

**CRITICAL VALUES OF RANKIN–SELBERG L -FUNCTIONS FOR $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$
AND THE SYMMETRIC CUBE L -FUNCTIONS FOR GL_2**

A. RAGHURAM

1. Introduction and statements of results

In a previous article [35] an algebraicity result for the central critical value for L -functions for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ over \mathbb{Q} was proved assuming the validity of a nonvanishing hypothesis involving archimedean integrals. The purpose of this article is to generalize [35, Thm. 1.1] for all critical values for L -functions for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ over any number field F while using the period relations of [37] and some additional inputs as will be explained below. Thanks to a recent preprint of Binyong Sun [44], the nonvanishing hypothesis has now been proved, and so one may claim that the results of this article are unconditional. Using such results for $\mathrm{GL}_3 \times \mathrm{GL}_2$, new unconditional algebraicity result for the special values of symmetric cube L -functions for GL_2 over F have been proved. Previously, algebraicity results for the critical values of symmetric cube L -functions for GL_2 have been known only in special cases: see Garrett–Harris [12], Kim–Shahidi [25], Grobner–Raghuram [17], and Januszewski [23].

1.1. L -functions for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$. Suppose \mathbb{A}_F is the ring of adèles of F . Let Π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Such a representation contributes to the cuspidal cohomology of $G_n := \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_n/F)$ with coefficients in a sheaf $\widetilde{\mathcal{M}}_\mu$ attached to an algebraic irreducible representation \mathcal{M}_μ of G_n with highest weight μ . This information will be denoted as $\Pi \in \mathrm{Coh}(G_n, \mu)$. Similarly, let Σ be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, and let $\Sigma \in \mathrm{Coh}(G_{n-1}, \lambda)$. Consider the Rankin–Selberg L -function $L(s, \Pi \times \Sigma)$ attached to such a pair of cohomological representations (Π, Σ) . Algebraicity results for the critical values of $L(s, \Pi \times \Sigma)$ are proved under a compatibility condition on the weights μ and λ .

Take a representation $\Pi \in \mathrm{Coh}(G_n, \mu^\vee)$ as above; henceforth, as in [35], working with the dual weight μ^\vee is only for convenience. Let $\Pi = \Pi_\infty \otimes \Pi_f$ be the usual decomposition of Π into its archimedean part Π_∞ and its finite part Π_f . The rationality field of Π is denoted $\mathbb{Q}(\Pi)$; it is a number field. For a given weight μ , the representation \mathcal{M}_μ^\vee is defined over a number field $\mathbb{Q}(\mu)$, and by Clozel [7], it is known that cuspidal cohomology has a $\mathbb{Q}(\mu)$ -structure; hence the realization of Π_f as a Hecke-summand in cuspidal cohomology in lowest possible degree has a $\mathbb{Q}(\Pi)$ -structure. The choice of lowest possible degree—as will be explained below—is absolutely crucial for this paper. On the other hand, the Whittaker model $\mathcal{W}(\Pi_f)$ of the finite part of the representation admits a $\mathbb{Q}(\Pi)$ -structure. By comparing these two $\mathbb{Q}(\Pi)$ -structures, in a previous article with Shahidi [37], certain periods $p^\epsilon(\Pi) \in \mathbb{C}^\times$ were defined and studied; here $\epsilon = (\epsilon_v)_{v \in S_r}$ is a collection of signs indexed by the set S_r of real places of F . These signs can be arbitrary if n is even, and are canonically determined by Π_∞ if n is odd; if the number of real places is r_1 , then there are 2^{r_1} such periods of Π if n is

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even and only one period if n is odd. For any $\sigma \in \text{Aut}(\mathbb{C})$, one knows that ${}^\sigma\Pi \in \text{Coh}(G_n, \sigma\mu^\vee)$; indeed, one defines periods $p^\epsilon({}^\sigma\Pi)$ simultaneously for all the conjugates of Π . The collection of periods $\{p^\epsilon({}^\sigma\Pi) : \sigma \in \text{Aut}(\mathbb{C})\}$ is well-defined as an element of $(\mathbb{Q}(\Pi) \otimes \mathbb{C})^\times / \mathbb{Q}(\Pi)^\times$. When F is totally real or a totally imaginary quadratic extension of a totally real field, then the constituents of the representation at infinity of ${}^\sigma\Pi$ are, up to signs, a permutation of the constituents of Π_∞ (see [11, Prop. 3.2]); however, in the case of a general number field this poses some additional difficulties involving a careful analysis at infinity. This issue is related to the set of possible weights that can support cuspidal cohomology for GL_n/F . We call them strongly pure weights; see 2.3.4. Henceforth, let $\mu \in X_{00}^+(T_n)$ stand for a dominant integral strongly pure weight and consider $\Pi \in \text{Coh}(G_n, \mu^\vee)$. The first main theorem of this article is the following:

Theorem 1.1. *Let $\mu \in X_{00}^+(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu^\vee)$ for $n \geq 2$. Similarly, let $\lambda \in X_{00}^+(T_{n-1})$ and $\Sigma \in \text{Coh}(G_{n-1}, \lambda^\vee)$. Assume that there is an integer j such that $\mathcal{M}_{\lambda+j} = \mathcal{M}_\lambda \otimes \det^j$ appears in the restriction to G_{n-1} of \mathcal{M}_{μ^\vee} , i.e.,*

$$(1.2) \quad \{j \in \mathbb{Z} : \text{Hom}_{G_{n-1}}(\mathcal{M}_{\mu^\vee}, \mathcal{M}_{\lambda+j}) \neq (0)\} \text{ is a non-empty set.}$$

Let $s = \frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z}$ be critical for $L_f(s, \Pi \times \Sigma)$ which is the finite part of the Rankin–Selberg L -function attached to the pair (Π, Σ) . Then, there exist signs $\epsilon = (\epsilon_v)_{v \in S_r}$ and $\eta = (\eta_v)_{v \in S_r}$ with $\eta_v = (-1)^n \epsilon_v$ attached to the data $(\Pi_\infty, \Sigma_\infty)$ as in 2.5.3.5, such that:

(1) if n is even, then there exists $p_\infty^{\epsilon, \eta}(\mu + m, \lambda) \in \mathbb{C}^\times$, such that for any $\sigma \in \text{Aut}(\mathbb{C})$ we have

$$\sigma \left(\frac{L_f(\frac{1}{2} + m, \Pi \times \Sigma)}{p^{\epsilon_m \epsilon}(\Pi) p^\eta(\Sigma) \mathcal{G}(\omega_{\Sigma_f}) p_\infty^{\epsilon, \eta}(\mu + m, \lambda)} \right) = \frac{L_f(\frac{1}{2} + m, {}^\sigma\Pi \times {}^\sigma\Sigma)}{p^{\epsilon_m \epsilon}({}^\sigma\Pi) p^\eta({}^\sigma\Sigma) \mathcal{G}(\omega_{\sigma\Sigma_f}) p_\infty^{\epsilon, \eta}(\sigma\mu + m, \sigma\lambda)},$$

where $\epsilon_m = (-1)^m$ and $\mathcal{G}(\omega_{\Sigma_f})$ is the Gauss sum attached to the central character of Σ . In particular, we have

$$L_f(\frac{1}{2} + m, \Pi \times \Sigma) \sim_{\mathbb{Q}(\Pi, \Sigma)} p^{\epsilon_m \epsilon}(\Pi) p^\eta(\Sigma) \mathcal{G}(\omega_{\Sigma_f}) p_\infty^{\epsilon, \eta}(\mu + m, \lambda),$$

where, by $\sim_{\mathbb{Q}(\Pi, \Sigma)}$, we mean up to an element of the number field $\mathbb{Q}(\Pi, \Sigma)$ which is the compositum of the rationality fields $\mathbb{Q}(\Pi)$ and $\mathbb{Q}(\Sigma)$ of Π and Σ respectively.

(2) If n is odd, then there exists $p_\infty^{\epsilon, \eta}(\mu, \lambda + m) \in \mathbb{C}^\times$, such that for any $\sigma \in \text{Aut}(\mathbb{C})$ we have

$$\sigma \left(\frac{L_f(\frac{1}{2} + m, \Pi \times \Sigma)}{p^\epsilon(\Pi) p^{\epsilon_m \eta}(\Sigma) \mathcal{G}(\omega_{\Sigma_f}) p_\infty^{\epsilon, \eta}(\mu, \lambda + m)} \right) = \frac{L_f(\frac{1}{2} + m, {}^\sigma\Pi \times {}^\sigma\Sigma)}{p^\epsilon({}^\sigma\Pi) p^{\epsilon_m \eta}({}^\sigma\Sigma) \mathcal{G}(\omega_{\sigma\Sigma_f}) p_\infty^{\epsilon, \eta}(\sigma\mu, \sigma\lambda + m)}.$$

In particular,

$$L_f(\frac{1}{2} + m, \Pi \times \Sigma) \sim_{\mathbb{Q}(\Pi, \Sigma)} p^\epsilon(\Pi) p^{\epsilon_m \eta}(\Sigma) \mathcal{G}(\omega_{\Sigma_f}) p_\infty^{\epsilon, \eta}(\mu, \lambda + m).$$

Various special cases have been known: For $n = 2$ and F totally real, which is the classical Hilbert modular situation, the above theorem is due to Shiumra [43, Thm. 4.3], and for a general F , it is due to Hida [22, Thm. I (i)]. For $F = \mathbb{Q}$ and $n \geq 2$, some variation of the above theorem has appeared in Kazhdan–Mazur–Schmidt [28], Mahnkopf [33], Kasten–Schmidt [27] and Raghuram [35]. Recently, Grobner–Harris [15] proved a similar result when F is an imaginary quadratic field.

The basic idea of the proof of Theorem 1.1 involves interpreting the L -value as a Rankin–Selberg integral involving two carefully chosen cusp forms; see Prop. 2.1. This integral is then interpreted as a map in cohomology stemming from Poincaré duality; see the diagram in 2.5.1 and then see the

main identity in Thm. 2.50. The reader is also referred to the introduction of [35] where this idea is explained in greater detail.

We now address an important additional ingredient needed for the above generalization from $F = \mathbb{Q}$ to any number field. Kasten and Schmidt observed in [27, Thm. 2.3] that when $F = \mathbb{Q}$, if the weights μ and λ satisfy (1.2), then for any $m \in \mathbb{Z}$ the point $\frac{1}{2} + m$ is critical for $L(s, \Pi \times \Sigma)$ if and only if \mathcal{M}'_μ contains $\mathcal{M}_{\lambda+m}$. This is a purely local statement involving cohomological representations of $\mathrm{GL}_n(\mathbb{R})$ and standard branching laws. What seems initially surprising, but rather natural after the fact, is that an identical statement (see Thm. 2.21) holds true for cohomological representations of $\mathrm{GL}_n(\mathbb{C})$ although the proof turns out to be combinatorially more challenging than for $\mathrm{GL}_n(\mathbb{R})$; the main steps in the proof of Thm. 2.21 are contained in Lem. 2.24, Cor. 2.30 and Prop. 2.31. (The reader should also look at Grobner–Harris [15, Lem. 4.7].) This particular observation, that we can deal with both real and complex cases on the same footing, was at the genesis of this article.

1.2. Relation with motivic periods and motivic L -functions. Let M be a pure motive over \mathbb{Q} with coefficients in a number field $\mathbb{Q}(M)$. Suppose M is critical, then a celebrated conjecture of Deligne [9, Conj. 2.8] relates the critical values of its L -function $L(s, M)$ to certain periods that arise out of a comparison of the Betti and de Rham realizations of the motive. One expects a cohomological cuspidal automorphic representation Π to correspond to a motive $M(\Pi)$; one of the properties of this correspondence is that the standard L -function $L(s, \Pi)$ is the motivic L -function $L(s, M(\Pi))$ up to a shift in the s -variable; see Clozel [7, Sect. 4]. With the current state of technology, it seems impossible to compare our periods $p^\epsilon(\Pi)$ with Deligne’s periods $c^\pm(M(\Pi))$. However, one may ask if the periods of a tensor product motive decompose in a way suggested by our theorem. Bhagwat [1] has given a description of Deligne’s periods c^\pm for the tensor product $M \otimes M'$, where M and M' are two pure motives over \mathbb{Q} all of whose nonzero Hodge numbers are one, in terms of the periods c^\pm and some other finer invariants attached to M and M' by Yoshida [48]. These period relations suggest that $p^\epsilon(\Pi)$ and $c^\pm(M(\Pi))$ are two very different kind of periods, and a direct comparison with Deligne’s conjecture seems out of reach; however, see Grobner–Harris [15]. Be that as it may, one can still claim that Thm. 1.1 is compatible with Deligne’s conjecture by considering the behavior of L -values under twisting by characters. Blasius [3] and Panchishkin [34] have independently studied the behavior of Deligne’s periods upon twisting the motive by a Dirichlet character (more generally by Artin motives). Using Deligne’s conjecture, they predict the behavior of critical values of motivic L -functions upon twisting by Dirichlet characters. For a critical motive over \mathbb{Q} , assumed to be simple and of rank $2r$, this prediction looks like $L(m, M \otimes \chi_f) \sim_{\mathbb{Q}(M, \chi)} L(m, M) \mathcal{G}(\chi_f)^r$. Applying this to the tensor product motive $M(\Pi) \otimes M(\Sigma)$, which has rank $n(n-1)$, we have the following compatibility with Deligne’s conjecture:

Corollary 1.3. *Let $\Pi \in \mathrm{Coh}(G_n, \mu^\vee)$ and $\Sigma \in \mathrm{Coh}(G_{n-1}, \lambda^\vee)$. Assume that the compatibility condition (1.2) holds. For any critical point $s = \frac{1}{2} + m$ of $L_f(s, \Pi \times \Sigma)$ and for any character $\chi : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ of finite order, and any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have*

$$\sigma \left(\frac{L_f(\frac{1}{2} + m, \Pi \times \Sigma \otimes \chi)}{\mathcal{G}(\chi)^{n(n-1)/2} L_f(\frac{1}{2} + m, \Pi \times \Sigma)} \right) = \frac{L_f(\frac{1}{2} + m, \sigma \Pi \times \sigma \Sigma \otimes \sigma \chi)}{\mathcal{G}(\sigma \chi)^{n(n-1)/2} L_f(\frac{1}{2} + m, \sigma \Pi \times \sigma \Sigma)}.$$

In the above corollary, the compatibility condition (1.2) has an important bearing, because, if not, then already for $\mathrm{GL}_2 \times \mathrm{GL}_1$ over an imaginary quadratic field, twisting by a finite order character can change the critical L -value by a quantity not lying in any field that is cyclotomic over the coefficient fields involved. (See, for example, Hida [22, Thm. I, (ii)] when the set A therein is not empty.) This phenomenon is further elaborated upon in 2.4.5.5.

It should be noted that Theorem 1.1 also gives some evidence toward a conjecture due to Gross [9, Conjecture 2.7(ii)] which says that the order of vanishing of a motivic L -function at a critical point is independent of which conjugate of the motive we are looking at, i.e., if M is critical, then $\text{ord}_{s=0}L(s, \sigma, M)$ is independent of the embedding $\sigma : \mathbb{Q}(M) \rightarrow \mathbb{C}$. We are unable to say anything about the order of vanishing, however, it follows from our theorem that the property of vanishing is indeed independent of which particular conjugate of the representation we consider.

Corollary 1.4. *Let $\Pi \in \text{Coh}(G_n, \mu^\vee)$ and $\Sigma \in \text{Coh}(G_{n-1}, \lambda^\vee)$. Assume that the compatibility condition (1.2) holds. For any critical point $\frac{1}{2} + m$ of $L_f(s, \Pi \times \Sigma)$ and $\sigma \in \text{Aut}(\mathbb{C})$ we have*

$$L(\tfrac{1}{2} + m, \Pi \times \Sigma) = 0 \iff L(\tfrac{1}{2} + m, \sigma\Pi \times \sigma\Sigma) = 0.$$

As in my paper with Wee Teck Gan [11], Thm. 1.1 implies an arithmeticity result for GL_{n-1} -periods of automorphic representations of $\text{GL}_n \times \text{GL}_{n-1}$. See [11, Thm. 7.1]. As explained in the introduction of that paper, this consequence is analogous to Gross's conjecture but for automorphic periods. We state this consequence as the following

Corollary 1.5. *Let $\Pi \in \text{Coh}(G_n, \mu^\vee)$ and $\Sigma \in \text{Coh}(G_{n-1}, \lambda^\vee)$ be as in Thm. 1.1. Suppose μ and λ satisfy (1.2). Consider the representation $\Pi \otimes \Sigma$ of $(G_n \times G_{n-1})(\mathbb{A})$. Let ΔG_{n-1} be the image of the diagonal embedding of G_{n-1} in $G_n \times G_{n-1}$. Then*

$$\Pi \otimes \Sigma \text{ is } \Delta G_{n-1} \text{ distinguished} \implies \sigma\Pi \otimes \sigma\Sigma \text{ is } \Delta G_{n-1} \text{ distinguished } \forall \sigma \in \text{Aut}(\mathbb{C}).$$

1.3. Symmetric power L -functions for GL_2 . Let π be a cohomological cuspidal automorphic representation of GL_2 over F . For any $r \geq 1$, let $\text{Sym}^r(\pi)$ denote the Langlands transfer of π corresponding to the homomorphism $\text{Sym}^r : \text{GL}_2(\mathbb{C}) \rightarrow \text{GL}_{r+1}(\mathbb{C})$ of L -groups; $\text{Sym}^r(\pi)$ is conjecturally an isobaric automorphic representation of GL_{r+1} over F . Such a transfer is known to exist for a general π for $r \leq 4$ by the works of Gelbart–Jacquet, Kim, and Kim–Shahidi, and for all r if π is dihedral. We define the r -th symmetric power L -function as the standard L -function of the r -th symmetric transfer of π , i.e., $L(s, \text{Sym}^r, \pi) = L(s, \text{Sym}^r(\pi))$; for more details and references, see [36, Sect. 3.1], [35, Sect. 5.1.1] and Sect. 3.1 below. (For some new instances of symmetric power functoriality, such as for $r = 6$ and $r = 8$, albeit with some restrictions on F , see Clozel–Thorne [8].)

Given a weight $\mu \in X_{00}^*(T_2)$ and an integer $r \geq 1$, we define its symmetric power transfer $\text{Sym}^r(\mu) \in X_{00}^*(T_{r+1})$ in Sect. 3.1.3. In Thm. 3.2 we prove

$$\pi \in \text{Coh}(G_2, \mu^\vee) \text{ and if } \text{Sym}^r(\pi) \text{ is cuspidal} \implies \text{Sym}^r(\pi) \in \text{Coh}(G_{r+1}, \text{Sym}^r(\mu)^\vee).$$

This says that symmetric power transfers preserves the property of being cohomological, generalizing our previous result [36, Thm. 5.5] and lending further evidence to the discussion in [36, Sect. 5.2] relating functoriality and the property of being cohomological. Similarly, we can define a weight $\det(\mu)$ so that $\omega_\pi \in \text{Coh}(G_1, \det(\mu)^\vee)$; see Sect. 3.1.2.

In [35] Rankin–Selberg theory for $\text{GL}_n \times \text{GL}_{n-1}$ over \mathbb{Q} was used to get results on special values of odd symmetric power L -functions attached to modular forms. We completely work out this idea in the case of symmetric cube L -functions for cusp forms for GL_2 over any number field. The starting point is the factorization:

$$L_f(s, \text{Sym}^2(\pi) \times \pi \otimes \xi) = L_f(s, \text{Sym}^3(\pi) \otimes \xi) \cdot L_f(s, \pi \otimes \omega_\pi \xi),$$

where ξ is a Hecke character of F of finite order. We know the special values of the left hand side (resp., the second factor on the right hand side) by the $\text{GL}_3 \times \text{GL}_2$ case (resp., $\text{GL}_2 \times \text{GL}_1$ case) of Thm. 1.1; we deduce a result for the special values of symmetric cube L -functions. Take a

half-integer $\frac{1}{2} + m$ which is critical for $L_f(s, \text{Sym}^3(\pi) \otimes \xi)$ and $L(s, \pi \otimes \omega_\pi \xi)$, then it is also critical for $L_f(s, \text{Sym}^2(\pi) \times \pi \otimes \xi)$; see Sect. 3.2.2 for a description of the set of critical points. We write

$$L_f\left(\frac{1}{2} + m, \text{Sym}^3(\pi) \otimes \xi\right) = \frac{L_f\left(\frac{1}{2} + m, \text{Sym}^2(\pi) \times \pi \otimes \xi\right)}{L_f\left(\frac{1}{2} + m, \pi \otimes \omega_\pi \xi\right)},$$

provided $L\left(\frac{1}{2} + m, \pi \otimes \omega_\pi \xi\right) \neq 0$. For all critical points, except possibly for the central critical point, the denominator is indeed nonzero. If $\mathbf{w} = \mathbf{w}(\mu)$ is the purity weight of μ , then $\pi = \pi^\circ \otimes |\cdot|^{\mathbf{w}/2}$ for a unitary cuspidal automorphic representation π° , and $L\left(\frac{1}{2} + m, \pi \otimes \omega_\pi \xi\right) = L\left(\frac{1}{2} + m + 3\mathbf{w}/2, \pi^\circ \otimes \omega_{\pi^\circ} \xi\right)$. Hence, $L\left(\frac{1}{2} + m, \pi \otimes \omega_\pi \xi\right) = 0$ can happen only when $m = -3\mathbf{w}/2$. (This is the center of symmetry in the unitary case; note the necessary condition that \mathbf{w} should be even.) In this particular case, we use the main theorem of Rohrlich [39] and introduce another twisting character making the L -value nonzero, i.e., after relabeling if necessary, we may take ξ to be such that $L\left((1 - 3\mathbf{w})/2, \pi \otimes \omega_\pi \xi\right) \neq 0$. Putting these together gives us the following

Theorem 1.6. *Let $\pi \in \text{Coh}(G_2, \mu^\vee)$ for $\mu \in X_{00}^*(T_2)$. Assume that the pairs of weights $(\text{Sym}^2(\mu), \mu)$ and $(\mu, \det(\mu))$ satisfy (1.2). Let $\xi : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ be a character of finite order; if \mathbf{w} is even, then take ξ such that $L\left((1 - 3\mathbf{w})/2, \pi \otimes \omega_\pi \xi\right) \neq 0$. Suppose $\frac{1}{2} + m$ is critical for $L_f(s, \text{Sym}^3(\pi) \otimes \xi)$ and $L(s, \pi \otimes \omega_\pi \xi)$. Then we have:*

$$L_f\left(\frac{1}{2} + m, \text{Sym}^3(\pi) \otimes \xi\right) \sim p^{\epsilon_+}(\text{Sym}^2(\pi)) \frac{p^{-\epsilon_m \epsilon_\xi(\pi)}}{p^{\epsilon_m \epsilon_\xi(\pi)}} \mathcal{G}(\xi)^2 \frac{p_{\infty}^{\epsilon_+, \epsilon_-}(\text{Sym}^2(\mu), \mu + m)}{p_{\infty}^{\epsilon_\xi, \epsilon_\xi}(\mu + m, \det(\mu))},$$

where, by \sim , we mean up to an element of $\mathbb{Q}(\pi, \xi)$; ϵ_+ is the sign which is + everywhere; $\epsilon_- = -\epsilon_+$, and ϵ_ξ is the signature of ξ as in Sect. 2.1.7. Furthermore, the ratio of the left hand side by the right hand side is equivariant under $\text{Aut}(\mathbb{C})$.

