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First passage time and escape time distributions for continuous time random walks

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Abstract. We consider an arbitrary continuous time random walk (CTRW) *via* unbiased nearest-neighbour jumps on a linear lattice. Solutions are presented for the distributions of the first passage time and the time of escape from a bounded region. A simple relation between the conditional probability function and the first passage time distribution is analysed. So is the structure of the relation between the characteristic functions of the first passage time and escape time distributions. The mean first passage time is shown to diverge for all (unbiased) CTRW's. The divergence of the mean escape time is related to that of the mean time between jumps. A class of CTRW's displaying a self-similar clustering behaviour in time is considered. The exponent characterising the divergence of the mean escape time is shown to be $(1 - H)$, where $H\,(0 < H < 1)$ is the fractal dimensionality of the CTRW.

Keywords. Continuous time random walk; first passage time; escape time; fractal random walks.

1. Introduction

The study of first passage times and times of escape from a given region (exit times) in random walks and diffusion processes finds application in a variety of physical problems (Montroll and Weiss 1965; Montroll and West 1979; Hänggi and Talkner 1981). Recently, the subject has evoked interest (Seshadri and West 1982) as a means of characterising fractal random walks (Hughes *et al* 1981, 1982). A vast literature exists on the classic first passage time problem for a variety of Markov processes (Pontryagin *et al* 1933; Darling and Siegert 1953; Stratonovich 1963; Weiss 1966; Montroll and Weiss 1965; Montroll 1969; Goel and Richter-Dyn 1974). The extension of such results to non-Markov processes in general, and to continuous-time random walks (CTRW's) in particular, would enable one to apply them to more complicated physical situations that incorporate, for instance, strong memory effects. This is the task carried out in this paper. All the known results for the Markov case are of course recovered as special cases of our general solutions. Some of the results obtained below for general CTRW's (specifically, results for the *mean* first passage and escape times) have also been found by Weiss (1981) in a different form (*viz.* formal expressions involving infinite sums) using the generalised master equation for a CTRW, assuming that the first waiting-time distribution is identically equal to the waiting-time distribution specifying the renewal process (that is, CTRW). We do not need this restriction in our approach.

We first obtain an exact solution for the characteristic function $\tilde{Q}$ of the first passage time distribution $Q(m, t \mid m_0)$ for a general continuous-time random walk on an infinite one-dimensional chain by two different methods. (For simplicity, we consider a symmetric walk *via* nearest-neighbour jumps; m, m_0 denote integers). In the first method, we construct an explicit solution for the conditional probability $P(m, t \mid m_0)$ for an arbitrary CTRW on an infinite chain and then use the reflection principle for symmetric random walks to obtain the first passage time distribution. The second is a direct 'first principles' calculation of $\tilde{Q}$. Next, the mean first passage time is shown to diverge for all CTRW's, as one would generally expect for a symmetric random walk on an infinite chain. Considering the entire class of CTRW's, we then show that there exists a certain simple relationship between Q and P, namely, $Q(m, t \mid m_0) = |m - m_0| P(m, t \mid m_0)/t$, only when the pausing time is exponentially distributed, in which case the random walk is a Markov chain. Surprisingly, however, there are even more general types of temporally correlated random walks for which this relationship is valid, and we present an explicit example of this.

We then consider the distribution of the time of escape from a given region (the 'exit time') and derive a compact expression for its characteristic function using the method of images. We show that the mean escape time from a bounded domain for a general CTRW is finite only when the first moment of the pausing time distribution exists. In this sense, even though the positional probability density spreads out in time from an initial sharp distribution, no long range diffusion can be said to occur for a random walk involving a pausing time distribution with a divergent first moment (*i.e.*, mean residence time at a site).

