\sigma \Pi_f)},
$$

rewriting which proves (1) of Theorem 3.1.

The proof of (2) of Theorem 3.1 is almost identical. Here we consider the diagram which is a triangular prism of $\mathrm{GL}_{2n}(\mathbb{A}_{F,f})$-module isomorphisms:

Now the composition $\tilde{\sigma} \circ \mathcal{T}_t^\epsilon(\Pi_f) \circ \Theta^\epsilon(\Pi_f)$ is written in two different ways, again while using the observation that the triangles at either end are commutative. This needs a little explanation: In the isomorphism (2.3) for top-degree relative periods, we can arrange it so that

$$(4.6) \qquad\qquad \mathcal{T}_t^\epsilon(\Pi_f) \circ \Theta^\epsilon(\Pi_f) = \Theta^{\epsilon'}(\Pi_f).$$

This follows from [4, (2)]. (But this time, in the notations of [4], the Hecke summand π_f appearing rationally in inner cohomology is identified with a rational structure $S(\Pi_f)_0$ for the Shalika model; after making this identification over $\mathbb{Q}(\Pi_f)$, (4.6) follows from [4, (2)].) We omit the rest of the details since they are absolutely similar to the proof of (1).

For the proof of (1) of Corollary 3.2, write the ratio

$$\frac{\Omega_b^\epsilon(\Pi_f \otimes \chi_f)}{\Omega_b^{\epsilon\epsilon_\chi}(\Pi_f)}$$

as a product of ratios:

$$\left(\frac{\Omega_b^\epsilon(\Pi_f \otimes \chi_f) p^{-\epsilon}(\Pi_f \otimes \chi_f)}{p^\epsilon(\Pi_f \otimes \chi_f)} \right) \left(\frac{p^\epsilon(\Pi_f \otimes \chi_f)}{p^{\epsilon\epsilon_\chi}(\Pi_f)\mathcal{G}(\chi_f)^{\frac{n(n-1)}{2}}} \right) \left(\frac{p^{-\epsilon\epsilon_\chi}(\Pi_f)\mathcal{G}(\chi_f)^{\frac{n(n-1)}{2}}}{p^{-\epsilon}(\Pi_f \otimes \chi_f)} \right) \left(\frac{p^{\epsilon\epsilon_\chi}(\Pi_f)}{p^{-\epsilon\epsilon_\chi}(\Pi_f)\Omega_b^{\epsilon\epsilon_\chi}(\Pi_f)} \right),$$

where $\mathcal{G}(\chi_f)$ is the Gauß sum of χ_f as defined in [6, §2]. The first and last factors are $\mathrm{Aut}(\mathbb{C})$-equivariant by (1) of Theorem 3.1, and the middle two factors are $\mathrm{Aut}(\mathbb{C})$-equivariant by the main theorem of my paper with Shahidi [9] on the behavior of Whittaker-Betti periods under twisting by algebraic Hecke characters, but as rewritten in [6, (2.2)]

For the proof of (2) of Corollary 3.2, we could rewrite the ratio $\Omega_t^\epsilon(\Pi_f \otimes \chi_f)/\Omega_t^{\epsilon\epsilon_\chi}(\Pi_f)$ as above in terms of the Shalika-Betti periods and then use [3, Thm. 5.2.1] about the behavior of these latter periods under twisting. However, we can also do this without recourse to Shalika models. First of all we define top-degree Whittaker-Betti periods, denoted $q^\epsilon(\Pi_f)$, by comparing $W(\Pi_f)$ with $H^t(\Pi_f \times \epsilon)$; see the last paragraph of [9]. Next, we prove a variation of (1) of Theorem 3.1, via an identical proof, to show that

$$(4.7) \qquad\qquad \frac{q^\epsilon(\Pi_f)}{q^{-\epsilon}(\Pi_f)} \frac{1}{\Omega_t^\epsilon(\Pi_f)}$$

is $\mathrm{Aut}(\mathbb{C})$-equivariant. Now (2) of Corollary 3.2 follows, exactly as in the proof of (1) of this corollary, by using (4.7) and [9, (4.6)] on the behavior of these $q^\epsilon(\Pi_f)$ under twisting.

Let me end this article by saying that I believe there should be some interesting relation between the relative periods $\Omega_b^\epsilon(\Pi_f)$ in bottom-degree and the relative periods $\Omega_t^\epsilon(\Pi_f)$ in top-degree, possibly replacing one of the Π by its contragredient representation $\Pi^\vee$, and that such a relation should follow from Poincaré duality. However, I have been unable to make this precise.

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