The proof of the above theorem is given in Sect. 3.2.4. Moreover, exactly as in [35, Sect. 5], we can get analogous results for higher odd symmetric power L -functions. For fifth and seventh symmetric power L -functions we would get partial results, and assuming Langlands's functoriality, we would get conditional results for all odd symmetric power L -functions. We omit the details as any interested reader can proceed as in Sect. 3.2.4.

Note to the reader: Although we have tried to make this article self-contained, anyone who wishes to verify details, will need to keep copies of [35], [36] and [37] by his/her side.

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2. Arithmetic properties of Rankin–Selberg L -functions

2.1. Some notation and preliminaries.

2.1.1. The base field. Let F be a number field of degree $d = [F : \mathbb{Q}]$ with ring of integers \mathcal{O} . For any place v we write F_v for the topological completion of F at v . Let S_∞ be the set of archimedean places of F . Let $S_\infty := S_r \cup S_c$, where S_r (resp., S_c) is the set of real (resp., complex) places. Let $\mathcal{E}_F = \text{Hom}(F, \mathbb{C})$ be the set of all embeddings of F as a field into \mathbb{C} . There is a canonical surjective map $\mathcal{E}_F \rightarrow S_\infty$, which is a bijection on the real embeddings and real places, and identifies a pair of complex conjugate embeddings $\{\iota_v, \bar{\iota}_v\}$ with the complex place v . For each $v \in S_r$, we fix an

isomorphism $F_v \cong \mathbb{R}$ which is canonical. Similarly for $v \in S_c$, we fix $F_v \cong \mathbb{C}$ given by (say) ι_v ; this choice is not canonical. Let $r_1 = |S_r|$ and $r_2 = |S_c|$; hence $d = r_1 + 2r_2$. If $v \notin S_\infty$, we let \mathcal{O}_v be the ring of integers of F_v , and \mathfrak{o}_v it's unique maximal ideal. Moreover, \mathbb{A}_F denotes the ring of adèles of F and $\mathbb{A}_{F,f}$ its finite part. The group of idèles of F will be denoted \mathbb{A}_F^\times and similarly, $\mathbb{A}_{F,f}^\times$ is the group of finite idèles. We will drop the subscript F when talking about \mathbb{Q} . Hence, \mathbb{A} is $\mathbb{A}_\mathbb{Q}$, etc. We use the local and global normalized absolute values and denote each of them by $|\cdot|$. Further, \mathfrak{D}_F stands for the absolute different of F , i.e., $\mathfrak{D}_F^{-1} = \{x \in F : Tr_{F/\mathbb{Q}}(x\mathcal{O}) \subset \mathbb{Z}\}$.

2.1.2. The groups $G_n \supset B_n \supset T_n \supset Z_n \supset S_n$. The algebraic group GL_n/F will be denoted as \underline{G}_n , and we put $G_n = R_{F/\mathbb{Q}}(\underline{G}_n)$. An F -group will be denoted by an underline and the corresponding \mathbb{Q} -group via Weil restriction of scalars will be denoted without the underline; hence for any \mathbb{Q} -algebra A the group of A -points of G_n is $G_n(A) = \underline{G}_n(A \otimes_\mathbb{Q} F)$. Let $\underline{B}_n = \underline{T}_n \underline{U}_n$ stand for the standard Borel subgroup of \underline{G}_n of all upper triangular matrices, where \underline{U}_n is the unipotent radical of \underline{B}_n , and \underline{T}_n the diagonal torus. The center of \underline{G}_n will be denoted by \underline{Z}_n . These groups define the corresponding \mathbb{Q} -groups $G_n \supset B_n = T_n U_n \supset Z_n$. Observe that Z_n is not \mathbb{Q} -split, and we let S_n be the maximal \mathbb{Q} -split torus in Z_n ; we have $S_n \cong \mathbb{G}_m$ over \mathbb{Q} .

2.1.3. The groups at infinity. Note that

$$G_{n,\infty} := G_n(\mathbb{R}) = \underline{G}_n(F \otimes_\mathbb{Q} \mathbb{R}) = \prod_{v \in S_\infty} \mathrm{GL}_n(F_v) \cong \prod_{v \in S_r} \mathrm{GL}_n(\mathbb{R}) \times \prod_{v \in S_c} \mathrm{GL}_n(\mathbb{C}).$$

Let \mathbb{K} denote either \mathbb{R} or \mathbb{C} . We have $Z_n(\mathbb{R}) = \prod_{v \in S_r} \mathbb{R}^\times \times \prod_{v \in S_c} \mathbb{C}^\times$, where each copy of \mathbb{K}^\times consists of nonzero scalar matrices in the corresponding copy of $\mathrm{GL}_n(\mathbb{K})$. The subgroup $S_n(\mathbb{R})$ of $Z_n(\mathbb{R})$ consists of \mathbb{R}^\times diagonally embedded in $\prod_{v \in S_r} \mathbb{R}^\times \times \prod_{v \in S_c} \mathbb{C}^\times$. Let $C_{n,\infty} = \prod_{v \in S_r} \mathrm{O}(n) \times \prod_{v \in S_c} \mathrm{U}(n)$ be the maximal compact subgroup of $G_n(\mathbb{R})$, and let

$$\begin{aligned} K_{n,\infty} &= S_n(\mathbb{R})C_{n,\infty} \\ &\cong \mathbb{R}^\times \left(\prod_{v \in S_r} \mathrm{O}(n) \times \prod_{v \in S_c} \mathrm{U}(n) \right) \\ &\cong \mathbb{R}_+^\times \left(\prod_{v \in S_r} \mathrm{O}(n) \times \prod_{v \in S_c} \mathrm{U}(n) \right) \cong S_n(\mathbb{R})^0 C_{n,\infty}. \end{aligned}$$

Let $K_{n,\infty}^0$ be the topological connected component of $K_{n,\infty}$. Hence

$$K_{n,\infty}^0 = S_n(\mathbb{R})^0 C_{n,\infty}^0 \cong \mathbb{R}_+^\times \left(\prod_{v \in S_r} \mathrm{SO}(n) \times \prod_{v \in S_c} \mathrm{U}(n) \right).$$

For any topological group \mathfrak{G} we will let $\pi_0(\mathfrak{G}) := \mathfrak{G}/\mathfrak{G}^0$ stand for the group of connected components. Inclusion of connected components induces the equality $\pi_0(K_{n,\infty}) = \pi_0(G_n(\mathbb{R}))$. Observe also that $\pi_0(K_{n,\infty}) \cong \prod_{v \in S_r} \{\pm 1\} \cong: \prod_{v \in S_r} \{\pm\}$. The matrix $\delta_n = \mathrm{diag}(-1, 1, \dots, 1)$ represents the nontrivial element in $\mathrm{O}(n)/\mathrm{SO}(n)$, and if n is odd, the scalar matrix -1_n also represents this nontrivial element. We identify $\pi_0(G_n(\mathbb{R}))$ inside $G_n(\mathbb{R})$ via the δ_n 's. The character group of $\pi_0(K_{n,\infty})$ is denoted $\pi_0(\widehat{K_{n,\infty}})$.

2.1.4. Lie algebras. The general notational principle we follow is that for a real Lie group G , we denote its Lie algebra by \mathfrak{g}^0 and the complexified Lie algebra by \mathfrak{g} , i.e., $\mathfrak{g} = \mathfrak{g}^0 \otimes_{\mathbb{R}} \mathbb{C}$. Hence, if G is the Lie group $\mathrm{GL}_n(\mathbb{R})$ then $\mathfrak{g}^0 = \mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. On the other hand, if G stands for the real Lie group $\mathrm{GL}_n(\mathbb{C})$ then $\mathfrak{g}^0 = \mathfrak{gl}_n(\mathbb{C})$ as a \mathbb{R} -Lie algebra, and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$. With this notational scheme, we have \mathfrak{g}_n , \mathfrak{b}_n , \mathfrak{t}_n and \mathfrak{k}_n denoting the complexified Lie algebras of $G_n(\mathbb{R})$, $B_n(\mathbb{R})$, $T_n(\mathbb{R})$ and $K_{n,\infty}^0$ respectively. For example, $\mathfrak{g}_n = \prod_{v \in S_r} \mathfrak{gl}_n(\mathbb{C}) \times \prod_{v \in S_c} (\mathfrak{gl}_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C})$.

2.1.5. Finite-dimensional representations. Consider $T_n(\mathbb{R}) = \underline{T}_n(F \otimes \mathbb{R}) \cong \prod_{v \in S_\infty} \underline{T}_n(F_v)$. We let $X^*(T_n)$ stand for the group of all algebraic characters of T_n , and let $X^+(T_n)$ stand for all those characters in $X^*(T_n)$ which are dominant with respect to B_n . A weight $\mu \in X^+(T_n)$ may be described as follows: $\mu = (\mu^\iota)_{\iota \in \mathcal{E}_F}$, where

- For $v \in S_r$, we have $\mu^v = (\mu_1^v, \dots, \mu_n^v)$, $\mu_i^v \in \mathbb{Z}$, $\mu_1^v \geq \dots \geq \mu_n^v$, and the character μ^v sends $t = \mathrm{diag}(t_1, \dots, t_n) \in \underline{T}_n(F_v)$ to $\prod_i t_i^{\mu_i^v}$.
- If $v \in S_c$ then μ^v is the pair $(\mu^{\iota v}, \mu^{\bar{\iota} v})$, with $\mu^{\iota v} = (\mu_1^{\iota v}, \dots, \mu_n^{\iota v})$, $\mu_i^{\iota v} \in \mathbb{Z}$, $\mu_1^{\iota v} \geq \dots \geq \mu_n^{\iota v}$; likewise $\mu^{\bar{\iota} v} = (\mu_1^{\bar{\iota} v}, \dots, \mu_n^{\bar{\iota} v})$ and $\mu_i^{\bar{\iota} v} \geq \dots \geq \mu_n^{\bar{\iota} v}$; the character μ^v is given by sending $t = \mathrm{diag}(z_1, \dots, z_n) \in \underline{T}_n(F_v)$ to $\prod_{i=1}^n z_i^{\mu_i^{\iota v}} \bar{z}_i^{\mu_i^{\bar{\iota} v}}$, where \bar{z}_i is the complex conjugate of z_i .

We also write $\mu = (\mu^v)_{v \in S_\infty}$, where $\mu^v = (\mu^{\iota v}, \mu^{\bar{\iota} v})$ for $v \in S_c$.

For $\mu \in X^+(T_n)$, we define a finite-dimensional complex representation $(\rho_\mu, \mathcal{M}_{\mu, \mathbb{C}})$ of $G_n(\mathbb{R})$ as follows. For $v \in S_r$, let $(\rho_{\mu^v}, \mathcal{M}_{\mu^v, \mathbb{C}})$ be the irreducible complex representation of $G_n(F_v) = \mathrm{GL}_n(\mathbb{R})$ with highest weight μ^v . For $v \in S_c$, let $(\rho_{\mu^v}, \mathcal{M}_{\mu^v, \mathbb{C}})$ be the complex representation of the real algebraic group $G(F_v) = \mathrm{GL}_n(\mathbb{C})$ defined as $\rho_{\mu^v}(g) = \rho_{\mu^{\iota v}}(g) \otimes \rho_{\mu^{\bar{\iota} v}}(\bar{g})$; here $\rho_{\mu^{\iota v}}$ (resp., $\rho_{\mu^{\bar{\iota} v}}$) is the irreducible representation of the complex group $\mathrm{GL}_n(\mathbb{C})$ with highest weight $\mu^{\iota v}$ (resp., $\mu^{\bar{\iota} v}$). Now we let $\rho_\mu = \otimes_{v \in S_\infty} \rho_{\mu^v}$.

Note that $\mathrm{Aut}(\mathbb{C})$ acts on $X^*(T_n)$ as follows: if $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $\mu \in X^*(T_n)$ then $\sigma\mu \in X^*(T_n)$ is defined as: $\sigma\mu = (\sigma\mu^\iota)_{\iota \in \mathcal{E}_F}$ where $\sigma\mu^\iota := \mu^{\sigma^{-1} \circ \iota}$. Define the rationality field $\mathbb{Q}(\mu)$ as the fixed field in \mathbb{C} under all those automorphisms σ which fix μ . Consider the representation $(\rho_{\sigma\mu}, \mathcal{M}_{\sigma\mu, \mathbb{C}})$ of $G_n(\mathbb{R})$ of highest weight $\sigma\mu$. Consider \mathbb{C} as a (\mathbb{C}, \mathbb{C}) -bimodule, where the left module structure is via σ and the right module structure is the usual multiplication in \mathbb{C} ; denote this bimodule as ${}_\sigma\mathbb{C}$. Then the canonical map $t : \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathbb{C}} {}_\sigma\mathbb{C}$ defined by $t(w) = w \otimes 1$ is a σ -linear isomorphism. Take $\mathcal{M} = \mathcal{M}_{\mu, \mathbb{C}}$ and denote by $({}^\sigma\rho_\mu, {}^\sigma\mathcal{M}_{\mu, \mathbb{C}})$ the representation of $G_n(\mathbb{R})$ where a $g \in G_n(\mathbb{R})$ acts on ${}^\sigma\mathcal{M}_{\mu, \mathbb{C}} = \mathcal{M}_{\mu, \mathbb{C}} \otimes {}_\sigma\mathbb{C}$ by ${}^\sigma\rho_\mu(g) = t \circ \rho_\mu(g) \circ t^{-1}$. Then ${}^\sigma\rho_\mu \simeq \rho_{\sigma\mu}$ as a representation of $G_n(F)$ (see [16, Lem. 7.1]) and this representation is defined over $\mathbb{Q}(\mu)$ which may be seen exactly as in Waldspurger [46, Prop. I.3]. For any extension $E/\mathbb{Q}(\mu)$ we will let $\mathcal{M}_{\mu, E} = \mathcal{M}_{\mu, \mathbb{Q}(\mu)} \otimes_{\mathbb{Q}(\mu)} E$ on which $G_n(F)$ acts via its action on the first factor.

2.1.6. Automorphic representations. Following Borel–Jacquet [6, §4.6], we say an irreducible representation of $G_n(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A}_F)$ is automorphic if it is isomorphic to an irreducible subquotient of the representation of $G_n(\mathbb{A})$ on its space of automorphic forms. We say an automorphic representation is cuspidal if it is a subrepresentation of the representation of $G_n(\mathbb{A})$ on the space of cusp forms $\mathcal{A}_{\mathrm{cusp}}(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) = \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}_F))$. The subspace of cusp forms realizing a cuspidal automorphic representation π will be denoted V_π . For an automorphic representation π of $G_n(\mathbb{A})$, we have $\pi = \pi_\infty \otimes \pi_f$, where π_∞ is a representation of $G_n(\mathbb{R})$, and $\pi_f = \otimes_{v \notin S_\infty} \pi_v$ is a representation of $G_n(\mathbb{A}_f)$. The central character of π will be denoted ω_π .

2.1.7. Algebraic Hecke characters. (References: Deligne [10, §5], Schappacher [40, Chapter 0], Waldspurger [46, I.5] or Weil [47].) A continuous homomorphism $\omega : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ is called a

Hecke character of F . An element $\alpha = \sum_{\iota \in \mathcal{E}_F} a_\iota \iota$, with $a_\iota \in \mathbb{Z}$ is called an *infinity type*. A Hecke character ω is called an algebraic Hecke character of infinity type α if

- for $v \in S_r$, $\omega_v(x) = x^{a_\iota v}$ for all $x \in \mathbb{R}_+^\times$;
- for $v \in S_c$, $\omega_v(z) = z^{a_\iota v} \bar{z}^{a_{\bar{\iota}v}}$ for all $z \in \mathbb{C}^\times$.

Weil gave the appellation ‘characters of type (A_0) ’ for such algebraic Hecke characters. The existence of an algebraic Hecke character ω with infinity type α implies the following purity constraint on α :

- (1) if S_r is not empty, i.e., if F has at least one real place, then the map from $\mathcal{E}_F \rightarrow \mathbb{Z}$ given by $\iota \mapsto a_\iota$ is constant; in this case, let $\mathbf{w}(\omega) := a_\iota$ for any ι .
- (2) if S_r is empty, i.e., if F is a totally imaginary field, then the map from $\mathcal{E}_F \times \text{Aut}(\mathbb{C}) \rightarrow \mathbb{Z}$ given by $(\iota, \sigma) \mapsto a_{\sigma\iota} + a_{\sigma\bar{\iota}}$ is constant; in this case, let $\mathbf{w}(\omega) := a_\iota + a_{\bar{\iota}}$ for any ι

In either case, we call $\mathbf{w}(\omega)$ the *purity weight* of ω .

Suppose that F has at least one real place, then we define the *signature* ϵ_ω of an algebraic Hecke character ω as follows: By the purity constraint, the character $\omega^\circ := \omega |^{-\mathbf{w}(\omega)}$ is a character of finite order. For $v \in S_r$, define

$$\epsilon_{\omega_v} = (-1)^{\mathbf{w}(\omega)} \omega_v^\circ(-1).$$

Now put $\epsilon_\omega = (\epsilon_{\omega_v})_{v \in S_r}$. The signature is an r_1 -tuple of signs indexed by real embeddings of F .

For each finite place v , and any smooth character $\chi_v : F_v^\times \rightarrow \mathbb{C}^\times$, define the rationality field $\mathbb{Q}(\chi_v)$ of χ_v as the field obtained by adjoining the values of ω_v to \mathbb{Q} . For an algebraic Hecke character ω , we define its *rationality field* $\mathbb{Q}(\omega)$ as the compositum of the fields $\mathbb{Q}(\omega_v)$ for all finite places v that are unramified for ω . It is a standard fact that $\mathbb{Q}(\omega)$ is a number field, and as Weil notes in [47], the field $\mathbb{Q}(\omega)$ need not contain the field F .

2.1.8. Additive characters and Gauss sums. We fix an additive character $\psi_{\mathbb{Q}}$ of $\mathbb{Q} \backslash \mathbb{A}$, as in Tate’s thesis, namely, $\psi_{\mathbb{Q}}(x) = e^{2\pi i \lambda(x)}$ with the λ as defined in [45, Sect. 2.2]. Next, we define a character ψ of $F \backslash \mathbb{A}_F$ by composing $\psi_{\mathbb{Q}}$ with the trace map from F to \mathbb{Q} : $\psi = \psi_{\mathbb{Q}} \circ \text{Tr}_{F/\mathbb{Q}}$. If $\mathfrak{D}_F = \prod_{\mathfrak{p}} \mathfrak{p}^{r_{\mathfrak{p}}}$ with the product running over all prime ideals \mathfrak{p} , and $\psi = \otimes_v \psi_v$, then the conductor of ψ_v at a finite place v is $\mathfrak{p}_v^{-r_{\mathfrak{p}}}$, i.e., ψ_v is trivial on $\mathfrak{p}_v^{-r_{\mathfrak{p}}}$ and nontrivial on $\mathfrak{p}_v^{-r_{\mathfrak{p}}-1}$. For any Hecke character χ of F , we define the Gauß sum $\mathcal{G}(\chi_f)$ of χ_f exactly as in [35, Sect. 2]; this depends on choice of ψ .

2.2. Rankin–Selberg L -functions: analytic aspects. This subsection is a brief summary of [35, Sect. 3.1]; see references therein for all the assertions made below.

2.2.1. Rankin–Selberg zeta integrals for $G_n \times G_{n-1}$. Let Π (resp., Σ) be a cuspidal automorphic representation of $G_n(\mathbb{A})$ (resp., $G_{n-1}(\mathbb{A})$). Let $\phi \in V_\Pi$ and $\phi' \in V_\Sigma$ be cusp forms. Consider

$$I(s, \phi, \phi') = \int_{G_{n-1}(\mathbb{Q}) \backslash G_{n-1}(\mathbb{A})} \phi(\iota(g)) \phi'(g) |\det(g)|^{s-1/2} dg.$$

The above integral converges for all $s \in \mathbb{C}$. Suppose that $w \in \mathcal{W}(\Pi, \psi)$ and $w' \in \mathcal{W}(\Sigma, \psi^{-1})$ are global Whittaker functions corresponding to ϕ and ϕ' , respectively. We have

$$I(s, \phi, \phi') = \Psi(s, w, w') := \int_{U_{n-1}(\mathbb{A}) \backslash G_{n-1}(\mathbb{A})} w(\iota(g)) w'(g) |\det(g)|^{s-1/2} dg.$$

The integral $\Psi(s, w, w')$ converges for $\text{Re}(s) \gg 0$. Let $w = \otimes w_v$ and $w' = \otimes w'_v$, then $\Psi(s, w, w') := \otimes \Psi_v(s, w_v, w'_v)$ for $\text{Re}(s) \gg 0$, where the local integral Ψ_v is given by a similar formula. Recall that the local integral $\Psi_v(s, w_v, w'_v)$ converges for $\text{Re}(s) \gg 0$ and has a meromorphic continuation

to all of \mathbb{C} . We will choose the local Whittaker functions carefully so that the integral $I(\frac{1}{2}, \phi, \phi')$ computes the special value $L_f(\frac{1}{2}, \Pi \times \Sigma)$ up to quantities which are $\text{Aut}(\mathbb{C})$ -equivariant.

2.2.2. Action of $\text{Aut}(\mathbb{C})$ on Whittaker models. Any $\sigma \in \text{Aut}(\mathbb{C})$ gives an element $t_\sigma \in \mathbb{A}_f^\times$. Given $w \in \mathcal{W}(\Pi_f, \psi_f)$, define ${}^\sigma w \in \mathcal{W}({}^\sigma \Pi_f, \psi_f)$ by ${}^\sigma w(g_f) = \sigma(w(t_{\sigma,n} g_f))$, $g_f \in G(\mathbb{A}_f)$, where $t_{\sigma,n} = \text{diag}(t_\sigma^{-(n-1)}, t_\sigma^{-(n-2)}, \dots, t_\sigma^{-1}, 1)$. For more details, see [35, Sect. 3.1] and [37, Sect. 3.2].