Finally, we turn to 'fractal' random walks, *i.e.*, walks which exhibit self-similar clustering. One such class is obtained in the case of a pausing-time distribution that is an infinite superposition of suitably-scaled exponentials (Shlesinger and Hughes 1981). For such 'temporally fractal' random walks, we show that the mean escape time diverges with an exponent that is related to the fractal (Hausdorff-Besicovitch) dimension of the walk. The scaling of the mean escape time thus provides a convenient index of the fractal dimensionality associated with the walk.

2. Formulas for the first passage time distribution

2.1 *The Siegert equation*

Let $P(m, t \mid m_0)$ denote the probability of finding the random walker at site m at time t given that she started from m_0 at time $t = 0$. Let $Q(m, t \mid m_0)\, dt$ be the probability of reaching m for the first time, in the time interval $(t, t + dt)$. For a temporally homogeneous *Markov* process, P and Q are related *via* the Siegert equation (Siegert 1951; Darling and Siegert 1953)

$$P(m_1, t \mid m_0) = \int_0^t P(m_1, t - t' \mid m) Q(m, t' \mid m_0)\, dt', \quad (m_0 < m \leqslant m_1). \tag{1}$$

Hence, in terms of the corresponding Laplace transforms (denoted by a tilde),

$$\tilde{Q}(m, u \mid m_0) = \tilde{P}(m_1, u \mid m_0)/\tilde{P}(m_1, u \mid m), \quad (m_0 < m \leqslant m_1), \tag{2}$$

where u is the transform variable. Using this simple relation, the first passage problem on finite or infinite Markov chains with specific reflecting or absorbing boundary conditions has been studied in detail (Darling and Siegert 1953; Goel and Richter-Dyn 1974; Montroll and West 1979; Khantha and Balakrishnan 1983). Equation (1) is based on a renewal principle that is *not valid* for non-Markov processes. In such cases, the first passage time problem must be solved by other methods.

2.2 *The method of images*

The problem of a first passage to the point m from a point $m_0 < m$ on an infinite chain is equivalent to that of a random walk in the restricted region $(-\infty, m)$ with an absorbing barrier at m. $Q(m, t \mid m_0)$ is given by (Montroll and West 1979)

$$Q(m, t \mid m_0) = -\frac{\mathrm{d}}{\mathrm{d}t} \sum_{m'=-\infty}^{m-1} \mathscr{P}_m(m', t \mid m_0), \tag{3}$$

where $\mathscr{P}_m(m', t \mid m_0)$ is the conditional probability of finding the random walker at m' at time t starting from m_0 at $t=0$, in the presence of an absorbing barrier at m. By invoking the method of images (Chandrasekhar 1943; Feller 1966; Montroll and West 1979), $\mathscr{P}_m(m', t \mid m_0)$ can be easily determined from $P(m', t \mid m_0)$ (the solution for random walk on an infinite chain) according to

$$\mathscr{P}_m(m', t \mid m_0) = P(m', t \mid m_0) - P(2m - m', t \mid m_0). \tag{4}$$

If we assume (without loss of generality) that the random walker starts from the origin, we have the formula (for $m \geqslant 1$)

$$Q(m, t \mid 0) = -\frac{\mathrm{d}}{\mathrm{d}t} \sum_{m'=-\infty}^{m-1} [P(m', t \mid 0) - P(2m - m', t \mid 0)]. \tag{5}$$

Making use of the initial condition on P, the characteristic function $\tilde{Q}$ is then

$$\tilde{Q}(m, u \mid 0) = 1 + u \sum_{m'=-\infty}^{m-1} [\tilde{P}(2m-m', u \mid 0) - \tilde{P}(m', u \mid 0)] \quad (m \geqslant 1). \tag{6}$$

We shall use this in the next section.

2.3 *A direct method*

There is an alternative way of obtaining Q (or $\tilde{Q}$) directly. This is closely related to the route we follow to calculate P (or $\tilde{P}$) itself. The jumps of the random walker may be regarded as being caused by a random sequence of pulses with a specified distribution. The actual location of the walker at time t depends only on the number of transition-causing pulses or 'steps' executed in the time interval t. Let $W(n, t)$ be the (normalized) probability that n pulses have occurred in $(0, t)$. Let $p_n(m)$ be the probability of reaching the point m from the point 0 in n steps. Then

$$P(m, t\,|\,0) = \sum_{n=0}^{\infty} W(n, t)\, p_n(m). \tag{7}$$

For the case at hand,

$$p_n(m) = \begin{cases} \dbinom{n}{\frac{n-m}{2}} 2^{-n} & \text{if } n = m \bmod 2 \text{ and } n \geqslant |m|, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The problem then reduces to specifying (or computing) $W(n, t)$ in a given physical situation and then performing the summation in (7) (Balakrishnan and Venkataraman 1981).

With the help of a simple geometric argument (again related to the reflection principle), it can be shown (Chandrasekhar 1943) that the probability of reaching m from the origin for the *first* time in precisely n steps is $(|m|/n)p_n(m)$. Let $\pi(n, t)\,\mathrm{d}t$ be the probability that, starting at $t = 0$, the nth pulse occurs in the time interval $(t, t + \mathrm{d}t)$. Then it is evident that

$$Q(m, t|0) = \sum_{n=0}^{\infty} \pi(n, t) \frac{|m|}{n} p_n(m)\,. \tag{9}$$

As before, then, the problem amounts to specifying (or computing) $\pi(n, t)$ and then carrying out the summation. For the characteristic function $\tilde{Q}(m, u|0)$, the transform $\tilde{\pi}(n, u)$ occurs on the right in (9). We shall compute $\tilde{\pi}(n, u)$ and evaluate the sum for a general CTRW in the next section.

3. Continuous time random walks

3.1 *Calculation of W and $\tilde{P}$*

A CTRW on the infinite chain occurs when $W(n, t)$ is generated by a renewal process. The latter is specified by a normalized pausing time density $\psi(t)$: if a step (event) has occurred at time t_0, the probability of the next one occurring in the interval $(t_0 + t, t_0 + t + \mathrm{d}t)$ is $\psi(t)\,\mathrm{d}t$. Starting from an arbitrary origin $t = 0$, the *first* pausing time distribution $\psi_0(t)$ could, in general, be distinct from $\psi(t)$, and may be specified independently. For an ongoing *equilibrium* renewal process, ψ_0 is related to ψ according to $\tau\,\tilde{\psi}_0(u) = 1 - \tilde{\psi}(u)$, where τ is the mean time between successive events (*i.e.*, the first moment of $\psi(t)$) (Feller 1966; Cox 1967; Kehr and Haus 1978;

Balakrishnan 1980). It is seen easily that, among such *equilibrium* renewal processes, $\psi_0 = \psi$ if and only if $\psi(t)$ is the exponential density $\lambda \exp(-\lambda t)$. ($W(n, t)$ is then a Poisson distribution and the random walk is a Markov chain). In general, however, physical applications *may* correspond to ordinary renewal processes rather than equilibrium ones, and the choice of ψ_0 may be dictated by physical considerations. For example, in the application of CTRW theory to hopping conduction in amorphous media (Lax and Scher 1977), the correct prescription happens to be $\psi_0(t) = \psi(t)$, even though $\psi(t)$ is not an exponential density in that problem. We shall work with an arbitrary normalized density $\psi_0(t)$ in what follows.

Corresponding to the pausing time densities $\psi_0(t)$ and $\psi(t)$, we have the 'survivor functions' (or holding-time distributions)

$$p_0(t) = 1 - \int_0^t \mathrm{d}t' \, \psi_0(t'), \quad p(t) = 1 - \int_0^t \mathrm{d}t' \, \psi(t') \tag{10}$$

Then, clearly,

$$\left.\begin{aligned} W(0, t) &= p_0(t), \\ W(n, t) &= \int_0^t \mathrm{d}t_n \ldots \int_0^{t_2} \mathrm{d}t_1 \, p(t-t_n)\, \psi(t_n - t_{n-1}) \ldots \psi(t_2 - t_1)\, \psi_0(t_1) \\ & \qquad\qquad\qquad\qquad\qquad\qquad (n \geqslant 1). \end{aligned}\right\} \tag{11}$$