2.2.3. Normalized new vectors. Just for this paragraph, let F be a non-archimedean local field, \mathcal{O}_F the ring of integers of F , and \mathcal{P}_F the maximal ideal of \mathcal{O}_F . Let (π, V) be an irreducible admissible generic representation of $\text{GL}_n(F)$. Let $K_n(m)$ be the ‘mirahoric subgroup’ of $\text{GL}_n(\mathcal{O}_F)$ consisting of all matrices whose last row is congruent to $(0, \dots, 0, *)$ modulo \mathcal{P}_F^m . Let $V_m := \{v \in V \mid \pi(k)v = w_\pi(k_{n,n})v, \forall k \in K_n(m)\}$. Let $\mathfrak{f}(\pi)$ be the least non-negative integer m for which $V_m \neq (0)$. One knows that $\mathfrak{f}(\pi)$ is the conductor of π and that $V_{\mathfrak{f}(\pi)}$ is one-dimensional. Any vector in $V_{\mathfrak{f}(\pi)}$ is called a *new vector* of π . Fix a nontrivial additive character ψ of F , and assume that $V = W(\pi, \psi)$ is the Whittaker model for π . If π is unramified, i.e., $\mathfrak{f}(\pi) = 0$, then we fix a specific new vector called the *spherical vector*, denoted w_π^{sp} , and normalized as $w_\pi^{\text{sp}}(1_n) = 1$. More generally, for any π , amongst all new vectors, there is a distinguished vector, called the *essential vector*, denoted w_π^{ess} , and characterized by the property that for any irreducible unramified generic representation ρ of $\text{GL}_{n-1}(F)$ one has

$$\Psi(s, w_\pi^{\text{ess}}, w_\rho^{\text{sp}}) = \int_{U_{n-1}(F) \backslash G_{n-1}(F)} w_\pi^{\text{ess}}(\iota(g)) w_\rho^{\text{sp}}(g) |\det(g)|^{s-1/2} dg = L(s, \pi \times \rho).$$

If π is unramified then $w_\pi^{\text{ess}} = w_\pi^{\text{sp}}$. In general, given π there exists $t_\pi \in T_n(F)$ such that a new vector for π is nonvanishing on t_π . Note that necessarily $t_\pi \in T_n^+(F)$, i.e., if $t_\pi = \text{diag}(t_1, t_2, \dots, t_n)$ then $t_i t_{i+1}^{-1} \in \mathcal{O}_F$ for all $1 \leq i \leq n-1$. We let w_π^0 be the new vector normalized such that $w_\pi^0(t_\pi) = 1$. If π is unramified then we may and will take $t_\pi = 1_n$, and so $w_\pi^0 = w_\pi^{\text{ess}} = w_\pi^{\text{sp}}$. For any $\sigma \in \text{Aut}(\mathbb{C})$ we may and will take $t_{\pi\sigma} = t_\pi$. Then ${}^\sigma w_\pi^0 = w_\pi^0$. Define $c_\pi \in \mathbb{C}^\times$ by $w_\pi^0 = c_\pi w_\pi^{\text{ess}}$, i.e., $c_\pi = w_\pi^{\text{ess}}(t_\pi)^{-1}$. For more details, see [35, Sect. 3.1.3].

2.2.4. Choice of Whittaker vectors and cusp forms. We now go back to global notation and choose global Whittaker vectors $w_\Pi = \otimes_v w_{\Pi,v} \in \mathcal{W}(\Pi, \psi)$ and $w_\Sigma = \otimes_v w_{\Sigma,v} \in \mathcal{W}(\Sigma, \psi^{-1})$ as follows. Let S_Σ be the set of finite places v where Σ_v is unramified.

- (1) If $v \notin S_\Sigma \cup S_\infty$, we let $w_{\Pi,v} = w_{\Pi,v}^0$, and $w_{\Sigma,v} = w_{\Sigma,v}^{\text{sp}}$.
- (2) If $v \in S_\Sigma$, we let $w_{\Sigma,v} = w_{\Sigma,v}^0$, and let $w_{\Pi,v}$ be the unique Whittaker function whose restriction to $\text{GL}_{n-1}(F_v)$ is supported on $\underline{U}_{n-1}(F_v) t_{\Sigma_v} K_{n-1}(\mathfrak{f}(\Sigma_v))$, and on this double coset it is given by $w_{\Pi,v}(ut_{\Sigma_v}k) = \psi(u) \omega_{\Sigma_v}^{-1}(k_{n-1,n-1})$, for all $u \in \underline{U}_{n-1}(F_v)$ and for all $k \in K_{n-1}(\mathfrak{f}(\Sigma_v))$.
- (3) If $v \in S_\infty$, we let $w_{\Pi,v}$ and $w_{\Sigma,v}$ be arbitrary nonzero vectors. (Later, these will be cohomological vectors.)

Let $w_{\Pi_f} = \otimes_{v \notin S_\infty} w_{\Pi,v}$, $w_{\Pi_\infty} = \otimes_{v \in S_\infty} w_{\Pi,v}$, and $w_\Pi = w_{\Pi_\infty} \otimes w_{\Pi_f}$. Similarly, let w_{Σ_f} , w_{Σ_∞} and w_Σ . Let ϕ_Π (resp., ϕ_Σ) be the cusp form corresponding to w_Π (resp., w_Σ).

2.2.5. Integral representation of the central L -value. For $\Re(s) \gg 0$, define $\Psi_\infty(s, w_{\Pi_\infty}, w_{\Sigma_\infty})$ to be $\prod_{v \in S_\infty} \Psi_v(s, w_{\Pi,v}, w_{\Sigma,v})$; this admits a meromorphic continuation to all of \mathbb{C} . One identifies local L -factors $L_v(s, \Pi_v \times \Sigma_v)$, and proves that $\Psi_v(s, w_{\Pi,v}, w_{\Sigma,v})/L_v(s, \Pi_v \times \Sigma_v)$ is entire. Later we will be taking $s = 1/2$ to be a critical point, which says that $L_v(\frac{1}{2}, \Pi_v \times \Sigma_v)$ is regular for all $v \in S_\infty$; hence, criticality of $s = 1/2$ gives that $\Psi_\infty(\frac{1}{2}, w_{\Pi_\infty}, w_{\Sigma_\infty})$ is finite.

Proposition 2.1. *We have*

$$I(\tfrac{1}{2}, \phi_\Pi, \phi_\Sigma) = \frac{\Psi_\infty(\tfrac{1}{2}, w_{\Pi_\infty}, w_{\Sigma_\infty}) \text{vol}(\Sigma) \prod_{v \notin S_\Sigma \cup \{\infty\}} c_{\Pi_v}}{\prod_{v \in S_\Sigma} L(\tfrac{1}{2}, \Pi_v \times \Sigma_v)} L_f(\tfrac{1}{2}, \Pi \times \Sigma),$$

where $\text{vol}(\Sigma) = \prod_{v \in S_\Sigma} \text{vol}(K_{n-1}(\mathfrak{f}(\Sigma_v))) \in \mathbb{Q}^*$.

Proof. See [35, Prop. 3.1]. □

The main theorem on critical values of Rankin–Selberg L -functions follows by interpreting the above proposition in cohomology.

2.3. Automorphic cohomology.

2.3.1. Locally symmetric spaces. (See Harder [18, 1.1].) Let K_f be an open-compact subgroup of $G_n(\mathbb{A}_f) = \text{GL}_n(\mathbb{A}_{F,f})$. Let us write $K_f = \prod_p K_p$ where each K_p is an open compact subgroup of $G_n(\mathbb{Q}_p)$ and for almost all p we have $K_p = \prod_{v|p} \text{GL}_n(\mathcal{O}_v)$. Define the double-coset space

$$S_{K_f}^{G_n} = G_n(\mathbb{Q}) \backslash G_n(\mathbb{A}) / K_{n,\infty}^0 K_f = \text{GL}_n(F) \backslash \text{GL}_n(\mathbb{A}_F) / K_{n,\infty}^0 K_f.$$

For brevity, let $K = K_{n,\infty}^0 K_f$, and define $X = G_n(\mathbb{A}) / K = G_n(\mathbb{R}) / K_{n,\infty}^0 \times G_n(\mathbb{A}_f) / K_f$, i.e., X is the product of the symmetric space $G_n(\mathbb{R}) / K_{n,\infty}^0$ with a totally disconnected space; any connected component of X is of the form $X_g = G_n(\mathbb{R})^0(g_\infty, g_f) K_f / K$ where $g = (g_\infty, g_f) \in G_n(\mathbb{A})$ with $g_\infty \in \pi_0(G_n(\mathbb{R})) \subset G_n(\mathbb{R})$; for the last inclusion see 2.1.3. The stabilizer of X_g inside $G_n(\mathbb{Q})$ is $\Gamma_g := \{\gamma \in G_n(\mathbb{Q}) : \gamma \in G_n(\mathbb{R})^0 \cap g_f K_f g_f^{-1}\}$. Any connected component of $S_{K_f}^{G_n}$ is of the form $\Gamma_g \backslash X_g \cong \Gamma_g \backslash G_n(\mathbb{R})^0 / K_{n,\infty}^0$. However, Γ_g does not act freely on X_g since $S_{n,\infty} \subset K_{n,\infty}$. Indeed, the stabilizer of every point in X_g contains a congruence subgroup Δ of $S_n(\mathcal{O}_F)$; this Δ is independent of the point in X_g , but the congruence conditions on Δ depend on K_f . The group $\bar{\Gamma}_g = \Gamma_g / \Delta$ acts freely on X_g and the quotient $\bar{\Gamma}_g \backslash X_g$ is a locally symmetric space. We will abuse terminology and sometimes refer to $S_{K_f}^{G_n}$ as a locally symmetric space of G_n with level structure K_f .

2.3.2. Sheaves on locally symmetric spaces, and their cohomology. (Reference: see Harder–Raghuram [19].) Given a dominant-integral weight $\mu \in X^+(T_n)$ and the associated representation $\mathcal{M}_{\mu,E}$, where E is an extension of $\mathbb{Q}(\mu)$, we get a sheaf $\widetilde{\mathcal{M}}_{\mu,E}$ of E -vector spaces on $S_{K_f}^{G_n}$ as follows: Let $\pi : G_n(\mathbb{A}) / K_{n,\infty}^0 K_f \rightarrow S_{K_f}^{G_n}$ be the canonical projection. For any open subset U of $S_{K_f}^{G_n}$ define the sections over U by:

$$\widetilde{\mathcal{M}}_\mu(U) := \left\{ s : \pi^{-1}(U) \rightarrow \mathcal{M}_{\mu,E} \mid \begin{array}{l} s \text{ is locally constant, and} \\ s(\gamma u) = \rho_\mu(\gamma) s(u), \text{ for all } \gamma \in G_n(\mathbb{Q}) \text{ and } u \in \pi^{-1}(U) \end{array} \right\}.$$

This defines a sheaf of complex vector spaces on $S_{K_f}^{G_n}$. Note that even if $\mathcal{M}_{\mu,E} \neq 0$ it is possible that the sheaf $\widetilde{\mathcal{M}}_{\mu,E} = 0$. (See Harder [18, 1.1.3].) Indeed, $\widetilde{\mathcal{M}}_{\mu,E} = 0$ unless the central character of ρ_μ has the infinity type of an algebraic Hecke character of F . Suppose $\mu = (\mu^i)_{i \in \mathcal{E}_F}$, then define $a_\iota(\mu) := \sum_{i=1}^n \mu_i^i$. The central character of ρ_μ has the infinity type of an algebraic Hecke character of F if and only if the map $\iota \mapsto a_\iota(\mu)$ satisfies either (1) or (2) of 2.1.7. Henceforth, we will assume that μ satisfies this condition.

We are interested in the sheaf cohomology groups

$$H^\bullet(S_{K_f}^{G_n}, \widetilde{\mathcal{M}}_{\mu,E}^\vee).$$

Here $\widetilde{\mathcal{M}}_{\mu,E}^\vee$ is the sheaf attached to the contragredient representation \mathcal{M}_μ^\vee of \mathcal{M}_μ . If $\mu^\vee = -w_0(\mu)$, where w_0 is the element of the Weyl group of longest length, then $\mathcal{M}_\mu^\vee = \mathcal{M}_{\mu^\vee}$. (This dualizing is only for convenience and is dictated by personal tastes. Dualizing here, avoids some negative signs elsewhere.) It is convenient to pass to the limit over all open-compact subgroups K_f and let $H^\bullet(S^{G_n}, \widetilde{\mathcal{M}}_{\mu,E}^\vee) := \varinjlim_{K_f} H^\bullet(S_{K_f}^{G_n}, \widetilde{\mathcal{M}}_{\mu,E}^\vee)$. There is an action of $\pi_0(G_{n,\infty}) \times G_n(\mathbb{A}_f)$ on $H^\bullet(S^{G_n}, \widetilde{\mathcal{M}}_{\mu,E}^\vee)$, and the cohomology of $S_{K_f}^{G_n}$ is obtained by taking invariants under K_f , i.e., $H^\bullet(S_{K_f}^{G_n}, \widetilde{\mathcal{M}}_{\mu,E}^\vee) = H^\bullet(S^{G_n}, \widetilde{\mathcal{M}}_{\mu,E}^\vee)^{K_f}$. See 2.3.5 below.

Working at a transcendental level, i.e., taking $E = \mathbb{C}$, we can compute the above sheaf cohomology via the de Rham complex, and then reinterpreting the de Rham complex in terms of the complex computing relative Lie algebra cohomology, we get the isomorphism:

$$H^\bullet(S^{G_n}, \widetilde{\mathcal{M}}_{\mu,\mathbb{C}}^\vee) \simeq H^\bullet(\mathfrak{g}_n, K_{n,\infty}^0; C^\infty(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) \otimes \mathcal{M}_{\mu,\mathbb{C}}^\vee).$$

With level structure K_f we have: $H^\bullet(S_{K_f}^{G_n}, \widetilde{\mathcal{M}}_{\mu,\mathbb{C}}^\vee) \simeq H^\bullet(\mathfrak{g}_n, K_{n,\infty}^0; C^\infty(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A}))^{K_f} \otimes \mathcal{M}_{\mu,\mathbb{C}}^\vee)$.

2.3.3. Cuspidal cohomology. The inclusion $C_{\text{cusp}}^\infty(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) \hookrightarrow C^\infty(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A}))$ of the space of smooth cusp forms in the space of all smooth functions induces, via results of Borel [5], an injection in cohomology; this defines cuspidal cohomology:

$$(2.2) \quad \begin{array}{ccc} H^\bullet(S^{G_n}, \widetilde{\mathcal{M}}_{\mu,\mathbb{C}}^\vee) & \longrightarrow & H^\bullet(\mathfrak{g}_n, K_{n,\infty}^0; C^\infty(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) \otimes \mathcal{M}_{\mu,\mathbb{C}}^\vee) \\ \uparrow & & \uparrow \\ H_{\text{cusp}}^\bullet(S^{G_n}, \widetilde{\mathcal{M}}_{\mu,\mathbb{C}}^\vee) & \longrightarrow & H^\bullet(\mathfrak{g}_n, K_{n,\infty}^0; C_{\text{cusp}}^\infty(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) \otimes \mathcal{M}_{\mu,\mathbb{C}}^\vee) \end{array}$$

Using the usual decomposition of the space of cusp forms into a direct sum of cuspidal automorphic representations, we get the following fundamental decomposition of $\pi_0(G_n(\mathbb{R})) \times G_n(\mathbb{A}_f)$ -modules:

$$(2.3) \quad H_{\text{cusp}}^\bullet(S^{G_n}, \widetilde{\mathcal{M}}_{\mu,\mathbb{C}}^\vee) = \bigoplus_{\Pi} H^\bullet(\mathfrak{g}_n, K_{n,\infty}^0; \Pi_\infty \otimes \mathcal{M}_{\mu,\mathbb{C}}^\vee) \otimes \Pi_f.$$

We say that Π contributes to the cuspidal cohomology of G_n with coefficients in $\mathcal{M}_{\mu,\mathbb{C}}^\vee$, and we write $\Pi \in \text{Coh}(G_n, \mu^\vee)$, if Π has a nonzero contribution to the above decomposition. Equivalently, if Π is a cuspidal automorphic representation whose representation at infinity Π_∞ after twisting by $\mathcal{M}_{\mu,\mathbb{C}}^\vee$ has nontrivial relative Lie algebra cohomology. With a level structure K_f , (2.3) takes the form:

$$(2.4) \quad H_{\text{cusp}}^\bullet(S_{K_f}^{G_n}, \widetilde{\mathcal{M}}_{\mu,\mathbb{C}}^\vee) = \bigoplus_{\Pi} H^\bullet(\mathfrak{g}_n, K_{n,\infty}^0; \Pi_\infty \otimes \mathcal{M}_{\mu,\mathbb{C}}^\vee) \otimes \Pi_f^{K_f}$$

We write $\Pi \in \text{Coh}(G_n, \mu^\vee, K_f)$ if a cuspidal automorphic representation Π contributes nontrivially to (2.4). Note that $\text{Coh}(G_n, \mu^\vee, K_f)$ is a finite set, and $\text{Coh}(G_n, \mu^\vee) = \cup_{K_f} \text{Coh}(G_n, \mu^\vee, K_f)$.

2.3.4. Purity. Let $\mu \in X^+(T_n)$ be a dominant integral weight satisfying the condition in (2.3.2). Suppose $\Pi \in \text{Coh}(G_n, \mu^\vee)$. The fact that μ^\vee supports cuspidal cohomology places some restrictions on μ . First of all, essential unitarity of Π , and in particular of Π_∞ gives, via Wigner's Lemma, essential self-duality of μ : there is an integer $w(\mu)$ such that

- (1) For $v \in S_r$ and $1 \leq i \leq n$ we have $\mu_i^{l_v} + \mu_{n-i+1}^{l_v} = w(\mu)$;
- (2) For $v \in S_c$ and $1 \leq i \leq n$ we have $\mu_i^{l_v} + \mu_{n-i+1}^{l_v} = w(\mu)$.

We will call such a weight μ as a *pure weight* and call $w(\mu)$ the *purity weight of μ* . Let $X_0^+(T_n)$ denote the set of dominant integral pure weights.

Furthermore, any $\Pi \in \text{Coh}(G_n, \mu^\vee)$ satisfies a purity condition (Clozel [7, Lem. 4.9]) which is a translation to the automorphic side of the phenomenon that the associated motive $M(\Pi)$ is pure—a condition on the Hodge types of $M(\Pi)$. For $v \in S_\infty$, let $r_L(\Pi_v)$ stand for the Langlands parameter of Π_v ; it is an n -dimensional semi-simple representation of the Weil group W_{F_v} of F_v ; $\mathbb{C}^\times \subset W_{F_v}$ as a subgroup of index at most 2. The representation $r_L(\Pi_v)|_{\mathbb{C}^\times}^{(1-n)/2}$ of \mathbb{C}^\times is a sum of n characters $z \mapsto z^p \bar{z}^q$ then $p, q \in \mathbb{Z}$. Purity says that there is an integer $w(\Pi)$ such that for any $v \in S_\infty$ all the exponents in $r_L(\Pi_v)|_{\mathbb{C}^\times}^{(1-n)/2}$ satisfy $p + q = w(\Pi)$. We will call $w(\Pi)$ the *purity weight of Π* , and it is related to the purity weight of μ via $w(\mu) = n - 1 + w(\Pi)$.

Finally, by Clozel's theorem that cuspidal cohomology has a rational structure, we get

$$\Pi \in \text{Coh}(G_n, \mu^\vee) \implies \sigma\Pi \in \text{Coh}(G_n, \sigma\mu^\vee), \quad \forall \sigma \in \text{Aut}(\mathbb{C}).$$

In particular, $\sigma\mu$ also satisfies the purity conditions (1) and (2) above. Note that $w(\mu) = w(\sigma\mu)$. Since this purity weight is going to appear frequently, we will often denote: $w := w(\mu) = w(\sigma\mu)$.

Definition 2.5. Let $\mu \in X^+(T_n)$ be a dominant integral weight satisfying the condition in (2.3.2). We say μ is *strongly pure* if $\sigma\mu$ is pure for all $\sigma \in \text{Aut}(\mathbb{C})$. Let $X_{00}^+(T_n)$ stand for the set of dominant integral strongly-pure weights. (Note that if a dominant integral weight μ , is such that for all $\sigma \in \text{Aut}(\mathbb{C})$, the weight $\sigma\mu$ satisfies the purity conditions (1) and (2) above, then necessarily, μ satisfies the the condition in (2.3.2).)

For any F , we have the following inclusions $X_{00}^+(T_n) \subset X_0^+(T_n) \subset X^+(T_n) \subset X^*(T_n)$ and in general they are all strict inclusions. If F is a totally real field or a CM field (totally imaginary quadratic extension of a totally real field) then μ is pure if and only if μ is strongly pure. However, this is not true in general; it is easy to give an example of a weight μ which is pure but not strongly pure when the base field is $F = \mathbb{Q}(2^{1/3})$ or it's Galois closure. For any number field, one may see that there are strongly pure weights μ . Take an integer b and integers $a_1 \geq a_2 \geq \dots \geq a_n$ such that $a_j + a_{n-j+1} = b$; now for each $\iota \in \mathcal{E}_F$ put $\mu^\iota = (a_1, \dots, a_n)$; then μ is strongly pure with $w(\mu) = b$. Such a weight may be called a *parallel weight*.

2.3.5. Hecke action Vs Galois action. We record here a fact, well-known to experts, but seemingly hard to find in the literature, on the relation between the action of $G_n(\mathbb{A}_f) \times \pi_0(G_{n,\infty})$ and the action of $\text{Aut}(\mathbb{C})$ on $H_{\text{cusp}}^\bullet(S^{G_n}, \widetilde{\mathcal{M}}_{\mu,\mathbb{C}})$. Given $\mu \in X_{00}^+(T_n)$ and $\sigma \in \text{Aut}(\mathbb{C})$, let's denote $T_\sigma : \mathcal{M}_\mu \rightarrow \mathcal{M}_{\sigma\mu} = \mathcal{M}_\mu \otimes_{\mathbb{C}} \sigma\mathbb{C}$ for the σ -linear $G_n(F)$ -isomorphism as in 2.1.5. Let $T_\sigma^* : \widetilde{\mathcal{M}}_\mu \rightarrow \widetilde{\mathcal{M}}_{\sigma\mu}$ denote the corresponding σ -linear isomorphism of sheaves on $S_{K_f}^{G_n}$, and $T_\sigma^\bullet : H^\bullet(S_{K_f}^{G_n}, \widetilde{\mathcal{M}}_\mu) \rightarrow H^\bullet(S_{K_f}^{G_n}, \widetilde{\mathcal{M}}_{\sigma\mu})$ the σ -linear isomorphism in cohomology. Then T_σ^\bullet preserves inner cohomology, and by Clozel [7], it also preserves cuspidal cohomology. Next, given $\underline{x} = (\underline{x}_f, \underline{x}_\infty) \in G_n(\mathbb{A}_f) \times G_{n,\infty}$, the map $\underline{g} \mapsto \underline{g}\underline{x}$ induces a homeomorphism $m_{\underline{x}} : S_{K_f}^{G_n} \rightarrow S_{\underline{x}_f^{-1}K_f\underline{x}_f}^{G_n}$, whose dependence on \underline{x}_∞ is only via its class in $\pi_0(G_{n,\infty})$. As sheaves on $S_{K_f}^{G_n}$, we have $m_{\underline{x}}^* \widetilde{\mathcal{M}}_\mu \simeq \widetilde{\mathcal{M}}_\mu$, where the left hand side is the pull-back via $m_{\underline{x}}$ of the sheaf $\widetilde{\mathcal{M}}_\mu$ on $S_{\underline{x}_f^{-1}K_f\underline{x}_f}^{G_n}$. Furthermore, we have $T_\sigma^*(m_{\underline{x}}^*(\widetilde{\mathcal{M}}_{\mu,\mathbb{C}})) = m_{\underline{x}}^*(T_\sigma^*(\widetilde{\mathcal{M}}_{\mu,\mathbb{C}}))$ as sheaves on $S_{K_f}^{G_n}$. Taking cohomology, and passing to the limit over all K_f , gives an equality $T_\sigma^\bullet \circ m_{\underline{x}}^\bullet = m_{\underline{x}}^\bullet \circ T_\sigma^\bullet$ of maps from $H^\bullet(S^{G_n}, \widetilde{\mathcal{M}}_{\mu,\mathbb{C}})$ to $H^\bullet(S^{G_n}, \widetilde{\mathcal{M}}_{\sigma\mu,\mathbb{C}})$. The same holds for cohomology with compact supports, and hence for inner and also for cuspidal cohomology via [7]. Therefore,

if $\Pi_f \otimes \epsilon$ is a $G_n(\mathbb{A}_f) \times \pi_0(G_{n,\infty})$ -module appearing in inner or cuspidal cohomology, then:

$$(2.6) \quad T_\sigma^\bullet(\Pi_f \otimes \epsilon) = {}^\sigma \Pi_f \otimes \epsilon,$$

since the image of ϵ is ± 1 and by the definition of ${}^\sigma \Pi_f$. See 2.5.2.5 below.

2.4. Archimedean considerations. Let $\mu \in X_{00}^+(T_n)$ be a strongly pure weight, and let $\Pi \in \text{Coh}(G_n, \mu^\vee)$. The purpose of this section is to write down explicitly the representation Π_∞ at infinity in terms of μ ; this is possible up to a sign; see Propositions 2.9, 2.11 and 2.14 below. We record a well-known conclusion—due to Clozel—on the possible degrees in which one has nontrivial cuspidal cohomology; see Prop. 2.15. This gives rise to an interesting numerology that ultimately permits us to give a cohomological interpretation to Rankin–Selberg L -values. Also, with local representations at hand, we compute the set of critical points of Rankin–Selberg L -functions. Since $\Pi_\infty = \otimes_{v \in S_\infty} \Pi_v$, the problem of describing Π_∞ is a purely local one. We begin by taking up real and complex places separately.

2.4.1. Cohomological representations of $\text{GL}_n(\mathbb{R})$. Fix a place $v \in S_r$, and since v is fixed, we drop it from our notations just for this subsection. For example, μ_v is just $\mu = (\mu_1, \dots, \mu_n)$, an n -tuple of integers with $\mu_1 \geq \dots \geq \mu_n$ and $\mu_i + \mu_{n-i+1} = \mathbf{w}$. We will now define the representation J_μ if n is even; and two representations J_μ^\pm if n is odd. For this we need to fix some notation for discrete series representations of $\text{GL}_2(\mathbb{R})$.

2.4.1.1. Discrete series for $\text{GL}_2(\mathbb{R})$. For any integer $l \geq 1$, let D_l stand for the discrete series representation with lowest non-negative K -type being the character $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mapsto e^{-i(l+1)\theta}$, and central character $a \mapsto \text{sgn}(a)^{l+1}$. Note the shift from l to $l+1$. The representation at infinity for a holomorphic elliptic modular cusp form of weight k is D_{k-1} . It is well-known that discrete series representations of $\text{GL}_2(\mathbb{R})$, possibly twisted by a half-integral power of absolute value, have nontrivial cohomology. For brevity, let $(\mathfrak{g}_2, K_2^0) := (\mathfrak{gl}_2, \text{SO}(2)Z_2(\mathbb{R})^0)$. For a dominant integral weight $\nu = (a, b)$, with integers $a \geq b$, the basic fact here is that there is a non-split exact sequence of (\mathfrak{g}_2, K_2^0) -modules:

$$(2.7) \quad 0 \rightarrow D_{a-b+1} \otimes | \cdot |_{\mathbb{R}}^{(a+b)/2} \rightarrow \text{Ind}_{B_2(\mathbb{R})}^{\text{GL}_2(\mathbb{R})} (\chi_{(a,a)} | \cdot |^{1/2} \otimes \chi_{(b,b)} | \cdot |^{-1/2}) \rightarrow \mathcal{M}_{\nu, \mathbb{C}} \rightarrow 0,$$

Moreover, $H^\bullet(\mathfrak{g}_2, K_2^0; (D_{a-b+1} \otimes | \cdot |_{\mathbb{R}}^{(a+b)/2}) \otimes \mathcal{M}_{\nu, \mathbb{C}}^\vee) \neq 0$ if and only if $\bullet = 1$, and that dimension of $H^1(\mathfrak{g}_2, K_2^0; (D_{a-b+1} \otimes | \cdot |_{\mathbb{R}}^{(a+b)/2}) \otimes \mathcal{M}_{\nu, \mathbb{C}}^\vee)$ is two, with both the characters of $\text{O}(2)/\text{SO}(2)$ appearing exactly once. (For more details see Raghuram–Tanabe [38, Sect. 3.1].)

2.4.1.2. Definition of J_μ when n is even. Given $\mu = (\mu_1, \dots, \mu_n)$ define an n -tuple $\ell = \ell(\mu) = (\ell_1, \dots, \ell_n)$ by $\ell = 2\mu + 2\mathbf{p}_n - \mathbf{w}$, i.e., we have

$$\ell_i = 2\mu_i + n - 2i + 1 - \mathbf{w} = \mu_i - \mu_{n-i+1} + n - 2i + 1, \quad 1 \leq i \leq n.$$

Observe that $\ell_1 > \ell_2 > \dots > \ell_{n/2} \geq 1$ and $\ell_i = -\ell_{n-i+1}$. Let P be the $(2, \dots, 2)$ -parabolic subgroup of $\text{GL}_n(\mathbb{R})$, i.e., P has the Levi quotient $L = \prod_{i=1}^{n/2} \text{GL}_2(\mathbb{R})$. Define the parabolically induced representation:

$$(2.8) \quad J_\mu := \text{Ind}_{P(\mathbb{R})}^{\text{GL}_n(\mathbb{R})} \left(D(\ell_1) |\det|^{\mathbf{w}/2} \otimes \dots \otimes D(\ell_{n/2}) |\det|^{\mathbf{w}/2} \right).$$

(This is a small change in notation from some of my earlier papers ([35], [36]) where J_μ , following Mahnkopf [33], was denoted $J(\mathbf{w}, \ell)$.) We will refer to the integers in ℓ as the cuspidal parameters of J_μ . It is well-known that J_μ is irreducible, essentially tempered and generic (being fully induced from essentially discrete series), and $H^\bullet(\mathfrak{gl}_n, \text{SO}(n); J_\mu \otimes \mathcal{M}_\mu^\vee) = H^\bullet(\mathfrak{gl}_n, \text{SO}(n)\mathbb{R}_+^\times; J_\mu \otimes \mathcal{M}_\mu^\vee) \neq 0$.

The following proposition describes the local component for a real place of a global cohomological cuspidal representation; it says that when n is even, the highest weight μ_v determines the isomorphism class of Π_v .

Proposition 2.9. *Let $\mu \in X_{00}^+(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu^\vee)$. Suppose n is even. Let $v \in S_r$ be a real place. Then $\Pi_v \cong J_{\mu_v}$.*

2.4.1.3. Definition of J_μ^ϵ when n is odd. When n is odd, for any sign ϵ , we define a representation J_μ^ϵ as follows. The cuspidal parameter ℓ is again defined by $\ell = 2\mu + 2\rho_n - \mathbf{w}$. This time, let P to be the $(2, \dots, 2, 1)$ -parabolic subgroup. Define

$$(2.10) \quad J_\mu^\epsilon := \text{Ind}_{P(\mathbb{R})}^{\text{GL}_n(\mathbb{R})} \left(D(\ell_1) |\det|^{\mathbf{w}/2} \otimes \cdots \otimes D(\ell_{n/2}) |\det|^{\mathbf{w}/2} \otimes \epsilon |\det|^{\mathbf{w}/2} \right).$$

It is well-known that J_μ^ϵ is irreducible, essentially tempered, generic and that the relative cohomology group $H^\bullet(\mathfrak{gl}_n, \text{SO}(n); J_\mu^\epsilon \otimes \mathcal{M}_\mu^\vee) = H^\bullet(\mathfrak{gl}_n, \text{SO}(n)\mathbb{R}_+^\times; J_\mu^\epsilon \otimes \mathcal{M}_\mu^\vee)$ is one-dimensional. Reverting to global notation, we have the following proposition which says that when n is odd, the highest weight μ_v and the sign of the central character of Π_v determine the isomorphism class of Π_v as a representation of $\text{GL}_n(\mathbb{R})$.

Proposition 2.11. *Let $\mu \in X_{00}^+(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu^\vee)$. Suppose n is odd. Let $v \in S_r$ be a real place. Then $\Pi_v \cong J_{\mu_v}^{\epsilon_v}$, where $\epsilon_v(-1) = \omega_{\Pi_v}(-1) \cdot (-1)^{(n-1)/2}$.*

The reader, who wishes to verify the above equality of signs, should note the following consequences of n being odd: (1) The purity weight \mathbf{w} is even since $\mathbf{w} = 2\mu_{(n+1)/2}$, and (2) The cuspidal parameters are even since $l_i = 2\mu_i + n - 2i + 1 - \mathbf{w}$.

2.4.2. Cohomological representations of $\text{GL}_n(\mathbb{C})$. Let $\mu \in X_{00}^+(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu^\vee)$. For a complex place v , μ_v is a pair of n -tuples $(\mu^{t_v}, \mu^{\bar{t}_v})$, where t_v is a complex embedding corresponding to v that has been noncanonically chosen and fixed; and \bar{t}_v is the conjugate embedding. Since $v \in S_c$ is fixed, we will drop it from our notations. Hence, $\mu = (\mu^t, \mu^{\bar{t}})$; we will further simplify our notation and write $\mu^t = (\mu_1, \dots, \mu_n)$ and $\mu^{\bar{t}} = (\mu_1^*, \dots, \mu_n^*)$. Recall from Sect. 2.3.4 that the integers in the pair $(\mu^t, \mu^{\bar{t}})$ are related by: $\mu_i^* + \mu_{n-i+1} = \mathbf{w}$. Hence, we have

$$\mu^t = (\mu_1, \dots, \mu_n), \quad \text{and} \quad \mu^{\bar{t}} = (\mathbf{w} - \mu_n, \dots, \mathbf{w} - \mu_1).$$

Define the cuspidal parameters as:

$$(2.12) \quad \begin{aligned} \mathbf{a} &:= \mu + \rho_n; & \mathbf{a} = (a_1, \dots, a_n) &:= \left(\mu_1 + \frac{n-1}{2}, \mu_2 + \frac{n-3}{2}, \dots, \mu_n - \frac{(n-1)}{2} \right) \\ \mathbf{b} &:= \mathbf{w} - \mu - \rho_n; & \mathbf{b} = (b_1, \dots, b_n) &:= \left(\mathbf{w} - \mu_1 - \frac{n-1}{2}, \mathbf{w} - \mu_2 - \frac{n-3}{2}, \dots, \mathbf{w} - \mu_n + \frac{(n-1)}{2} \right). \end{aligned}$$

Now define the representation J_μ to be induced from the Borel subgroup $B(\mathbb{C})$ of $\text{GL}_n(\mathbb{C})$ as:

$$(2.13) \quad J_\mu := \text{Ind}_{B(\mathbb{C})}^{\text{GL}_n(\mathbb{C})} \left(z^{a_1} \bar{z}^{b_1} \otimes \cdots \otimes z^{a_n} \bar{z}^{b_n} \right).$$

where, for any half-integers a, b , by $z^a \bar{z}^b$ we mean the character of \mathbb{C}^\times which sends z to $z^a \bar{z}^b$. It is well-known that J_μ is irreducible, essentially tempered, generic and $H^\bullet(\mathfrak{gl}_n, \text{U}(n); J_\mu \otimes \mathcal{M}_\mu^\vee) \neq 0$. Reverting to global notation, we have:

Proposition 2.14. *Let $\mu \in X_{00}^+(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu^\vee)$. Let $v \in S_c$, i.e., v is a complex place. Then $\Pi_v \cong J_{\mu_v}$.*

2.4.3. The cuspidal range. In this subsection we record well-known bounds for the possible degrees in which there can be nonzero cuspidal cohomology. These bounds depend only on the rank n of GL_n , and the numbers r_1 and r_2 of real and complex embeddings of F .

Proposition 2.15. *Define the following numbers:*

The bottom degrees:

$$b_n^{\mathbb{R}} := \left\lfloor \frac{n^2}{4} \right\rfloor, \quad b_n^{\mathbb{C}} := n(n-1)/2.$$

The top degrees:

$$t_n^{\mathbb{R}} = b_n^{\mathbb{R}} + \left\lfloor \frac{n-1}{2} \right\rfloor, \quad t_n^{\mathbb{C}} = b_n^{\mathbb{C}} + n - 1.$$

Now define the bottom degree and top degree for the group $G_n = R_{F/\mathbb{Q}}(\mathrm{GL}_n/F)$ as:

$$b_n^F = r_1 b_n^{\mathbb{R}} + r_2 b_n^{\mathbb{C}}, \quad t_n^F = r_1 t_n^{\mathbb{R}} + r_2 t_n^{\mathbb{C}}, \quad \tilde{t}_n^F = t_n^F + [F : \mathbb{Q}] - 1$$

Let $\mu \in X_{00}^+(T_n)$. Then $H_{\mathrm{cusp}}^{\bullet}(S^{G_n}, \widetilde{\mathcal{M}}_{\mu}^{\vee}) \neq 0 \iff b_n^F \leq \bullet \leq \tilde{t}_n^F$.

Proof. Given $\mu \in X_{00}^+(T_n)$, and a $\Pi \in \mathrm{Coh}(G_n, \mu^{\vee})$, we know that $\Pi_{\infty} = \otimes_{v \in S_{\infty}} J_{\mu_v}^{\epsilon_v}$. (The symbol ϵ_v is a nonempty condition only for n odd and $v \in S_r$.) For $v \in S_{\infty}$, let $\tilde{G}_{n,v} = \{g_v \in \mathrm{GL}_n(F_v) : |\det(g_v)| = 1\}$ and let $\tilde{\mathfrak{g}}_{n,v}$ be the Lie algebra of $\tilde{G}_{n,v}$. Note that $\mathfrak{g}_{n,v} = \tilde{\mathfrak{g}}_{n,v} \oplus \mathfrak{z}_{n,v}$ and summing over $v \in S_{\infty}$, we have $\mathfrak{g}_n = \tilde{\mathfrak{g}}_n + \mathfrak{z}_n$. By Wigner's Lemma we have:

$$H^{\bullet}(\mathfrak{g}_n, K_{n,\infty}^0; \Pi_{\infty} \otimes \mathcal{M}_{\mu,\mathbb{C}}^{\vee}) = H^{\bullet}(\tilde{\mathfrak{g}}_n, C_{n,\infty}^0; \Pi_{\infty} \otimes \mathcal{M}_{\mu,\mathbb{C}}^{\vee}) \otimes \wedge^{\bullet}(\mathfrak{z}_n/\mathfrak{s}_n).$$

The term $\wedge^{\bullet}(\mathfrak{z}_n/\mathfrak{s}_n)$ accounts for the difference between \tilde{t}_n^F and t_n^F . By the Künneth formula we get

$$H^q(\tilde{\mathfrak{g}}_n, C_{n,\infty}^0; \Pi_{\infty} \otimes \mathcal{M}_{\mu,\mathbb{C}}^{\vee}) = \bigotimes_{\sum_v q_v = q} H^{q_v}(\tilde{\mathfrak{g}}_{n,v}, C_{n,v}^0; \Pi_v \otimes \mathcal{M}_{\mu_v,\mathbb{C}}^{\vee}).$$

From Clozel [7, Lemme 3.14], we get $H^{q_v}(\tilde{\mathfrak{g}}_{n,v}, C_{n,v}^0; \Pi_v \otimes \mathcal{M}_{\mu_v,\mathbb{C}}^{\vee}) \neq 0$ if and only if $b_n^{F_v} \leq q_v \leq t_n^{F_v}$. \square

Motivated by the numerical coincidence to be discussed below, we will focus our attention exclusively on cohomology in degree $\bullet = b_n^F$. (In contrast, see my paper with Grobner [17] where we considered top-degree cuspidal cohomology.)

Corollary 2.16. *Let $\Pi \in \mathrm{Coh}(G_n, \mu^{\vee})$ for $\mu \in X_{00}^+(T_n)$.*

- (1) *If n is even then every character $\epsilon = (\epsilon_v)_{v \in S_r}$ of $\pi_0(K_{n,\infty}) = K_{n,\infty}/K_{n,\infty}^0$ appears once in $H^{b_n^F}(\mathfrak{g}_n, K_{n,\infty}^0; \Pi_{\infty} \otimes \mathcal{M}_{\mu,\mathbb{C}}^{\vee})$; hence cuspidal cohomology in degree b_n^F decomposes as*

$$H_{\mathrm{cusp}}^{b_n^F}(S^{G_n}, \widetilde{\mathcal{M}}_{\mu}^{\vee}) = \bigoplus_{\Pi} \bigoplus_{\epsilon} \epsilon \otimes \Pi_f,$$

where Π runs over $\mathrm{Coh}(G_n, \mu^{\vee})$ and ϵ runs over all possible characters of $\pi_0(K_{n,\infty})$. Hence, each Π_f appears 2^{r_1} times in cuspidal cohomology in degree b_n^F .

- (2) *If n is odd then $H^{b_n^F}(\mathfrak{g}_n, K_{n,\infty}^0; \Pi_{\infty} \otimes \mathcal{M}_{\mu,\mathbb{C}}^{\vee})$ is one-dimensional and the character of $\pi_0(K_{n,\infty})$ which appears is denoted as $\epsilon_{\Pi_{\infty}}$. In this case, cuspidal cohomology decomposes as*

$$H_{\mathrm{cusp}}^{b_n^F}(S^{G_n}, \widetilde{\mathcal{M}}_{\mu}^{\vee}) = \bigoplus_{\Pi} \epsilon_{\Pi_{\infty}} \otimes \Pi_f,$$

where Π runs over $\mathrm{Coh}(G_n, \mu^{\vee})$; and each Π_f appears once.

As in [37], we say that an r_1 -tuple of signs $\epsilon = (\epsilon_v)_{v \in S_r}$ is *permissible* for μ if $\epsilon = \epsilon_{\Pi_\infty}$ for some $\Pi \in \text{Coh}(G_n, \mu^\vee)$ when n is odd, and is any of the possible 2^{r_1} signatures when n is even. For any n , given $\mu \in X_{00}^+(T_n)$, $\Pi \in \text{Coh}(G_n, \mu^\vee)$, and given a permissible signature ϵ for μ , the $\epsilon \otimes \Pi_f$ isotypic component in cuspidal cohomology in degree b_n^F , via the isomorphism in (2.2), can be expressed in terms of the space V_Π of cusp forms realizing Π as

$$H_{\text{cusp}}^{b_n^F}(S^{G_n}, \widetilde{\mathcal{M}}_{\mu, \mathbb{C}}^\vee)(\epsilon \otimes \Pi_f) = H^{b_n^F}(\mathfrak{g}_n, K_{n, \infty}^0; V_\Pi \otimes \mathcal{M}_{\mu, \mathbb{C}}^\vee)(\epsilon),$$

where the right hand side is the ϵ -isotypic component in $H^{b_n^F}(\mathfrak{g}_n, K_{n, \infty}^0; V_\Pi \otimes \mathcal{M}_{\mu, \mathbb{C}}^\vee)$ for the action of $K_{n, \infty}/K_{n, \infty}^0$.

2.4.4. A numerical coincidence. We record a relation between the numbers b_n^F and b_{n-1}^F and the dimension of a locally symmetric space for G_{n-1} which is crucial for giving a cohomological interpretation to the Rankin–Selberg theory for $\text{GL}_n \times \text{GL}_{n-1}$. Define

$$\tilde{S}_{K_f}^{G_n} := G_n(\mathbb{Q}) \backslash G_n(\mathbb{A}) / C_{n, \infty}^0 K_f = \text{GL}_n(F) \backslash \text{GL}_n(\mathbb{A}) / C_{n, \infty}^0 K_f$$

where K_f is an open compact subgroup of $G_n(\mathbb{A}_f)$, and $C_{n, \infty}^0 = \prod_{v \in S_r} \text{SO}(n) \times \prod_{v \in S_c} \text{U}(n)$ is the connected component of the identity in the maximal compact subgroup $C_{n, \infty}$ of $G_n(\mathbb{R})$. Since $K_{n, \infty}^0 = S_n(\mathbb{R})^0 C_{n, \infty}^0$, see Sect. 2.1.3 for our notations, we get a canonical fibration ϕ given by:

$$\begin{array}{ccc} \tilde{S}_{K_f}^{G_n} & = & G_n(\mathbb{Q}) \backslash G_n(\mathbb{A}) / C_{n, \infty}^0 K_f \\ & & \downarrow \phi \\ S_{K_f}^{G_n} & = & G_n(\mathbb{Q}) \backslash G_n(\mathbb{A}) / K_{n, \infty}^0 K_f \end{array}$$

Proposition 2.17. *Let $n \geq 2$. Let b_n^F be the bottom degree for G_n as defined in Prop. 2.15. Then*

$$b_n^F + b_{n-1}^F = \dim(\tilde{S}_{R_f}^{G_{n-1}}),$$

for any open compact subgroup R_f of $G_{n-1}(\mathbb{A}_f)$.

Proof. Note that

$$\begin{aligned} \dim(\tilde{S}_{R_f}^{G_{n-1}}) &= \dim(G_{n-1}(\mathbb{R})^0 / C_{n, \infty}^0) \\ &= r_1 \cdot \dim(\text{GL}_{n-1}(\mathbb{R})^0 / \text{SO}(n-1)) + r_2 \cdot \dim(\text{GL}_{n-1}(\mathbb{C}) / U(n-1)). \end{aligned}$$

The proof follows if we check that

$$\begin{aligned} b_n^{\mathbb{R}} + b_{n-1}^{\mathbb{R}} &= \dim(\text{GL}_{n-1}(\mathbb{R})^0 / \text{SO}(n-1)), \\ b_n^{\mathbb{C}} + b_{n-1}^{\mathbb{C}} &= \dim(\text{GL}_{n-1}(\mathbb{C}) / U(n-1)). \end{aligned}$$

These are easy to verify using the definitions of $b_n^{\mathbb{R}}$ and $b_n^{\mathbb{C}}$; for example, $b_n^{\mathbb{C}} + b_{n-1}^{\mathbb{C}} = n(n-1)/2 + (n-1)(n-2)/2 = (n-1)^2 = \dim(\text{GL}_{n-1}(\mathbb{C}) / U(n-1))$. \square

For $F = \mathbb{Q}$ and $n = 3$, this kind of numerology is apparent in Schmidt [41] and Mahnkopf [32]. For $F = \mathbb{Q}$ and general n , see Kazhdan, Mazur and Schmidt [28, Table on p. 100]; in this situation, the numerology was cleverly exploited by Mahnkopf [33], and so was used in my paper [35]. For general n and F , this was recently used by Januszewski [23] in his study of modular symbols.