Hence

$$\left.\begin{aligned} \tilde{W}(0, u) &= u^{-1} [1 - \tilde{\psi}_0(u)] \\ \tilde{W}(n, u) &= u^{-1} \tilde{\psi}_0(u) [1 - \tilde{\psi}_0(u)] [\tilde{\psi}(u)]^{n-1}, \quad n \geqslant 1. \end{aligned}\right\} \tag{12}$$

The Laplace transform of the conditional probability, $\tilde{P}(m, u \mid 0)$, is then found by inserting (12) and (8) in (the Laplace transform of) (7). A summation of the type $\sum x^n p_n(m)$ arises. Using the result given in the Appendix, we obtain finally

$$\tilde{P}(m, u \mid 0) = \frac{1}{u}\left(1 - \frac{\tilde{\psi}_0}{\tilde{\psi}}\right) \delta_{m,0} + \frac{\tilde{\psi}_0 (1 - \tilde{\psi})}{u \tilde{\psi} (1 - \tilde{\psi}^2)^{1/2}} \left[\frac{1 - (1 - \tilde{\psi}^2)^{1/2}}{\tilde{\psi}}\right]^{|m|} \quad (m = 0, \pm 1, \pm 2, \ldots). \tag{13}$$

This is a special case of the more general result found elsewhere (Balakrishnan and Venkataraman 1981). It is convenient to introduce the variable

$$\xi(u) = \operatorname{arc\,sech} \tilde{\psi}(u). \tag{14}$$

Then $\widetilde{P}$ has the compact form

$$\widetilde{P}(m, u \mid 0) = (1/u) \left[\left(1 - \frac{\widetilde{\psi}_0}{\widetilde{\psi}}\right) \delta_{m,0} + (\widetilde{\psi}_0/\widetilde{\psi}) \tanh(\xi/2) \exp(-\mid m \mid \xi) \right] \quad (m = 0, \pm 1, \ldots) \tag{15}$$

3.2 *Calculation of* $\widetilde{Q}$

Substituting the result (13) in formula (6) obtained by the method of images, we get after simplification,

$$\widetilde{Q}(m, u \mid 0) = \left(\frac{\widetilde{\psi}_0}{\widetilde{\psi}}\right) \left[\frac{1 - (1 - \widetilde{\psi}^2)^{1/2}}{\widetilde{\psi}}\right]^m = (\widetilde{\psi}_0/\widetilde{\psi}) \exp(-m\xi), \ (m \geqslant 1). \tag{16}$$

This is the characteristic function of the first passage time distribution for an arbitrary continuous time random walk *via* unbiased nearest-neighbour jumps on an infinite chain. As both $\psi_0(t)$ and $\psi(t)$ are normalized, $\widetilde{\psi}_0(0) = \widetilde{\psi}(0) = 1$; hence $Q(m, t \mid 0)$ is properly normalized; its integral form $t = 0$ to ∞ is equal to unity. As the random walk is unbiased, it is evident that $Q(m, t \mid 0) = Q(-m, t \mid 0)$, so that the exponent in (16) may be replaced by $|m|$, making the result valid for all non-zero m.