2.4.5. Critical points and compatibility of coefficient systems.

2.4.5.1. **Definition of the critical set.** Consider the Rankin–Selberg L -function $L(s, \Pi \times \Sigma)$ where Π (resp., Σ) is a cuspidal automorphic representation $\mathrm{GL}_n(\mathbb{A}_F)$ (resp., $\mathrm{GL}_{n-1}(\mathbb{A}_f)$).

Definition 2.18. We say that $s_0 = \frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z}$ is critical for $L(s, \Pi \times \Sigma)$ if both $L_\infty(s, \Pi_\infty \times \Sigma_\infty)$ and $L_\infty(1 - s, \Pi_\infty^\vee \times \Sigma_\infty^\vee)$ are regular at $s = s_0$, i.e., both the L -factors at infinity on either side of the functional equation are holomorphic at $s = s_0$.

By definition, we only look at half-integers, i.e, the critical set is a subset of $\frac{1}{2} + \mathbb{Z}$. This has to do with the so-called motivic normalization: that if a cuspidal representation Θ of GL_r corresponds to a motive M then under this correspondence, $L(s + \frac{1-r}{2}, \Theta) = L(s, M)$. On the motivic side, one always looks for critical points amongst integers; see Deligne [9]. Transcribing to the automorphic side, one looks for critical points amongst $\frac{r-1}{2} + \mathbb{Z}$; in particular, if r is even then we look for critical points in $\frac{1}{2} + \mathbb{Z}$. In our situation, assuming Langlands’s functoriality, we have $\Theta = \Pi \boxtimes \Sigma$, which is (usually) a cuspidal representation of $\mathrm{GL}_r(\mathbb{A}_F)$ with $r = n(n-1)$; in particular r is even; hence the critical set for $L(s, \Pi \times \Sigma)$ consists of only half-integers. Another easy point to note is that given a particular half-integer $s_0 = \frac{1}{2} + m$, to check whether s_0 is critical or not is an entirely local calculation because $L_\infty(s_0, \Pi_\infty \times \Sigma_\infty) = \prod_{v \in S_\infty} L_v(s_0, \Pi_v \times \Sigma_v)$ and local L -factors are nonvanishing everywhere.

2.4.5.2. **Branching rule for the pair** $(\mathrm{GL}_n(\mathbb{C}), \mathrm{GL}_{n-1}(\mathbb{C}))$. Working with local notations, let $\mu = (\mu_1, \dots, \mu_n)$ be a dominant integral weight for $\mathrm{GL}_n(\mathbb{C})$ and \mathcal{M}_μ be the irreducible finite-dimensional representation of the algebraic Lie group $\mathrm{GL}_n(\mathbb{C})$ with highest weight μ . Similarly, we have $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ and \mathcal{M}_λ for $\mathrm{GL}_{n-1}(\mathbb{C})$. The following branching rule is well-known (see, for example, Goodman-Wallach [14, Thm. 8.1.1]):

Proposition 2.19. *The representation \mathcal{M}_λ appears in the restriction to $\mathrm{GL}_{n-1}(\mathbb{C})$ of the representation \mathcal{M}_μ , i.e., $\mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{C})}(\mathcal{M}_\lambda, \mathcal{M}_\mu) \neq 0$ if and only if*

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \mu_n.$$

In this situation, \mathcal{M}_λ appears with multiplicity one in \mathcal{M}_μ . The conditions on the weights μ and λ captured by the above inequalities will be denoted $\mu \succ \lambda$, and we say μ interlaces λ .

Remember that we dualized the coefficient systems. Let us restate the above proposition in the form that we will need it later on. Given $\mu = (\mu_1, \dots, \mu_n)$ recall that $\mu^\vee = (-\mu_n, \dots, -\mu_1)$, and that the contragredient representation \mathcal{M}_μ^\vee of \mathcal{M}_μ is \mathcal{M}_{μ^\vee} . Similarly, for λ and \mathcal{M}_λ .

Corollary 2.20. *$\mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{C})}(\mathcal{M}_\mu^\vee \otimes \mathcal{M}_\lambda^\vee, \mathbb{C}) \neq 0$ if and only if $\mu^\vee \succ \lambda$, i.e.,*

$$-\mu_n \geq \lambda_1 \geq -\mu_{n-1} \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq -\mu_1.$$

In this situation, $\mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{C})}(\mathcal{M}_\mu^\vee \otimes \mathcal{M}_\lambda^\vee, \mathbb{C})$ is a one-dimensional space.

Proof. $\mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{C})}(\mathcal{M}_\mu^\vee \otimes \mathcal{M}_\lambda^\vee, \mathbb{C}) = \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{C})}(\mathcal{M}_\mu^\vee, \mathcal{M}_\lambda)$. □

We will also write $\mathcal{M}_\lambda \hookrightarrow \mathcal{M}_\mu$ to say that \mathcal{M}_λ appears (with multiplicity one) in \mathcal{M}_μ . Given λ and an $m \in \mathbb{Z}$, the representation $\mathcal{M}_\lambda \otimes \det^m$ corresponds to the weight $\lambda + m$. In global notations; given a weight $\lambda = (\lambda^t)_{t \in \mathcal{E}_F}$, the weight $\lambda + m$ is simply $(\lambda^t + m)_{t \in \mathcal{E}_F}$.

2.4.5.3. Critical set and compatibility. The main result of this section is the following theorem which relates the critical set to a condition on the coefficient systems.

Theorem 2.21. *Let $\mu \in X_{00}^+(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu^\vee)$. Similarly, let $\lambda \in X_{00}^+(T_{n-1})$ and $\Sigma \in \text{Coh}(G_{n-1}, \lambda^\vee)$. Assume that there is an integer m_0 such that*

$$M_{\lambda+m_0} \hookrightarrow M_\mu^\vee, \quad \text{that is } \mu^\vee \succ \lambda + m_0.$$

Then we have

$$\{m \in \mathbb{Z} : \mu^\vee \succ \lambda + m\} = \left\{m \in \mathbb{Z} : \frac{1}{2} + m \text{ is critical for } L(s, \Pi \times \Sigma)\right\}.$$

(In particular, the critical set is a finite set.)

Proof. Let us begin by noting that the assertion is of a purely local nature. If $m \in \mathbb{Z}$ then $\mu^\vee \succ \lambda + m$ if and only if for every $v \in S_\infty$ we have $\mu_v^\vee \succ \lambda_v + m$. On the other hand, $\frac{1}{2} + m$ is critical for $L(s, \Pi \times \Sigma)$ if and only if both $L_v(s, \Pi_v \times \Sigma_v)$ and $L_v(1-s, \Pi_v^\vee \times \Sigma_v^\vee)$ are holomorphic at $s = \frac{1}{2} + m$ for every $v \in S_\infty$. We will consider the real and complex places separately.

For $v \in S_r$, this has been observed by Kasten and Schmidt [27, Theorem 2.3]. What is remarkable is that their observation for a real place in fact goes through for a complex place also; see also Grobner–Harris [15, Lem. 4.7]. We very briefly summarize the proof of [27, Theorem 2.3], so that the reader may compare the similarities and the differences in the combinatorics in the real and complex cases. Fix a real place $v \in S_r$ and we drop it from the notations. The hypothesis $\mu^\vee \succ \lambda$ means

$$(2.22) \quad -\mu_n \geq \lambda_1 \geq \mu_{n-1} \geq \dots \geq \lambda_{n-1} \geq -\mu_1.$$

If ℓ and ℓ' are the cuspidal parameters of μ and λ (see 2.4.1) then define the cuspidal width between Π and Σ as:

$$c_{\Pi, \Sigma} := \min\{|\ell_i - \ell'_j| : 1 \leq i \leq n, 1 \leq j \leq n-1\}.$$

Verify that $s = 1/2$ is critical if and only if

$$(2.23) \quad -(c_{\Pi, \Sigma} - 1) \leq (\mathbf{w} + \mathbf{w}') \leq c_{\Pi, \Sigma} - 1.$$

Verify that (2.22) \implies (2.23), i.e., $\mu^\vee \succ \lambda$ implies that $s = 1/2$ is critical. During this verification one observes that the condition $\mu^\vee \succ \lambda$ in fact implies

$$\ell_1 > \ell'_1 > \ell_2 > \ell'_2 > \dots > \ell'_{n-1} > \ell'_n.$$

These inequalities are then used to verify that suppose $\mu^\vee \succ \lambda + m_0$ for some m_0 and $\frac{1}{2} + m$ is critical for $L(s, \Pi \times \Sigma)$ then $\mu^\vee \succ \lambda + m$. For more details the reader is referred to [27]. Now we take the complex case, which is combinatorially more complicated.

For the rest of this proof, we assume $v \in S_c$, i.e., that v is a complex place. Since we have now fixed $v \in S_c$, we drop it and use local notations as in 2.4.2. Recall: $\mu = (\mu^\ell, \mu^{\bar{\ell}})$ with $\mu^\ell = (\mu_1, \dots, \mu_n)$ and $\mu^{\bar{\ell}} = (\mu_1^*, \dots, \mu_n^*)$; the purity condition gives $\mu_i^* + \mu_{n-i+1} = \mathbf{w}$, where \mathbf{w} is the purity weight of the global μ ; the cuspidal parameters are:

$$(a_1, \dots, a_n) = \left(\mu_1 + \frac{n-1}{2}, \dots, \mu_n - \frac{(n-1)}{2}\right) \quad \text{and} \quad (b_1, \dots, b_n) = (\mathbf{w} - a_1, \dots, \mathbf{w} - a_n);$$

the local representation $\Pi_v = J_\mu := \text{Ind}(z^{a_1} \bar{z}^{b_1} \otimes \dots \otimes z^{a_n} \bar{z}^{b_n})$. Note that $a_i + b_i = \mathbf{w}$ and $a_i - b_i = 2\mu_i - \mathbf{w} + n - 2i + 1$. Similarly, we have $\lambda = (\lambda^\ell, \lambda^{\bar{\ell}})$ with $\lambda^\ell = (\lambda_1, \dots, \lambda_{n-1})$ and $\lambda^{\bar{\ell}} = (\lambda_1^*, \dots, \lambda_{n-1}^*)$; purity gives $\lambda_j^* + \lambda_{n-j} = \mathbf{w}'$, where $\mathbf{w}' = \mathbf{w}(\lambda)$ is the purity weight of λ ; the cuspidal parameters are:

$$(a'_1, \dots, a'_{n-1}) = \left(\lambda_1 + \frac{n-2}{2}, \dots, \lambda_{n-1} - \frac{(n-2)}{2}\right) \quad \text{and} \quad (b'_1, \dots, b'_{n-1}) = (\mathbf{w}' - a'_1, \dots, \mathbf{w}' - a'_{n-1});$$

the local representation $\Sigma_v = J_\lambda := \text{Ind} \left(z^{a'_1} \bar{z}^{b'_1} \otimes \cdots \otimes z^{a'_{n-1}} \bar{z}^{b'_{n-1}} \right)$. Note that $a'_j + b'_j = w'$ and $a'_j - b'_j = 2\lambda'_j - w' + n - 2j$.

It suffices to prove the theorem for a pair (Π, Σ) such that $\mu^\vee \succ \lambda$. This is because, suppose we have a pair (Π_1, Σ_1) such that $\mu_1^\vee \succ \lambda_1 + m_0$ for an integer m_0 , then we can consider $\Pi = \Pi_1$ and $\Sigma = \Sigma_1 \otimes | \cdot |^{m_0}$; for these representations the weights are $\mu = \mu_1$ and $\lambda = \lambda_1 + m_0$. It is easy to see that if the theorem holds for (Π, Σ) then it holds for (Π_1, Σ_1) . Henceforth we assume therefore that $\mu^\vee \succ \lambda$. We have:

Lemma 2.24. *If $\mu^\vee \succ \lambda$ then $s = \frac{1}{2}$ is critical for $L(s, \Pi \times \Sigma)$.*

Proof. The hypothesis $\mu^\vee \succ \lambda$, means $\mu^{\vee v} \succ \lambda^v$ and $\mu^{\bar{v}} \succ \lambda^{\bar{v}}$. By Cor.2.20 and the formulae for $\mu_i^* = w - \mu_{n-i+1}$ and $\lambda_j^* = w' - \lambda_{n-j}$ we have:

$$(2.25) \quad \begin{aligned} -\mu_n &\geq \lambda_1 \geq -\mu_{n-1} \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq -\mu_1, \\ w - \mu_n &\geq \lambda_1 - w' \geq w - \mu_{n-1} \geq \lambda_2 - w' \geq \cdots \geq \lambda_{n-1} - w' \geq w - \mu_1. \end{aligned}$$

Using the formulae in Knapp [29, Sect. 4], for any complex place v as above, we have

$$L_v(s, \Pi_v \times \Sigma_v) = \prod_{i,j} L \left(s, z^{a_i + a'_j} \bar{z}^{b_i + b'_j} \right) \sim \prod_{i,j} \Gamma \left(s + \frac{a_i + a'_j + b_i + b'_j}{2} + \frac{|a_i - b_i + a'_j - b'_j|}{2} \right),$$

where, by \sim , we mean up to a nonzero exponential factor which is irrelevant in computing the critical set. In terms of the data going in to μ and λ , we have:

$$(2.26) \quad L_v(s, \Pi_v \times \Sigma_v) = \prod_{1 \leq i, j \leq n} \Gamma \left(s + \frac{w + w'}{2} + \frac{|2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (w + w')|}{2} \right),$$

$$(2.27) \quad L_v(s, \Pi_v^\vee \times \Sigma_v^\vee) = \prod_{1 \leq i, j \leq n} \Gamma \left(s - \frac{w + w'}{2} + \frac{|2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (w + w')|}{2} \right).$$

It follows that $s = 1/2$ is critical if every factor on the right hand side above is regular at $s = 1/2$, i.e., if both the inequalities

$$(2.28) \quad \frac{1 + (w + w') + |2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (w + w')|}{2} \geq 1,$$

$$(2.29) \quad \frac{1 - (w + w') + |2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (w + w')|}{2} \geq 1.$$

are satisfied. Consider two cases:

Case 1: $i + j \geq n + 1$. By (2.25), we have $\mu_i + \lambda_j \leq 0$ and $\mu_i + \lambda_j \leq w + w'$. Hence

$$|2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (w + w')| = -2\mu_i - 2\lambda_j - 2n + 2i + 2j - 1 + (w + w').$$

For a given j , this quantity is minimized if $i = n + 1 - j$. Hence (2.28) is satisfied since

$$\frac{1 + (w + w') - 2\mu_{n+1-j} - 2\lambda_j + 1 + (w + w')}{2} = -\mu_{n+1-j} - \lambda_j + (w + w') + 1 \geq 1.$$

Similarly, (2.29) is satisfied since

$$\frac{1 - (w + w') - 2\mu_{n+1-j} - 2\lambda_j + 1 + (w + w')}{2} = 1 - \mu_{n+1-j} - \lambda_j \geq 1.$$

Case 2: $i + j \leq n$. By (2.25), we have $\mu_i + \lambda_j \geq 0$ and $\mu_i + \lambda_j \geq \mathbf{w} + \mathbf{w}'$. Hence

$$|2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (\mathbf{w} + \mathbf{w}')| = 2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (\mathbf{w} + \mathbf{w}').$$

For a given j , this quantity is minimized if $i = n - j$. Hence (2.28) is satisfied since

$$\frac{1 + (\mathbf{w} + \mathbf{w}') + 2\mu_{n-j} + 2\lambda_j + 1 - (\mathbf{w} + \mathbf{w}')}{2} = 1 + \mu_{n-j} + \lambda_j \geq 1.$$

Similarly, (2.29) is satisfied since

$$\frac{1 - (\mathbf{w} + \mathbf{w}') + 2\mu_{n-j} + 2\lambda_j + 1 - (\mathbf{w} + \mathbf{w}')}{2} = 1 + \mu_{n-j} + \lambda_j - (\mathbf{w} + \mathbf{w}') \geq 1.$$

□

Corollary 2.30.

$$\{m \in \mathbb{Z} : \mu^\vee \succ \lambda + m\} \subset \left\{m \in \mathbb{Z} : \frac{1}{2} + m \text{ is critical for } L(s, \Pi \times \Sigma)\right\}.$$

Proof. If m is such that $\mu^\vee \succ \lambda + m$ then by Lem. 2.24, $s = 1/2$ is critical for $L(s, \Pi \times \Sigma(m))$ where $\Sigma(m) = \Sigma \otimes |\cdot|^m$. But, $L(\frac{1}{2}, \Pi \times \Sigma(m)) = L(\frac{1}{2} + m, \Pi \times \Sigma)$. □

Proposition 2.31. *Assume $\mu^\vee \succ \lambda$. Define the numbers:*

$$\begin{aligned} m_{\mu,\lambda}^+ &:= \min \left\{ \{-\mu_{n+1-j} - \lambda_j : 1 \leq j \leq n-1\} \cup \{\mu_{n-j} + \lambda_j - (\mathbf{w} + \mathbf{w}') : 1 \leq j \leq n-1\} \right\}, \\ m_{\mu,\lambda}^- &:= \max \left\{ \{-\mu_{n-j} - \lambda_j : 1 \leq j \leq n-1\} \cup \{\mu_{n+1-j} + \lambda_j - (\mathbf{w} + \mathbf{w}') : 1 \leq j \leq n-1\} \right\}. \end{aligned}$$

Observe that $m_{\mu,\lambda}^- \leq 0 \leq m_{\mu,\lambda}^+$. We have

- (1) $\{m \in \mathbb{Z} : \mu^\vee \succ \lambda + m\} = \left\{m \in \mathbb{Z} : m_{\mu,\lambda}^- \leq m \leq m_{\mu,\lambda}^+\right\}$; and
- (2) $\left\{m \in \mathbb{Z} : \frac{1}{2} + m \text{ is critical for } L(s, \Pi \times \Sigma)\right\} = \left\{m \in \mathbb{Z} : m_{\mu,\lambda}^- \leq m \leq m_{\mu,\lambda}^+\right\}$.

Proof. That $m_{\mu,\lambda}^- \leq 0 \leq m_{\mu,\lambda}^+$ follows from (2.25). To prove (1), suppose $\mu^\vee \succ \lambda + m$, then applying (2.25) to the case of $\lambda + m$ we get:

$$-\mu_n \geq \lambda_1 + m \geq -\mu_{n-1} \geq \lambda_2 + m \geq \cdots \geq \lambda_{n-1} + m \geq -\mu_1,$$

$$\mathbf{w} - \mu_n \geq \lambda_1 - \mathbf{w}' - m \geq \mathbf{w} - \mu_{n-1} \geq \lambda_2 - \mathbf{w}' - m \geq \cdots \geq \lambda_{n-1} - \mathbf{w}' - m \geq \mathbf{w} - \mu_1.$$

From these inequalities, we deduce for all $1 \leq j \leq n-1$:

$$(2.32) \quad \begin{aligned} \mu_{n+1-j} + \lambda_j - \mathbf{w} - \mathbf{w}' &\leq m \leq -\mu_{n+1-j} - \lambda_j, \\ -\mu_{n-j} - \lambda_j &\leq m \leq \mu_{n-j} + \lambda_j - \mathbf{w} - \mathbf{w}'. \end{aligned}$$

Hence $m_{\mu,\lambda}^- \leq m \leq m_{\mu,\lambda}^+$. This entire chain of reasoning is reversible, which proves (1).

To prove (2), from Cor. 2.30 and (1), it suffices to show that if $\frac{1}{2} + m$ is critical then the integer m satisfies: $m_{\mu,\lambda}^- \leq m \leq m_{\mu,\lambda}^+$. Suppose then that $\frac{1}{2} + m$ is critical, then from (2.26) and (2.27) it follows that

$$(2.33) \quad m + \frac{1 + (\mathbf{w} + \mathbf{w}') + |2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (\mathbf{w} + \mathbf{w}')|}{2} \geq 1,$$

$$(2.34) \quad -m + \frac{1 - (\mathbf{w} + \mathbf{w}') + |2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (\mathbf{w} + \mathbf{w}')|}{2} \geq 1.$$

As in the proof of Lem. 2.24 consider two cases:

Case 1: $i + j \geq n + 1$. The hypothesis $\mu^\vee \succ \lambda$ gives the inequalities (2.25) from which we get as before $\mu_i + \lambda_j \leq 0$ and $\mu_i + \lambda_j \leq \mathbf{w} + \mathbf{w}'$. Hence

$$|2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (\mathbf{w} + \mathbf{w}')| = -2\mu_i - 2\lambda_j - 2n + 2i + 2j - 1 + (\mathbf{w} + \mathbf{w}').$$

In this case, (2.33) holds if and only if $m \geq \mu_{n+1-j} + \lambda_j - (\mathbf{w} + \mathbf{w}')$, and (2.34) holds if and only if $m \leq -\mu_{n+1-j} - \lambda_j$.

Case 2: $i + j \leq n$. In this case, exactly as above, (2.33) and (2.34) hold if and only if $-\mu_{n-j} - \lambda_j \leq m \leq \mu_{n-j} + \lambda_j - (\mathbf{w} + \mathbf{w}')$.

Putting both cases together, we see that $\frac{1}{2} + m$ is critical implies that $m_{\mu, \lambda}^- \leq m \leq m_{\mu, \lambda}^+$, which concludes the proof of the proposition. \square

This also concludes the proof of Thm. 2.21. \square

Let's revert to global notations, and for future reference, record the set of all critical points:

Corollary 2.35. *Let $\mu \in X_{00}^+(T_n)$ (resp., $\lambda \in X_{00}^+(T_{n-1})$) and $\Pi \in \text{Coh}(G_n, \mu^\vee)$ (resp., $\Sigma \in \text{Coh}(G_{n-1}, \lambda^\vee)$). Assume $\mu^\vee \succ \lambda$. Define the numbers:*

- (1) For $v \in S_r$ which corresponds to $\iota_v \in \mathcal{E}_F$:
 - $m_{\mu^{\iota_v}, \lambda^{\iota_v}}^+ = \min_{1 \leq j \leq n-1} \left\{ -\mu_{n+1-j}^{\iota_v} - \lambda_j^{\iota_v} \right\}$;
 - $m_{\mu^{\iota_v}, \lambda^{\iota_v}}^- := \max_{1 \leq j \leq n-1} \left\{ -\mu_{n-j}^{\iota_v} - \lambda_j^{\iota_v} \right\}$.
- (2) For $v \in S_c$ which corresponds to a pair of embeddings $\{\iota_v \bar{\iota}_v\}$:
 - $m_{\mu^{\iota_v}, \lambda^{\iota_v}}^+ = \min \left\{ \left\{ -\mu_{n+1-j}^{\iota_v} - \lambda_j^{\iota_v} \right\}_{1 \leq j \leq n-1} \cup \left\{ \mu_{n-j}^{\iota_v} + \lambda_j^{\iota_v} - (\mathbf{w} + \mathbf{w}') \right\}_{1 \leq j \leq n-1} \right\}$;
 - $m_{\mu^{\iota_v}, \lambda^{\iota_v}}^- := \max \left\{ \left\{ -\mu_{n-j}^{\iota_v} - \lambda_j^{\iota_v} \right\}_{1 \leq j \leq n-1} \cup \left\{ \mu_{n+1-j}^{\iota_v} + \lambda_j^{\iota_v} - (\mathbf{w} + \mathbf{w}') \right\}_{1 \leq j \leq n-1} \right\}$.
- (3) Now define
 - $m^+(\mu, \lambda) := \min\{v \in S_\infty : m_{\mu^{\iota_v}, \lambda^{\iota_v}}^+\}$;
 - $m^-(\mu, \lambda) := \max\{v \in S_\infty : m_{\mu^{\iota_v}, \lambda^{\iota_v}}^-\}$.

Then, $\frac{1}{2} + m$ is critical for $L(s, \Pi \times \Sigma)$ if and only if $m^-(\mu, \lambda) \leq m \leq m^+(\mu, \lambda)$.

2.4.5.4. Consequences of compatibility of coefficient systems - I. Let us consider the hypothesis that there is an integer m_0 such that $M_{\lambda+m_0} \hookrightarrow M_\mu^\vee$, in the statement of Thm. 2.21. Suppose $F = \mathbb{Q}$. Observe that this hypothesis is always satisfied for $\text{GL}_2 \times \text{GL}_1$; take $\lambda_1 = -\mu_2$. But for $n \geq 3$ without that hypothesis, the conclusion of Theorem 2.21 need not hold. For example, consider $\text{GL}_3 \times \text{GL}_2$ over \mathbb{Q} . Take $\mu = (0, 0, 0)$; then $\mathbf{w} = 0$; $\ell = (2, 0, -2)$. Take $\lambda = (1, -1)$; then $\mathbf{w}' = 0$; $\ell' = (3, -3)$. Then $C_{\Pi, \Sigma} = 1$; hence $\frac{1}{2}$ is critical since $1 - C_{\Pi, \Sigma} \leq \mathbf{w} + \mathbf{w}' \leq C_{\Pi, \Sigma} - 1$. However, there is no integer m such that $M_{\lambda+m} \hookrightarrow M_\mu^\vee$. In other words, there are Rankin–Selberg L -functions $L(s, \Pi \times \Sigma)$ which have critical values, but the underlying sheaves are not compatible; in such situations we are unable to say anything about these critical values. Furthermore, the methods of this paper say that when the sheaves are compatible, we can prove a theorem for every critical value, and only for critical values, but we are unable to say anything about noncritical values.

2.4.5.5. Consequences of compatibility of coefficient systems - II. When the base field F has a complex place, there is yet another consequence of the hypothesis that $\mu^\vee \succ \lambda + m$. This issue is already seen by working with $\mathrm{GL}_2 \times \mathrm{GL}_1$. A consequence of the discussion to follow will be that our Thm. 1.1 in the case $\mathrm{GL}_2 \times \mathrm{GL}_1$ is exactly Hida's [22, Thm. I] for critical values of type \emptyset . We will follow Hida [22] for most of 2.4.5.5.

Suppose M is a pure motive of rank 2 over F with coefficients in a field E . Recall that $\mathcal{E}_F = \mathrm{Hom}(F, \mathbb{C})$. Let's write $\mathcal{E}_r = \mathrm{Hom}(F, \mathbb{R}) \subset \mathcal{E}_F$ for the set of real embeddings, and $\mathcal{E}_c = \mathcal{E} \setminus \mathcal{E}_r$ for the set of complex embeddings. The Hodge types of M are of the form

$$\mathcal{H}_\iota(M) = \{(p_\iota, q_{\bar{\iota}}), (q_\iota, p_{\bar{\iota}})\}, \quad \forall \iota \in \mathcal{E}_F.$$

The set of all Hodge types is denoted $\mathcal{H}(M) = \cup_\iota \mathcal{H}_\iota(M)$. Make the following assumptions:

- (Purity) There exists w such that $p_{\bar{\iota}} + q_\iota = q_{\bar{\iota}} + p_\iota = w, \forall \iota \in \mathcal{E}_F$.
- (No middle Hodge type) $(p, q) \in \mathcal{H}(M) \implies p \neq q$.
- (Regularity) $p_\iota \neq q_\iota$ for all $\iota \in \mathcal{E}_F$, hence all nonzero Hodge numbers are 1. Without loss of generality we may also assume that $p_\iota > q_\iota$ for all ι .

The motive conjecturally attached to $\pi \in \mathrm{Coh}(\mathrm{GL}_2/F, \mu)$ satisfies the above properties; this motive is proven to exist by Scholl [42] for $F = \mathbb{Q}$ and by Blasius–Rogawski [4] for F totally real.

Now let χ be an algebraic Hecke character of F , and suppose the infinity type of χ is written as $\infty(\chi) = j = (j_\iota)_\iota$. Let $M(\chi)$ be the rank 1 motive attached to χ . The Hodge types of $\mathbb{M} = M \otimes M(\chi)$ are of the form:

$$\mathcal{H}_\iota(\mathbb{M}) = \{(p_\iota - j_\iota, q_\iota - j_\iota), (q_\iota - j_\iota, p_\iota - j_\iota)\}, \quad \forall \iota \in \mathcal{E}_F.$$

Recall the Γ -factors for \mathbb{M} , defined as:

$$L_\infty(s, \mathbb{M}) = \prod_\iota L_\iota(s, \mathbb{M}), \quad \text{where } L_\iota(s, \mathbb{M}) = \prod_{(p, q) \in \mathcal{H}_\iota(\mathbb{M}), p < q} \Gamma_{\mathbb{C}}(s - p),$$

and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$. For any subset S of \mathcal{E}_F , for example S could be $\{\iota, \bar{\iota}\}$ for a fixed complex embedding ι , we let $L_S(s, \mathbb{M}) = \prod_{\iota \in S} L_\iota(s, \mathbb{M})$. Let \mathbb{M}^\vee be the dual motive; note that $(p, q) \in \mathcal{H}(\mathbb{M}) \iff (-p, -q) \in \mathcal{H}(\mathbb{M}^\vee)$. We say \mathbb{M} is critical if both $L_\infty(s, \mathbb{M})$ and $L_\infty(1 - s, \mathbb{M}^\vee)$ are finite at $s = 0$. Suppose, $\mathbb{M} = M \otimes M(\chi)$ is critical, then we deduce:

- (1) $\iota \in \mathcal{E}_r, q_\iota + 1 \leq j_\iota \leq p_\iota$.
- (2) $\iota \in \mathcal{E}_c$, then one and only one of the following three cases are possible:
 - (a) $p_\iota - j_\iota < q_{\bar{\iota}} - j_{\bar{\iota}}$. Then we also have $q_\iota - j_\iota < p_{\bar{\iota}} - j_{\bar{\iota}}$. Looking at the contribution from $L_{\{\iota, \bar{\iota}\}}(s, \mathbb{M})$ towards criticality, we get the inequalities $j_\iota \geq p_\iota + 1$ and $j_{\bar{\iota}} \leq q_{\bar{\iota}}$.
 - (b) $p_{\bar{\iota}} - j_{\bar{\iota}} < q_\iota - j_\iota$. This is similar to (a) with $\bar{\iota}$ instead of ι .
 - (c) $p_\iota - j_\iota > q_{\bar{\iota}} - j_{\bar{\iota}}$ and $p_{\bar{\iota}} - j_{\bar{\iota}} > q_\iota - j_\iota$. In this case we see that criticality gives $q_\iota + 1 \leq j_\iota \leq p_\iota$, which is similar to the case of a real embedding.

Therefore, if \mathbb{M} is critical then there exist sets A and T with $A \subset \mathcal{E}_c$ and $\mathcal{E}_r \subset T$ such that $\mathcal{E}_F = T \cup A \cup \bar{A}$ as a disjoint union; furthermore,

$$\iota \in T \iff q_\iota + 1 \leq j_\iota \leq p_\iota, \quad \iota \in A \iff j_\iota \geq p_\iota + 1 \text{ and } j_{\bar{\iota}} \leq q_{\bar{\iota}}.$$

In this situation, following Hida, we say \mathbb{M} is *critical of type A*, or that $s = 0$ is critical of type A for $L(s, \mathbb{M})$. The nature of the algebraicity results for a critical value depends on this set A ; compare the two different parts of Hida [22, Theorem I]. We now show that in our situation, the compatibility of coefficient systems implies that we are looking at critical points of type $A = \emptyset$, the empty set. In the proposition below, $M(1) = M \otimes \mathbb{Q}(1)$ denotes a Tate twist, which is a harmless artifice resulting from the vagaries of the so-called motivic normalization.

Proposition 2.36. *Let $\pi \in \text{Coh}(G_2, \mu^\vee)$ and $\chi \in \text{Coh}(G_1, \lambda^\vee)$. Let $M(\pi)$ (resp., $M(\chi)$) be the motive attached to π (resp., χ). Suppose $\mu^\vee \succ \lambda$, then $(M(\pi) \otimes M(\chi))(1)$ is critical of type \emptyset .*

Proof. Suppose $\mu = (\mu^\iota)$ with $\mu^\iota = (\mu_1^\iota, \mu_2^\iota)$, then by comparing our coefficient system \mathcal{M}_μ^\vee with the coefficient system $L(\kappa; \mathbb{C})$ in Hida [22, p. 433] and from the recipe for Hodge types of $M(\pi)$ in terms of the coefficient system as in [22, Conj. 0.1, (i)], we see that $p_\iota = -\mu_2^\iota + 1, q_\iota = -\mu_1^\iota$. Also, the infinity type of χ is given by $j := \infty(\chi) = \lambda$. Applying the preceding discussion, and the well-known fact about Tate twists: if $(p, q) \in \mathcal{H}(M)$ then $(p-1, q-1) \in \mathcal{H}(M(1))$, we see that the Hodge types of $\mathbb{M} := (M(\pi) \otimes M(\chi))(1)$ are given by $\mathcal{H}_\iota(\mathbb{M}) = \{(-\mu_2^\iota - \lambda^\iota, -\mu_1^\iota - \lambda^\iota - 1), (-\mu_1^\iota - \lambda^\iota - 1, -\mu_2^\iota - \lambda^\iota)\}$. Compatibility of the coefficient systems ($\mu^\vee \succ \lambda$) translates to: $-\mu_2^\iota \geq \lambda^\iota \geq -\mu_1^\iota, \forall \iota \in \mathcal{E}_F$. It is easy to see that these inequalities preclude the possibilities in (2)(a) and (2)(b) above. For example, for (2)(a), we will need $-\mu_2^\iota - \lambda^\iota < -\mu_1^\iota - \lambda^\iota - 1$, but from the above inequalities under compatibility we will then have: $0 \leq -\mu_2^\iota - \lambda^\iota < -\mu_1^\iota - \lambda^\iota - 1 \leq -1$, which is absurd. Similarly, (2)(b) is not possible. It is easy now to see that \mathbb{M} is critical of type \emptyset . \square

2.5. The proof of Theorem 1.1.

2.5.1. The main idea behind a cohomological interpretation of $L(\frac{1}{2}, \Pi \times \Sigma)$. We interpret the Rankin–Selberg integral $I(\frac{1}{2}, \phi_\Pi, \phi_\Sigma)$ in terms of Poincaré duality. More precisely, the vector w_{Π_f} will correspond to a cohomology class ϑ_Π in degree b_n^F , and similarly w_{Σ_f} will correspond to a class ϑ_Σ in degree b_{n-1}^F . These classes, after dividing by certain periods, have good rationality properties. Pull back $\iota^* \vartheta_\Pi$ along the proper map $\iota: \tilde{S}^{G_{n-1}} \rightarrow S^{G_n}$, and wedge (or cup) with ϑ_Σ , to give a top degree class (by Prop. 2.17) on $\tilde{S}^{G_{n-1}}$ with coefficients in a tensor product sheaf. Now if the constituent sheaves are compatible ($\mu^\vee \succ \lambda$), which by 2.4.5 is the same as saying $s = 1/2$ is critical, then we get a top-degree class on $\tilde{S}^{G_{n-1}}$ with constant coefficients. Apply Poincaré duality, i.e., fix an orientation on $\tilde{S}^{G_{n-1}}$ and integrate. One realizes then that this is essentially the Rankin–Selberg integral in Prop. 2.1. Interpreting that integral, and hence the critical L -value $L_f(\frac{1}{2}, \Pi \times \Sigma)$ in cohomology, permits us to study its arithmetic properties, since Poincaré duality is Galois equivariant. All this may be summarized in the diagram:

$$\begin{array}{ccc}
\mathcal{W}(\Pi_f) \times \mathcal{W}(\Sigma_f) & \longrightarrow & H^{b_n^F}(\mathfrak{g}_n, K_{n,\infty}^0; V_\Pi \otimes \mathcal{M}_{\mu,\mathbb{C}}^\vee(\epsilon)) \times H^{b_{n-1}^F}(\mathfrak{g}_{n-1}, K_{n-1,\infty}^0; V_\Sigma \otimes \mathcal{M}_{\lambda,\mathbb{C}}^\vee(\eta)) \\
\downarrow & & \downarrow \\
\mathbb{C} & & H_{\text{cusp}}^{b_n^F}(S^{G_n}, \widetilde{\mathcal{M}}_{\mu,\mathbb{C}}^\vee) \times H_{\text{cusp}}^{b_{n-1}^F}(S^{G_{n-1}}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}^\vee) \\
\uparrow f & & \downarrow \\
H_1^{\text{top}}(\tilde{S}^{G_{n-1}}, \mathbb{C}) & \xleftarrow{\mathcal{T}^*} & H_1^{b_n^F}(S^{G_n}, \widetilde{\mathcal{M}}_{\mu,\mathbb{C}}^\vee) \times H_1^{b_{n-1}^F}(S^{G_{n-1}}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}^\vee) \\
& & \downarrow \iota^* \times \phi^* \\
& & H_1^{b_n^F}(\tilde{S}^{G_{n-1}}, \iota^* \widetilde{\mathcal{M}}_{\mu,\mathbb{C}}^\vee) \times H_1^{b_{n-1}^F}(\tilde{S}^{G_{n-1}}, \phi^* \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}^\vee) \\
& & \downarrow \wedge \\
& & H_1^{d_{n-1}^F}(\tilde{S}^{G_{n-1}}, \iota^* \widetilde{\mathcal{M}}_{\mu,\mathbb{C}}^\vee \otimes \phi^* \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}^\vee)
\end{array}$$

2.5.2. Review of certain periods and period relations. We briefly review the definition of certain periods attached to $\Pi \in \text{Coh}(G_n, \mu)$, and their behavior upon twisting by Π by algebraic Hecke characters. The reader is referred to my paper with Shahidi [37] for all the details.

2.5.2.1. Comparing Whittaker models and cohomological representations. Given any $\Pi \in \text{Coh}(G_n, \mu^\vee)$ and a permissible signature ϵ for Π , fix a generator $[\Pi_\infty]^\epsilon$ of the one-dimensional space $H^{b_n^F}(\mathfrak{g}_n, K_{n,\infty}^0; \mathcal{W}(\Pi_\infty) \otimes \mathcal{M}_{\mu,\mathbb{C}}^\vee)(\epsilon)$. We have the following comparison isomorphism:

$$\mathcal{F}_\Pi^\epsilon : \mathcal{W}(\Pi_f) \longrightarrow H^{b_n^F}(\mathfrak{g}_n, K_{n,\infty}^0; V_\Pi \otimes \mathcal{M}_{\mu,\mathbb{C}}^\vee)(\epsilon).$$

See [37, Sect. 3.3].

2.5.2.2. Rationality fields and rational structures. Given $\mu \in X_{00}^+(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu^\vee)$, define the number field $\mathbb{Q}(\mu)$ as in 2.1.5, and the rationality field $\mathbb{Q}(\Pi_f)$ as in [16, Sect. 7.1]. Now define $\mathbb{Q}(\Pi)$ as the compositum of $\mathbb{Q}(\mu)$ and $\mathbb{Q}(\Pi_f)$. The Whittaker model $\mathcal{W}(\Pi_f)$ of the finite part of Π admits a $\mathbb{Q}(\Pi_f)$ -structure, and hence a $\mathbb{Q}(\Pi)$ -structure; see [37, Sect. 3.2]. This rational structure is generated by normalized new vectors. The cohomological model $H^{b_n^F}(\mathfrak{g}_n, K_{n,\infty}^0; V_\Pi \otimes \mathcal{M}_{\mu,\mathbb{C}}^\vee)(\epsilon)$ admits a $\mathbb{Q}(\Pi)$ -structure arising from purely geometric considerations, since the sheaf $\widetilde{\mathcal{M}}_{\mu,\mathbb{C}}$ has a $\mathbb{Q}(\mu)$ -structure $\widetilde{\mathcal{M}}_{\mu,\mathbb{Q}(\mu)}$; see [37, Sect. 3.3].

2.5.2.3. Definition of the periods. The isomorphism \mathcal{F}_Π^ϵ need not preserve rational structures on either side. Each side is an irreducible representation space for the action of $G_n(\mathbb{A}_f)$ and rational structures being unique up to homotheties, we can adjust the isomorphism \mathcal{F}_Π^ϵ by a scalar—which is the period—so as to preserve rational structures. There is a nonzero complex number $p^\epsilon(\Pi)$ attached to the datum $(\Pi_f, \epsilon, [\Pi_\infty]^\epsilon)$ such that the normalized map $\mathcal{F}_{\Pi,0}^\epsilon := p^\epsilon(\Pi)^{-1} \mathcal{F}_\Pi^\epsilon$ is $\text{Aut}(\mathbb{C})$ -equivariant, i.e., the following diagram commutes:

$$(2.37) \quad \begin{array}{ccc} \mathcal{W}(\Pi_f) & \xrightarrow{\mathcal{F}_{\Pi,0}^\epsilon} & H^{b_n^F}(\mathfrak{g}_n, K_{n,\infty}^0; V_\Pi \otimes \mathcal{M}_{\mu,\mathbb{C}}^\vee)(\epsilon) \\ \sigma \downarrow & & \downarrow \sigma \\ \mathcal{W}(\sigma\Pi_f) & \xrightarrow{\mathcal{F}_{\sigma\Pi,0}^{\sigma\epsilon}} & H^{b_n^F}(\mathfrak{g}_n, K_{n,\infty}^0; V_{\sigma\Pi} \otimes \mathcal{M}_{\sigma\mu,\mathbb{C}}^\vee)(\epsilon) \end{array}$$

The complex number $p^\epsilon(\Pi)$ is well-defined up to multiplication by elements of $\mathbb{Q}(\Pi)^\times$. The collection $\{p^\epsilon(\sigma\Pi) : \sigma \in \text{Aut}(\mathbb{C})\}$ is well-defined in $(\mathbb{Q}(\Pi) \otimes \mathbb{C})^\times / \mathbb{Q}(\Pi)^\times$. In terms of the un-normalized maps, we can write the above commutative diagram as

$$(2.38) \quad \sigma \circ \mathcal{F}_\Pi^\epsilon = \left(\frac{\sigma(p^\epsilon(\Pi))}{p^\epsilon(\sigma\Pi)} \right) \mathcal{F}_{\sigma\Pi}^\epsilon \circ \sigma.$$

See [37, Def./Prop. 3.3].

2.5.2.4. Behaviour of the periods under twisting. Let ξ be an algebraic Hecke character with signature ϵ_ξ as in 2.1.7, and Gauß sum $\mathcal{G}(\xi_f)$ as in 2.1.8. Then [37, Thm. 4.1] says that

$$(2.39) \quad \sigma \left(\frac{p^{\epsilon-\epsilon_\xi}(\Pi_f \otimes \xi_f)}{\mathcal{G}(\xi_f)^{n(n-1)/2} p^\epsilon(\Pi_f)} \right) = \left(\frac{p^{\epsilon-\epsilon_\xi}(\sigma\Pi_f \otimes \sigma\xi_f)}{\mathcal{G}(\sigma\xi_f)^{n(n-1)/2} p^\epsilon(\sigma\Pi_f)} \right).$$

2.5.2.5. **Remarks on [37].** We take this opportunity to correct a small mistake in the definition of periods given in [37, Def./Prop. 3.3], which is that the right vertical arrow in (2.37) above was in [37] erroneously taken as mapping into the σ_ϵ -isotypic component; this arrow is exactly the T_σ^\bullet in (2.6) which maps $\Pi_f \otimes \epsilon$ to ${}^\sigma\Pi_f \otimes \epsilon$. It should be noted that, in general, ${}^\sigma\epsilon$ is not even defined, since σ need not stabilize the set of real embeddings of F .

Next, in [37, Thm. 4.1], we used the signature $\epsilon\sigma_\xi$. This is fine, but is potentially misleading, unless the reader realizes that when the base field F has a real place (it is only then that we have these signatures to worry about), for any algebraic Hecke character such as ξ , we have

$$\epsilon\xi = \epsilon\sigma_\xi.$$

This may be seen by showing, exactly as in Gan–Raghuram [11, (3.7)], that for any archimedean place v we have $(\sigma\xi)_v = \xi_v$; in *loc. cit.* the base field was totally real, but the same argument works, if F has at least one real place: by purity (see 2.1.7) we have $\xi = |\cdot|^{\mathbf{w}} \cdot \xi^0$ for an integer \mathbf{w} and a finite order character ξ^0 , which implies that $\sigma\xi = |\cdot|^{\mathbf{w}} \cdot \sigma \circ \xi^0$. Since ξ_v^0 is a finite order character of \mathbb{R}^\times or \mathbb{C}^\times , we conclude that it is quadratic, hence σ -invariant, whence $(\sigma\xi)_v = \xi_v$.

2.5.3. The cohomology classes and the global pairing.

2.5.3.1. **The cohomology classes.** Recall our choice of Whittaker vectors $w_\Pi = w_{\Pi_f} \otimes w_{\Pi_\infty} \in \mathcal{W}(\Pi, \psi)$ and $w_\Sigma = w_{\Sigma_f} \otimes w_{\Sigma_\infty} \in \mathcal{W}(\Sigma, \psi^{-1})$ from 2.2.4. Define

$$(2.40) \quad \vartheta_\Pi^\epsilon := \mathcal{F}_\Pi^\epsilon(w_{\Pi_f}) \quad \text{and} \quad \vartheta_\Sigma^\eta := \mathcal{F}_\Sigma^\eta(w_{\Sigma_f}).$$

These classes are transcendental, whereas the normalized classes

$$(2.41) \quad \vartheta_{\Pi,0}^\epsilon := \mathcal{F}_{\Pi,0}^\epsilon(w_{\Pi_f}) = p^\epsilon(\Pi_f)^{-1}\vartheta_\Pi^\epsilon \quad \text{and} \quad \vartheta_{\Sigma,0}^\eta := \mathcal{F}_{\Sigma,0}^\eta(w_{\Sigma_f}) = p^\eta(\Sigma_f)^{-1}\vartheta_\Sigma^\eta.$$

are rational, i.e., $\vartheta_{\Pi,0}^\epsilon \in H_1^{b_F}(S^{G_n}, \widetilde{\mathcal{M}}_\mu^\vee)$ and $\vartheta_{\Sigma,0}^\eta \in H_1^{b_F}(S^{G_{n-1}}, \widetilde{\mathcal{M}}_\lambda^\vee)$.

2.5.3.2. **Compatibility of the sheaves and the map \mathcal{T} .** The hypothesis (1.2) on the weights μ and λ , implies via Thm. 2.21 that the critical set is non-empty. In particular, $s = 1/2$ is critical, and hence $\mu^\vee \succ \lambda$. Apply the branching rule Cor. 2.20 at every archimedean place; for $v \in S_\infty$, fix a nonzero $\mathcal{T}_v \in \text{Hom}_{\text{GL}_{n-1}(F_v)}(\mathcal{M}_{\mu_v}^\vee \otimes \mathcal{M}_{\lambda_v}^\vee, \mathbb{1})$, where $\mathbb{1}$ is the trivial representation. Define

$$(2.42) \quad \otimes_{v \in S_\infty} \mathcal{T}_v =: \mathcal{T} \in \text{Hom}_{G_{n-1}}(\mathcal{M}_\mu^\vee \otimes \mathcal{M}_\lambda^\vee, \mathbb{1}).$$

2.5.3.3. **The orientation class.** Consider $\tilde{S}_{R_f}^{G_{n-1}}$, for an open-compact subgroup $R_f \in G_{n-1}(\mathbb{A}_f)$, as defined in 2.4.4. From the description in 2.3.1, the connected components of $\tilde{S}_{R_f}^{G_{n-1}}$ are of the form $\Gamma_i \backslash G_{n-1}(\mathbb{R})^0 / C_{n-1,\infty}^0$. We fix an orientation on $G_{n-1}(\mathbb{R})^0 / C_{n-1,\infty}^0$, push it down to all such connected components, and then take the sum over the index i (as in Γ_i) to get the orientation class on $\tilde{S}_{R_f}^{G_{n-1}}$; which we denote as $[\tilde{S}_{R_f}^{G_{n-1}}]$. For each $v \in S_r$, define $\delta_v \in \pi_0(K_{n-1,\infty})$ to be the element which is trivial at all places other than v , and at v it is $\delta_v = \text{diag}(-1, 1, 1, \dots, 1)$; see 2.1.3. Then $\pi_0(K_{n-1,\infty})$ is generated by $\{\delta_v : v \in S_r\}$.

Lemma 2.43. *The group $\pi_0(K_{n-1,\infty})$ acts on the orientation class $[\tilde{S}_{R_f}^{G_{n-1}}]$ via:*

$$\delta_v^* \cdot [\tilde{S}_{R_f}^{G_{n-1}}] = (-1)^n [\tilde{S}_{R_f}^{G_{n-1}}], \quad \forall v \in S_r.$$

Brief sketch of proof. Let $\{X_1, \dots, X_m\}$ be an ordered basis for $\mathfrak{gl}_{n-1}/\mathfrak{so}(n-1)$ consisting of:

$$\{H_i = E_{ii} : 1 \leq i \leq n-1\} \cup \{X_{ij} = E_{ij} + E_{ji} : 1 \leq i < j \leq n-1\},$$

where E_{ij} is the matrix with 1 in the (i, j) -th entry and 0 elsewhere. Then $\delta = \text{diag}(-1, 1, 1, \dots, 1)$ fixes each H_i and each X_{ij} with $i \neq 1$, and takes X_{1j} to $-X_{1j}$. Hence $\delta^*(X_1 \wedge \dots \wedge X_m) = (-1)^{n-2}(X_1 \wedge \dots \wedge X_m)$. The proof consists in applying this remark to δ as a given δ_v , which has no effect on the basis vectors coming from places other than v . (See also [35, Lem. 3.4].) \square

2.5.3.4. The global pairing. Referring to the diagram in 2.5.1, we wish to compute the pairing

$$(2.44) \quad \langle \vartheta_{\Pi}^{\epsilon}, \vartheta_{\Sigma}^{\eta} \rangle := \int_{[\tilde{S}_{R_f}^{G_{n-1}}]} \mathcal{T}^* (\iota^* \vartheta_{\Pi}^{\epsilon} \wedge \phi^* \vartheta_{\Sigma}^{\eta}), \quad \text{or} \quad \langle \vartheta_{\Pi,0}^{\epsilon}, \vartheta_{\Sigma,0}^{\eta} \rangle = \frac{\langle \vartheta_{\Pi}^{\epsilon}, \vartheta_{\Sigma}^{\eta} \rangle}{p^{\epsilon}(\Pi) p^{\eta}(\Sigma)}.$$

2.5.3.5. The choice of signs ϵ and η . Recall that $\epsilon = (\epsilon_v)_{v \in S_r}$ and $\eta = (\eta_v)_{v \in S_r}$. It is clear from Lem. 2.43 that we can expect to have a nonzero pairing in (2.44) only when

$$\epsilon_v = (-1)^n \eta_v, \quad \forall v \in S_r.$$

Further, by parity constraints as in 2.4.3, we know that ϵ or η is uniquely determined by Π or Σ , depending on whether n is odd or even. To summarize, for every $v \in S_r$, we have

- (1) $\epsilon_v = (-1)^n \eta_v$.
- (2)
 - If n is odd then let $\epsilon_v = \omega_{\Pi_v}(-1) \cdot (-1)^{w(\mu)/2}$;
 - if n is even then let $\eta_v = \omega_{\Sigma_v}(-1) \cdot (-1)^{w(\lambda)/2}$.

2.5.3.6. The pairing at infinity. As in [35, Sect. 3.2.5], the computation of the pairing in (2.44) involves a certain archimedean contribution. Recall from 2.5.2.1, the generator $[\Pi_{\infty}]^{\epsilon}$ of the one-dimensional space $H^{b_n^F}(\mathfrak{g}_n, K_{n,\infty}^0; \mathcal{W}(\Pi_{\infty}) \otimes \mathcal{M}_{\mu,\mathbb{C}}^{\vee})(\epsilon)$, and similarly, we have $[\Sigma_{\infty}]^{\eta}$ generating the one-dimensional $H^{b_{n-1}^F}(\mathfrak{g}_{n-1}, K_{n-1,\infty}^0; \mathcal{W}(\Sigma_{\infty}) \otimes \mathcal{M}_{\lambda,\mathbb{C}}^{\vee})(\eta)$. To compute the pairing at infinity, fix a basis $\{\mathbf{y}_j : 1 \leq j \leq d_{n-1}^F\}$ for $(\mathfrak{g}_{n-1,\infty}/\mathfrak{k}_{n-1,\infty})^*$ such that $\{\mathbf{y}_j : 1 \leq j \leq d_{n-1}^F - 1\}$ is a basis for $(\mathfrak{g}_{n-1,\infty}/\mathfrak{k}_{n-1,\infty})^*$. Next, fix a basis $\{\mathbf{x}_i : 1 \leq i \leq d_n^F - 1\}$ for $(\mathfrak{g}_{n,\infty}/\mathfrak{k}_{n,\infty})^*$, such that $\iota^* \mathbf{x}_j = \mathbf{y}_j$ for all $1 \leq j \leq \dim(\mathfrak{g}_{n-1,\infty}/\mathfrak{k}_{n-1,\infty})^* = d_{n-1}^F - 1$, and $\iota^* \mathbf{x}_i = 0$ if $i \geq d_{n-1}^F$. We further note that $\mathbf{y}_1 \wedge \mathbf{y}_2 \wedge \dots \wedge \mathbf{y}_{d_{n-1}^F}$ corresponds to a $G_{n-1}(\mathbb{R})^0$ -invariant measure on $\tilde{S}_{R_f}^{G_{n-1}}$. Let $\{m_{\alpha}^{\vee}\}$ (resp., $\{m_{\beta}^{\vee}\}$) be a \mathbb{Q} -basis for \mathcal{M}_{μ}^{\vee} (resp., $\mathcal{M}_{\lambda}^{\vee}$). The class $[\Pi_{\infty}]^{\epsilon}$ is represented by a $K_{n,\infty}^0$ -invariant element in $\wedge^{b_n}(\mathfrak{g}_{n,\infty}/\mathfrak{k}_{n,\infty})^* \otimes \mathcal{W}(\Pi_{\infty}) \otimes \mathcal{M}_{\mu,\mathbb{C}}^{\vee}$ which we write as

$$(2.45) \quad [\Pi_{\infty}]^{\epsilon} = \sum_{\mathbf{i}=i_1 < \dots < i_{b_n^F}} \sum_{\alpha} \mathbf{x}_{\mathbf{i}} \otimes w_{\infty,\mathbf{i},\alpha} \otimes m_{\alpha}^{\vee},$$

where $w_{\infty,\mathbf{i},\alpha} \in \mathcal{W}(\Pi_{\infty}, \psi_{\infty})$. Thinking in terms of the Künneth theorem for relative Lie algebra cohomology, the choice of basis $\{\mathbf{x}_{\mathbf{i}}\}$ and $\{m_{\alpha}^{\vee}\}$ can be made so that we have:

$$[\Pi_{\infty}]^{\epsilon} = \otimes_{v \in S_{\infty}} [\Pi_v]^{\epsilon_v}.$$

(It is understood that for $v \in S_c$ there is no sign ϵ_v .) Similarly, $[\Sigma_{\infty}]^{\eta}$ is represented by a $K_{n-1,\infty}^0$ -invariant element in $\wedge^{b_{n-1}^F}(\mathfrak{g}_{n-1,\infty}/\mathfrak{k}_{n-1,\infty})^* \otimes \mathcal{W}(\Sigma_{\infty}) \otimes \mathcal{M}_{\lambda}^{\vee}$ which we write as:

$$(2.46) \quad [\Sigma_{\infty}]^{\eta} = \sum_{\mathbf{j}=j_1 < \dots < j_{b_{n-1}^F}} \sum_{\beta} \mathbf{y}_{\mathbf{j}} \otimes w_{\infty,\mathbf{j},\beta} \otimes m_{\beta}^{\vee},$$

with $w_{\infty, \mathbf{j}, \beta} \in \mathcal{W}(\Sigma_{\infty}, \psi_{\infty}^{-1})$. Again, we have

$$[\Sigma_{\infty}]^{\eta} = \otimes_{v \in S_{\infty}} [\Sigma_v]^{\eta_v}.$$

We now define a pairing at infinity by

$$(2.47) \quad \langle [\Pi_{\infty}]^{\epsilon}, [\Sigma_{\infty}]^{\eta} \rangle = \sum_{\mathbf{i}, \mathbf{j}} s(\mathbf{i}, \mathbf{j}) \sum_{\alpha, \beta} \mathcal{T}(m_{\beta}^{\vee} \otimes m_{\alpha}^{\vee}) \Psi_{\infty}(1/2, w_{\infty, \mathbf{i}, \alpha}, w_{\infty, \mathbf{j}, \beta}),$$

where $s(\mathbf{i}, \mathbf{j}) \in \{0, -1, 1\}$ is defined by $\iota^* \mathbf{x}_{\mathbf{i}} \wedge \mathbf{y}_{\mathbf{j}} = s(\mathbf{i}, \mathbf{j}) \mathbf{y}_1 \wedge \mathbf{y}_2 \wedge \cdots \wedge \mathbf{y}_{d_{n-1}^F}$. Recall that $\Psi_{\infty}(1/2, w_{\infty, \mathbf{i}, \alpha}, w_{\infty, \mathbf{j}, \beta})$ is defined only after meromorphic continuation. However, the assumption on μ and λ in (1.2) guarantees that $s = 1/2$ is critical which ensures that the integrals $\Psi_{\infty}(1/2, w_{\infty, \mathbf{i}, \alpha}, w_{\infty, \mathbf{j}, \beta})$ are all finite, hence $\langle [\Pi_{\infty}], [\Sigma_{\infty}] \rangle$ is finite. Furthermore, computing this pairing is a purely local problem since

$$\langle [\Pi_{\infty}]^{\epsilon}, [\Sigma_{\infty}]^{\eta} \rangle = \prod_{v \in S_{\infty}} \langle [\Pi_v]^{\epsilon_v}, [\Sigma_v]^{\eta_v} \rangle.$$

Binyong Sun [44] has recently proved that the local pairings are all nonzero giving us the following

Theorem 2.48. $\langle [\Pi_{\infty}]^{\epsilon}, [\Sigma_{\infty}]^{\eta} \rangle \neq 0$.

The quantity $\langle [\Pi_{\infty}]^{\epsilon}, [\Sigma_{\infty}]^{\eta} \rangle$ depends only on the weights μ and λ , and the signs ϵ and η which are determined as in 2.5.3.5. Now define:

$$(2.49) \quad p_{\infty}^{\epsilon, \eta}(\mu, \lambda) := \frac{1}{\langle [\Pi_{\infty}]^{\epsilon}, [\Sigma_{\infty}]^{\eta} \rangle}.$$

Ultimately, one should expect $p_{\infty}^{\epsilon, \eta}(\mu, \lambda)$ to be a power of $(2\pi i)$.

2.5.4. The main identity. The following theorem is a generalization of [35, Thm. 3.12].

Theorem 2.50 (Main Identity). *Let $\mu \in X_{00}^+(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu^{\vee})$. Let $\lambda \in X_{00}^+(T_{n-1})$ and $\Sigma \in \text{Coh}(G_{n-1}, \lambda^{\vee})$. Assume that μ and λ satisfy $\mu^{\vee} \succ \lambda$; in particular, $s = 1/2$ is critical for $L_f(s, \Pi \times \Sigma)$. Take the signs ϵ, η as in 2.5.3.5. Let $\vartheta_{\Pi, 0}^{\epsilon}$ and $\vartheta_{\Sigma, 0}^{\eta}$ be the normalized classes defined in (2.41). Then*

$$\frac{L_f(\frac{1}{2}, \Pi \times \Sigma)}{p^{\epsilon}(\Pi) p^{\eta}(\Sigma) p_{\infty}^{\epsilon, \eta}(\mu, \lambda)} = \frac{\prod_{v \in S_{\Sigma}} L(\frac{1}{2}, \Pi_v \times \Sigma_v)}{\text{vol}(\Sigma) \prod_{v \notin S_{\Sigma} \cup S_{\infty}} c_{\Pi_v}} \langle \vartheta_{\Pi, 0}^{\epsilon}, \vartheta_{\Sigma, 0}^{\eta} \rangle,$$

where the pairing on the right hand side is defined in (2.44), the nonzero rational number $\text{vol}(\Sigma)$ is as in Prop. 2.1, and c_{Π_v} is defined in 2.2.3.

Since $c_{\Pi_v} = 1$ for an unramified places v , the infinite product in the denominator of the right hand side is in fact a finite product.

Proof. We will only adumbrate the proof, since the proof of [35, Thm. 3.12] goes through *mutatis mutandis*. Using the definition of normalized classes in (2.41), we see that it is enough to show:

$$\langle \vartheta_{\Pi}^{\epsilon}, \vartheta_{\Sigma}^{\eta} \rangle = \frac{L_f(\frac{1}{2}, \Pi \times \Sigma) \cdot \text{vol}(\Sigma) \cdot \prod_{v \notin S_{\Sigma} \cup S_{\infty}} c_{\Pi_v}}{\prod_{v \in S_{\Sigma}} L(\frac{1}{2}, \Pi_v \times \Sigma_v) \cdot p_{\infty}^{\epsilon, \eta}(\mu, \lambda)}.$$

Using (2.40) and (2.45) we may write the cohomology classes as:

$$\vartheta_{\Pi}^{\epsilon} = \sum_{\mathbf{i}} \sum_{\alpha} \mathbf{x}_{\mathbf{i}} \otimes \phi_{\mathbf{i}, \alpha} \otimes m_{\alpha}^{\vee},$$

where the cusp form $\phi_{i,\alpha}$, in the ψ -Whittaker model of Π , looks like $w_{\Pi_f} \otimes w_{\infty,i,\alpha}$. Similarly,

$$\vartheta_{\Sigma}^{\eta} = \sum_{\mathbf{j}} \sum_{\beta} \mathbf{y}_{\mathbf{j}} \otimes \phi_{\mathbf{j},\beta} \otimes m_{\beta}^{\vee},$$

where the cusp form $\phi_{\mathbf{j},\beta}$, in the ψ^{-1} -Whittaker model of Σ , looks like $w_{\Sigma_f} \otimes w_{\infty,\mathbf{j},\beta}$. Computing the global pairing as defined in (2.44), in the situation where the signatures are as prescribed in 2.5.3.5, while using the above expressions for $\vartheta_{\Pi}^{\epsilon}$ and $\vartheta_{\Sigma}^{\eta}$, we get

$$\langle \vartheta_{\Pi}^{\epsilon}, \vartheta_{\Sigma}^{\eta} \rangle = \sum_{\mathbf{i}, \mathbf{j}} \sum_{\alpha, \beta} s(\mathbf{i}, \mathbf{j}) \mathcal{T}(m_{\beta}^{\vee} \otimes m_{\alpha}^{\vee}) I(\frac{1}{2}, \phi_{i,\alpha}, \phi_{\mathbf{j},\beta}).$$

Using Prop. 2.1 and the pairing at infinity (2.47) this may be re-written as

$$\langle \vartheta_{\Pi}^{\epsilon}, \vartheta_{\Sigma}^{\eta} \rangle = \frac{L_f(\frac{1}{2}, \Pi \times \Sigma) \cdot \text{vol}(\Sigma) \cdot \prod_{v \notin S_{\Sigma} \cup S_{\infty}} c_{\Pi_v}}{\prod_{v \in S_{\Sigma}} L(\frac{1}{2}, \Pi_v \times \Sigma_v)} \langle [\Pi_{\infty}]^{\epsilon}, [\Sigma_{\infty}]^{\eta} \rangle.$$

The main identity now follows from (2.49). \square

2.5.5. The proof of Theorem 1.1.

2.5.5.1. Central critical value. Suppose $\mu^{\vee} \succ \lambda$ then $s = 1/2$ is critical. The proof of [35, Thm. 1.1] goes through *mutatis mutandis* giving an algebraicity result for the central critical value; the proof entails verifying that the right hand side of the main identity in Thm. 2.50 above is Galois equivariant, i.e., well-behaved under the action of $\sigma \in \text{Aut}(\mathbb{C})$. This is the essence of [35, Sect. 3.3], all of which appropriately generalizes to the situation of this paper and so we merely state the necessary observations to make while omitting the details.

- The Poincaré duality pairing $\langle \cdot, \cdot \rangle$ is Galois-equivariant; see [35, Prop. 3.14].
- The Galois equivariance of the classes $\vartheta_{\Pi,0}^{\epsilon}$ and $\vartheta_{\Sigma,0}^{\eta}$, is somewhat more subtle, because of our specific choice of finite Whittaker vectors w_{Π_f} and w_{Σ_f} . Using [35, Prop. 3.15] and [35, Cor. 3.14], both of which easily go through in our setting, we get:

$$\sigma \vartheta_{\Pi,0}^{\epsilon} = \frac{\sigma(\mathcal{G}(\omega_{\Sigma_f}))}{\mathcal{G}(\omega_{\Sigma_f^{\sigma}})} \vartheta_{\Pi^{\sigma},0}^{\epsilon}, \quad \text{and} \quad \sigma \vartheta_{\Sigma,0}^{\eta} = \vartheta_{\Sigma^{\sigma},0}^{\eta}.$$

- Noting that [35, Prop. 3.17] works for any local field F_v we get that local L -values are suitably rational, and moreover, we have:

$$\sigma(L(1/2, \Pi_v \times \Sigma_v)) = L(1/2, \Pi_v^{\sigma} \times \Sigma_v^{\sigma}).$$

- Finally, exactly as in [35, Prop. 3.18], we have $\sigma(c_{\Pi_v}) = c_{\Pi_v^{\sigma}}$.

2.5.5.2. All critical values. Now suppose μ and λ satisfy (1.2), and suppose $\frac{1}{2} + m$ is critical. Then we take a suitable Tate twist and consider a situation when $\frac{1}{2}$ is critical, and then apply the period relations in 2.5.2.4. The reader should bear in mind that Thm. 2.21 says that the set of possible Tate-twists we can take, subject to the restriction imposed by the compatibility condition (1.2), is exactly the set of all critical points for $L(s, \Pi \times \Sigma)$. However, the parity of n will play a role, because this will affect the recipe for signs ϵ and η as in 2.5.3.5. We argue as follows:

Suppose n is even, then we absorb the m into the representation of GL_n as:

$$L_f(\frac{1}{2} + m, \Pi \times \Sigma) = L_f(\frac{1}{2}, \Pi(m) \times \Sigma) \sim p^{\epsilon}(\Pi(m)) p^{\eta}(\Sigma) \mathcal{G}(\omega_{\Sigma_f}) p_{\infty}^{\epsilon, \eta}(\mu + m, \lambda).$$

Using (2.39) we can write this as $p^{\epsilon m \epsilon}(\Pi) p^{\eta}(\Sigma) \mathcal{G}(\omega_{\Sigma_f}) p_{\infty}^{\epsilon, \eta}(\mu + m, \lambda)$. Suppose n is odd, then we absorb the m into the representation of GL_{n-1} , and argue similarly.

3. Symmetric power L -functions

3.1. Symmetric power transfers are cohomological.

3.1.1. Definition of $\text{Sym}^r(\pi)$ and $L(s, \text{Sym}^r(\pi) \otimes \chi)$. Let π be a cohomological cuspidal automorphic representation of GL_2 over F . Let $\text{Sym}^r : \text{GL}_2(\mathbb{C}) \rightarrow \text{GL}_{r+1}(\mathbb{C})$ be the r -th symmetric power of the standard representation of $\text{GL}_2(\mathbb{C})$. By the local Langlands correspondence, see Harris–Taylor [20] and Henniart [21] for the non-archimedean places and Langlands [31] for the archimedean places, the local transfer $\text{Sym}^r(\pi_v)$ is defined as an irreducible admissible representation of $\text{GL}_{r+1}(F_v)$. Now define $\text{Sym}^r(\pi) := \bigotimes'_v \text{Sym}^r(\pi_v)$ which is a well-defined irreducible admissible representation of $\text{GL}_{r+1}(\mathbb{A}_F)$. Langlands’s functoriality predicts that $\text{Sym}^r(\pi)$ is an isobaric automorphic representation of $\text{GL}_{r+1}(\mathbb{A}_F)$; this is known for $r \leq 4$ by Gelbart–Jacquet [13], Kim–Shahidi [26], and Kim [24] for a general π , and it is known for all r if π is dihedral. Define the r -th symmetric power L -function of π as the standard L -function of the r -th symmetric power transfer of π ; we may also introduce a twisting Hecke character χ . We are interested in the special values of such twisted symmetric power L -functions of π : $L(s, \text{Sym}^r(\pi) \otimes \chi)$. As in [35] we approach odd symmetric power L -functions inductively. To get started, consider the case when $r = 1$.

3.1.2. L -functions for $\text{GL}_2 \times \text{GL}_1$. Let’s explicate the $n = 2$ case of Thm. 1.1. The reader should note that this particular case is not new; see, for example, Shimura [43] when F is totally real, and Hida [22] in the general case, although our notations are rather different.

Let $\pi \in \text{Coh}(G_2, \mu^\vee)$ with $\mu \in X_{00}^+(T_2)$. Let $\mu = (\mu^t)_{t \in \mathcal{E}_F}$, and $\mu^t = (a^t, b^t) \in \mathbb{Z}^2$ with $a^t \geq b^t$. Let $\mathbf{w} = \mathbf{w}(\mu)$ be the purity weight of μ . Recall: for any $v \in S_r$, $\mathbf{w} = a^t + b^t$, and for any $v \in S_c$, if $\mu^{lv} = (a^{lv}, b^{lv})$ then $\mu^{lv} = (\mathbf{w} - b^{lv}, \mathbf{w} - a^{lv})$.

Let $\lambda \in \mathbb{Z}$ be thought of as an element $\lambda = (\lambda^t)_{t \in \mathcal{E}_F} \in X_{00}^+(T_1)$, where each $\lambda^t = \lambda$. The corresponding representation \mathcal{M}_λ of G_1 is \det^λ . The purity weight is $\mathbf{w}(\lambda) = 2\lambda$. Let $\chi \in \text{Coh}(G_1, \lambda^\vee)$, then $\chi = \chi^\circ \otimes |\cdot|^\lambda$, where $\chi^\circ : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ is a character of finite order. For any $v \in S_r$, define $\epsilon_{\chi, v} = (-1)^\lambda \cdot \chi_v^\circ(-1)$; the signature of χ is $\epsilon_\chi = (\epsilon_{\chi, v})_{v \in S_r}$.

The compatibility condition (1.2) for the pair (μ, λ) is satisfied if there is an integer j such that $-b^t \geq \lambda + j \geq -a^t$ for all t ; this is best considered in two cases:

- (1) $v \in S_r$: here we have $a^t - \mathbf{w} \geq \lambda + j \geq -a^t$. Note a consequence of these inequalities is that $a^t - \mathbf{w} \geq -[\mathbf{w}/2] \geq -a^t$.
- (2) $v \in S_c$: here we have $-b^t \geq \lambda + j \geq -a^t$ and $a^t - \mathbf{w} \geq \lambda + j \geq b^t - \mathbf{w}$. A consequence of both these inequalities are: $-b^t \geq -[\mathbf{w}/2] \geq -a^t$ and $a^t - \mathbf{w} \geq -[\mathbf{w}/2] \geq b^t - \mathbf{w}$.

It is important to appreciate the possibility that the compatibility condition (1.2) need not always be satisfied. For example, take F to be imaginary quadratic extension of \mathbb{Q} , take $a^t = b^t = 0$ and $\mathbf{w} \neq 0$; then clearly the necessary condition in (2) is not satisfied. Furthermore, the reader can check in this situation that $L(s, \pi)$ doesn’t have any critical points. (As explained in 2.4.5.5, even when F is an imaginary quadratic extension and say $\mathcal{E}_F = \{t, \bar{t}\}$, as in [22] we could conceivably have critical points of type $A = \{t\}$, however, such a critical point is outside the purview of the compatibility condition (1.2).) At the other extreme, if F is totally real, then (1.2) is satisfied for $j = -\lambda - [\mathbf{w}/2] = -[(\mathbf{w}(\lambda) + \mathbf{w}(\mu))/2]$; and by Thm. 2.21, there are critical points for $L(s, \pi \otimes \chi)$. Finally, let’s note that for any number field with at least one real place, if μ is a parallel weight then (1.2) is satisfied and $L(s, \pi \otimes \chi)$ has critical points. Assume henceforth that (1.2) holds for (μ, λ) .

Let $\frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z}$ be any critical point for $L(s, \pi \times \chi)$. By Thm. 2.21 this is the same as $m \in \mathbb{Z}$ such that $\mu^\vee \succ \lambda + m$. The sign $\eta = \epsilon_\chi$ as given above, and since $n = 2$, $\epsilon = \eta$. Further, χ being on

GL_1 , we have $p^\eta(\chi) \sim 1$. The statement of Thm. 1.1 gives:

$$L_f\left(\frac{1}{2} + m, \pi \otimes \chi\right) \sim p^{\epsilon_m \epsilon_\chi}(\pi) \mathcal{G}(\chi) p_\infty^{\epsilon_\chi}(\mu + m, \lambda),$$

and more generally, the ratio of the left hand side by the right hand side is $\mathrm{Aut}(\mathbb{C})$ -equivariant. When F is totally real, the quantity $p_\infty^{\epsilon_\chi}(\mu + m, \lambda)$ has been explicitly calculated in [38, Prop. 3.24]. In general, one expects, as in the totally real case, that this quantity is a rational multiple of an integral power of $2\pi i$.

We will apply the above discussion to the situation $L(s, \pi \times \omega_\pi \xi)$ where ω_π is the central character of π and ξ is a finite order Hecke character of F . Given μ as above, define $\det(\mu)$ by $\det(\mu)^\iota = a^\iota + b^\iota$; then $w(\det(\mu)) = 2w(\mu)$. We have:

$$(3.1) \quad \pi \in \mathrm{Coh}(G_2, \mu^\vee) \implies \omega_\pi \in \mathrm{Coh}(G_1, \det(\mu)^\vee).$$

The latter also implies that $\omega_\pi \xi \in \mathrm{Coh}(G_1, \det(\mu)^\vee)$. As above, the compatibility condition (1.2) for the pair $(\mu, \det(\mu))$ need not be satisfied in general.

3.1.3. Transferring the weights μ . Let $\mu \in X_{00}^+(T_2)$. As above, suppose $\mu = (\mu^\iota)_{\iota \in \mathcal{E}_F}$, with $\mu^\iota = (a^\iota, b^\iota) \in \mathbb{Z}^2$ and $a^\iota \geq b^\iota$. Define a weight $\mathrm{Sym}^r(\mu) = (\mathrm{Sym}^r(\mu^\iota))_{\iota \in \mathcal{E}_F} \in X^+(T_{r+1})$, where for each ι we have:

$$\mathrm{Sym}^r(\mu)^\iota := \mathrm{Sym}^r(\mu^\iota) := (ra^\iota, (r-1)a^\iota + b^\iota, \dots, a^\iota + (r-1)b^\iota, rb^\iota).$$

If $w = w(\mu)$ be the purity weight of μ , then it is easy to check that $\mathrm{Sym}^r(\mu)$ is also pure and its purity weight is $w(\mathrm{Sym}^r(\mu)) = rw$. Furthermore, one checks that $\mathrm{Sym}^r(\mu)$ is strongly pure, i.e., $\mathrm{Sym}^r(\mu) \in X_{00}^+(T_{r+1})$. Also, if μ is a parallel weight, then so is $\mathrm{Sym}^r(\mu)$.

3.1.4. Symmetric power transfer preserves the property of being cohomological. The following theorem is a generalization of our result with Shahidi [36, Thm. 5.5] and is a variation of Labesse–Schwermer [30, Prop. 5.4].

Theorem 3.2. *Let $\mu \in X_{00}^+(T_2)$ and $\pi \in \mathrm{Coh}(G_2, \mu^\vee)$. Suppose $\mathrm{Sym}^r(\pi)$ is a cuspidal automorphic representation of G_{r+1} , then $\mathrm{Sym}^r(\pi) \in \mathrm{Coh}(G_{r+1}, \mathrm{Sym}^r(\mu)^\vee)$.*

Proof. The proof is purely local, as it suffices to check that for $v \in S_\infty$, the representation $\mathrm{Sym}^r(\pi)_v$ has nontrivial relative Lie algebra cohomology after twisting by $\mathcal{M}_{\mathrm{Sym}^r(\mu_v)^\vee}$. Consider two cases:

$v \in S_r$: Let's suppress the $\iota_v = \iota$ and write $\mu^\iota = (a, b)$. Then $\pi_v = D(a - b + 1) | \cdot |^{w/2}$, whose Langlands parameter we denote for brevity as $I(\xi_l)(w/2)$ with $l = a - b + 1$; it is, up to twisting by $| \cdot |^{w/2}$, the two-dimensional representation of the Weil group $W_\mathbb{R}$ of \mathbb{R} which is induced from the Weil group $W_\mathbb{C} = \mathbb{C}^\times$ of \mathbb{C} by the character ξ_l given by $z = re^{i\theta} \mapsto e^{il\theta}$. A pleasant exercise gives:

$$\mathrm{Sym}^r(I(\xi_l)) = \begin{cases} I(\xi_l) \oplus I(\xi_{3l}) \oplus \cdots \oplus I(\xi_{rl}), & \text{if } r \text{ is odd,} \\ \mathrm{sgn}^{r/2} \oplus I(\xi_{2l}) \oplus I(\xi_{4l}) \oplus \cdots \oplus I(\xi_{rl}), & \text{if } r \text{ is even.} \end{cases}$$

Also, $\mathrm{Sym}^r(I(\xi_l)(w/2)) = \mathrm{Sym}^r(I(\xi_l))(rw/2)$. By the local Langlands correspondence for $\mathrm{GL}_{r+1}(\mathbb{R})$ (see Knapp [29]), we see that $\mathrm{Sym}^r(\pi_v)$ is the representation in (2.8) or (2.10) exactly when the corresponding highest weight is taken as $\mathrm{Sym}^r(\mu)$ when defined as in Sect. 3.1.3.

$v \in S_c$: Suppressing again the notations for $\{\iota_v, \bar{\iota}_v\}$, we write $\mu_v = \{(a, b), (w - b, w - a)\}$. Then, the Langlands parameter of $\pi_v = J_{\mu_v}$, see (2.13), is $\xi(a + \frac{1}{2}, w - a - \frac{1}{2}) \oplus \xi(b - \frac{1}{2}, w - b + \frac{1}{2})$, where for half-integers p, q by $\xi(p, q)$ is meant the character of $W_\mathbb{C}$ given by $z \mapsto z^p \bar{z}^q$. Then the Langlands parameter of $\mathrm{Sym}^r(\pi_v)$ is given by:

$$\xi(ra + \frac{r}{2}, rw - ra - \frac{r}{2}) \oplus \xi((r-1)a + b + \frac{r-2}{2}, rw - (r-1)a - b - \frac{r-2}{2}) \oplus \cdots \oplus \xi(rb - \frac{r}{2}, rw - rb + \frac{r}{2}).$$

Hence, $\text{Sym}^r(\pi_v)$, via (2.12) and (2.13), corresponds to the highest weight $\text{Sym}^r(\mu_v)$. \square

Thm. 3.2 and (3.1) lend further evidence to the discussion in [36, Sect. 5.2] relating functoriality and the property of being cohomological.

3.2. Special values of Symmetric power L -functions.

3.2.1. A factorization of L -functions. Let π be a cuspidal automorphic representation of $G_2(\mathbb{A})$, and suppose $\text{Sym}^r(\pi)$ is automorphic for all r , then our main idea behind special values of symmetric power L -functions is:

$$L_f(s, \text{Sym}^r(\pi) \times \text{Sym}^{r-1}(\pi)) = \prod_{a=1}^r L_f(s, \text{Sym}^{2a-1}(\pi) \otimes \omega_\pi^{r-a});$$

see [35, Cor. 5.2]. Now suppose that $\pi \in \text{Coh}(G_2, \mu^\vee)$ and suppose that the symmetric power transfers are all cuspidal, then we apply Thm. 1.1 to get algebraicity results for the values of $L_f(s, \text{Sym}^r(\pi) \times \text{Sym}^{r-1}(\pi))$, and inductively, we get special value results for odd symmetric power L -functions on the right hand side. As can be seen in [35, Prop. 5.4], where the base field F was \mathbb{Q} , carrying out this exercise can be combinatorially tedious. In the rest of this article, we carry through the above idea for symmetric cube L -functions of π . Algebraicity results for the critical values of symmetric cube L -functions are available in the literature in various special cases; see Garrett–Harris [12, Thm. 6.2], Kim–Shahidi [25, Prop. 4.1], Grobner–Raghuram [17, Cor. 8.1.2], and Januszewski [23, Sect. 6]. The following results are new when F is a general number field and the representation π is cohomological with respect to a general strongly pure coefficient system μ .

3.2.2. Critical points for symmetric cube L -functions.

Proposition 3.3. *Let $\pi \in \text{Coh}(G_2, \mu^\vee)$ with $\mu \in X_{00}^+(T_2)$. Let $\mu = (\mu^t)_{t \in \mathcal{E}_F}$, and $\mu^t = (a^t, b^t) \in \mathbb{Z}^2$ with $a^t \geq b^t$. Let $\mathfrak{w} = \mathfrak{w}(\mu)$ be the purity weight of μ . Let χ be any Hecke character of F of finite order. Then, $\frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z}$ is critical for $L(s, \text{Sym}^3(\pi) \otimes \chi)$ if and only if m satisfies the inequalities in (3.4) and (3.5) below. For $v \in S_r$ and $\iota_v \in \mathcal{E}_F$ the corresponding embedding, we have*

$$(3.4) \quad -2a^{\iota_v} - b^{\iota_v} \leq m \leq -2b^{\iota_v} - a^{\iota_v}.$$

For $v \in S_c$ and $\{\iota_v, \bar{\iota}_v\}$ the corresponding pair of embeddings, suppose α_v stands for the minimum of $\{|6a^{\iota_v} - 3\mathfrak{w} + 3|, |4a^{\iota_v} + 2b^{\iota_v} - 3\mathfrak{w} + 1|, |2a^{\iota_v} + 4b^{\iota_v} - 3\mathfrak{w} - 1|, |6b^{\iota_v} - 3\mathfrak{w} - 3|\}$, then

$$(3.5) \quad \frac{1 - 3\mathfrak{w} - \alpha_v}{2} \leq m \leq \frac{-1 - 3\mathfrak{w} + \alpha_v}{2}.$$

Proof. The proof is a somewhat tedious exercise, and we merely sketch it leaving all the details to the reader. Consider two cases:

$v \in S_r$: suppressing the superscript ι_v , let $\mu_v = (a, b)$. Then up to nonzero constants and exponential functions, we have

$$L(s, \text{Sym}^3(\pi_v)) = \Gamma\left(s + \frac{3\mathfrak{w}}{2} + \frac{a - b + 1}{2}\right) \Gamma\left(s + \frac{3\mathfrak{w}}{2} + \frac{3(a - b + 1)}{2}\right).$$

We leave it to the reader to check that $L(s, \text{Sym}^3(\pi_v))$ and $L(1 - s, \text{Sym}^3(\pi_v)^\vee)$ are regular at $s = \frac{1}{2} + m$ if and only if m satisfies (3.4).

$v \in S_c$: suppressing the superscripts ι_v and $\bar{\iota}_v$, up to nonzero constants and exponential functions, we have

$$L(s, \text{Sym}^3(\pi_v)) = \Gamma\left(s + \frac{3w}{2} + \frac{|6a - 3w + 3|}{2}\right) \Gamma\left(s + \frac{3w}{2} + \frac{|4a + 2b - 3w + 1|}{2}\right) \\ \cdot \Gamma\left(s + \frac{3w}{2} + \frac{|2a + 4b - 3w - 1|}{2}\right) \Gamma\left(s + \frac{3w}{2} + \frac{|6b - 3w - 3|}{2}\right).$$

We leave it to the reader to check that $L(s, \text{Sym}^3(\pi_v))$ and $L(1 - s, \text{Sym}^3(\pi_v)^\vee)$ are regular at $s = \frac{1}{2} + m$ if and only if m satisfies (3.5). \square

The following special case of a parallel weight when F has at least one real place is interesting, and the reader can just as well consider only this case:

Corollary 3.6. *Let $\pi \in \text{Coh}(G_2, \mu^\vee)$ with $\mu \in X_{00}^+(T_2)$ being a parallel weight $\mu = (a, b) \in \mathbb{Z}^2$ with $a \geq b$. Suppose $S_r \neq \emptyset$, i.e., F is not totally imaginary then the purity weight is $w = w(\mu) = a + b$ and furthermore, the set*

$$\left\{\frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z} : -2a - b \leq m \leq -a - 2b\right\}$$

is the critical set for $L(s, \text{Sym}^3(\pi) \otimes \chi)$, $L(s, \pi \otimes \omega_\pi \chi)$ and $L(s, \text{Sym}^2(\pi) \times \pi \otimes \chi)$.

Proof. The inequalities in (3.4) and (3.5) boil down to $-2a - b \leq m \leq -a - 2b$ which takes care of $L(s, \text{Sym}^3(\pi) \otimes \chi)$. The critical sets for $L(s, \pi \otimes \omega_\pi \chi)$ and $L(s, \text{Sym}^2(\pi) \times \pi \otimes \chi)$ follow from $n = 3$ and $n = 2$ cases respectively of Thm. 2.21; we leave the details to the reader. \square

3.2.3. The compatibility condition (1.2). Let $\pi \in \text{Coh}(G_2, \mu^\vee)$ with $\mu \in X_{00}^+(T_2)$. Consider two successive symmetric power transfers of π . We address the question of whether Thm. 1.1 is applicable to $L(s, \text{Sym}^r(\pi) \times \text{Sym}^{r-1}(\pi))$, i.e., after applying Thm. 3.2, we ask whether the transferred weights $\text{Sym}^r(\mu)$ and $\text{Sym}^{r-1}(\mu)$ satisfy the compatibility condition (1.2). For this we are seeking an integer j such that for all $\iota \in \mathcal{E}_F$, we should have $\text{Sym}^r(\mu^\iota)^\vee \succ \text{Sym}^{r-1}(\mu^\iota) + j$, i.e.,

$$-rb^\iota \geq (r-1)a^\iota + j \geq -(r-1)b^\iota - a^\iota \geq (r-2)a^\iota + b^\iota + j \geq \dots \geq (r-1)b^\iota + j \geq -ra^\iota.$$

Consider two cases: if ι corresponds to a real place then we want

$$-ra^\iota - (r-1)b^\iota \leq j \leq -(r-1)a^\iota - rb^\iota,$$

and if ι corresponds to a complex place, then not only do we want the above (at ι) but also at the conjugate embedding, since $(a^\iota, b^\iota) = (w - b^\iota, w - a^\iota)$, we would also want

$$-(2r-1)w + (r-1)a^\iota + rb^\iota \leq j \leq -(2r-1)w + ra^\iota + (r-1)b^\iota.$$

Such an integer j may not exist, because, if j exists, then both the above inequalities give the necessary condition:

$$2ra^\iota + 2(r-1)b^\iota \geq (2r-1)w \geq 2(r-1)a^\iota + 2rb^\iota,$$

which need not be satisfied. (For example, take F to be an imaginary quadratic extension of \mathbb{Q} , and take $a^\iota = b^\iota$, $w \neq 2a^\iota$, and any $r \geq 1$.)

On the other hand, suppose F is not totally imaginary, i.e., $S_r \neq \emptyset$, then $w = a^\iota + b^\iota$ and all the above conditions are equivalent to $-rw + b^\iota \leq j \leq -rw + a^\iota$. For brevity, let $j' = j + rw$. Then we are seeking j' such that $b^\iota \leq j' \leq a^\iota$ for all ι ; this is possible because $b^\iota \leq w/2 \leq a^\iota$; take $j' = \lfloor w/2 \rfloor$.

3.2.4. Proof of Theorem 1.6. The proof, as explained in the introduction, is to explicate Thm. 1.1 for $L_f(\frac{1}{2} + m, \pi \otimes \omega_\pi \xi)$ and $L_f(\frac{1}{2} + m, \text{Sym}^2(\pi) \times \pi \otimes \xi)$ which we take up in the following two paragraphs:

3.2.4.1. Thm. 1.1 for $L_f(\frac{1}{2} + m, \pi \otimes \omega_\pi \xi)$. If $\pi \in \text{Coh}(G_2, \mu^\vee)$ then $\omega_\pi \xi \in \text{Coh}(G_1, \det(\mu)^\vee)$. We are in the situation when $n = 2$, hence $\epsilon = \eta$ and

$$\eta_v = \omega_{\pi_v}(-1) \cdot \xi_v(-1) \cdot (-1)^{w(\det(\mu))/2} = \xi_v(-1).$$

Hence $\eta = \epsilon_\xi$. Thm. 1.1 takes the form:

$$(3.7) \quad \begin{aligned} L_f(\tfrac{1}{2} + m, \pi \otimes \omega_\pi \xi) &\sim p^{\epsilon_m \epsilon_\xi}(\pi) \mathcal{G}(\omega_\pi \xi) p^{\epsilon_\xi, \epsilon_\xi}(\mu + m, \det(\mu)) \\ &\sim p^{\epsilon_m \epsilon_\xi}(\pi) \mathcal{G}(\omega_\pi) \mathcal{G}(\xi) p^{\epsilon_\xi, \epsilon_\xi}(\mu + m, \det(\mu)). \end{aligned}$$

3.2.4.2. Thm. 1.1 for $L_f(\frac{1}{2} + m, \text{Sym}^2(\pi) \times \pi \otimes \xi)$. Here, $n = 3$, hence $\eta = -\epsilon$ and

$$\epsilon_v = \omega_{\text{Sym}^2(\pi_v)}(-1) \cdot (-1)^{w(\text{Sym}^2(\mu))/2} = \omega_{\pi_v}^3(-1) \cdot (-1)^w = 1.$$

Hence $\epsilon = \epsilon_+$ and $\eta = -\epsilon_+ =: \epsilon_-$.

$$(3.8) \quad \begin{aligned} L_f(\tfrac{1}{2} + m, \text{Sym}^2(\pi) \times \pi \otimes \xi) &\sim p^{\epsilon_+}(\text{Sym}^2(\pi)) \cdot p^{-\epsilon_m}(\pi \otimes \xi) \cdot \mathcal{G}(\omega_{\pi \otimes \xi}) \cdot p_{\infty}^{\epsilon_+, \epsilon_-}(\text{Sym}^2(\mu), \mu + m) \\ &\sim p^{\epsilon_+}(\text{Sym}^2(\pi)) \cdot p^{-\epsilon_m \epsilon_\xi}(\pi) \mathcal{G}(\xi) \cdot \mathcal{G}(\omega_\pi) \mathcal{G}(\xi)^2 \cdot p_{\infty}^{\epsilon_+, \epsilon_-}(\text{Sym}^2(\mu), \mu + m). \end{aligned}$$

Proof of Theorem 1.6 follows from (3.7) and (3.8).

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A. RAGHURAM, INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH, DR.HOMI BHABHA ROAD, PASHAN,
PUNE 411008, INDIA.

E-mail address: raghuram@iiserpune.ac.in