3.3 *Direct calculation of* $\widetilde{\pi}$ *and* $\widetilde{Q}$

Considering the definitions of $W(n, t)$ and $\pi(n, t)$ given in § 2.3, it is evident that, *for a renewal process,*

$$W(n, t) = \int_0^t \mathrm{d}t' \, \pi(n, t') \, p(t - t'), \ (n \geqslant 1), \tag{17}$$

where $p(t)$ is the holding time distribution defined in (10). Hence,

$$\widetilde{\pi}(n, u) = \widetilde{\psi}_0(u) (\widetilde{\psi}(u))^{n-1} \ (n \geqslant 1). \tag{18}$$

If this is substituted in (the Laplace transform of) the 'direct' formula of (9) for Q, we are left with a sum of the type $\Sigma \widetilde{\psi}^n p_n(m)/n$. It is easily seen that $|\widetilde{\psi}(u)| < 1$ for all Re $u > 0$, so that the result given in the Appendix may be used to obtain

$$\widetilde{Q}(m, u \mid 0) = \left(\frac{\widetilde{\psi}_0}{\widetilde{\psi}}\right) \left[\frac{1 - (1 - \widetilde{\psi}^2)^{1/2}}{\widetilde{\psi}}\right]^{|m|} = (\widetilde{\psi}_0 \mid \widetilde{\psi}) \exp(-\mid m \mid \xi) \tag{19}$$

for $m = \pm 1, \pm 2, \ldots$.

3.4 *Mean first passage time*

The mean first passage time from the origin to the site m is equal to

$$\langle t(m)\rangle = \int_0^\infty t\, Q\,(m, t\,|\,0)\,\mathrm{d}t$$

$$= -\,[\partial\, \tilde{Q}\,(m, u\,|\,0)/\partial u]_{u=0}, \tag{20}$$

as $Q\,(m, t\,|\,0)$ is already normalized to unity. Using (19) we find

$$\langle t(m)\rangle \sim \lim_{u\to 0}\;(1-\tilde{\psi}^2(u))^{-1/2} = \infty, \tag{21}$$

for all symmetric continuous time random walks on an infinite chain.(The special case $\tilde{\psi} = \lambda/(u+\lambda)$ corresponds to Polya's classic result (Polya 1921)).

3.5 *A simple relation between P and Q*

As stated earlier, $W(n, t)$ is a Poisson distribution and the random walk is a Markov chain when $\psi_0 = \psi = \lambda \exp\,(-\,\lambda t)$. The transforms $\tilde{P}$ and $\tilde{Q}$ can be inverted in this case, to yield the well-known results (Feller 1966)

$$\left.\begin{aligned} P\,(m, t\,|\,0) &= \exp\,(-\,\lambda\, t)\; I_m\,(\lambda\, t),\\ Q\,(m, t\,|\,0) &= |m|\; t^{-1} \exp\,(-\,\lambda\, t)\; I_m\,(\lambda\, t), \end{aligned}\right\} \tag{22}$$

where I_m is the modified Bessel function of order m. For this simplest of random walks, therefore, we have the interesting connection

$$Q\,(m, t\,|\,0) = (|m|\,/t)\; P\,(m, t\,|\,0). \tag{23}$$

Are there other random walks for which this relationship is satisfied?

Equation (23) is equivalent to

$$\frac{\partial}{\partial u}\tilde{Q}\,(m, u\,|\,0) + |\,m\,|\;\tilde{P}\,(m, u\,|\,0) = 0 \tag{24}$$

for $m = \pm 1, \pm 2, \ldots.$ Using (13) and (19) for an arbitrary CTRW, we find that (24) requires that

$$\frac{\mathrm{d}}{\mathrm{d}u}\ln\,(\tilde{\psi}_0/\tilde{\psi}) + |\,m\,|\left(\frac{1}{u}\tanh\frac{\xi}{2} - \frac{\mathrm{d}\xi}{\mathrm{d}u}\right) = 0, \tag{25}$$

for every non-zero integral value of m. Hence, we *must* have $\psi_0 = \psi$, and further, $\tanh\,(\xi/2) = u\,(\mathrm{d}\,\xi/\mathrm{d}u)$. It is shown easily that this last condition is satisfied only by the functional form $\tilde{\psi}(u) = \lambda/(u+\lambda)$, *i.e.* only in the Markovian case specified by (22).

Remarkably enough, there do exist random walks that are even more strongly correlated (temporally) than a CTRW, and which display property (23). An explicit example is provided by the geometric distribution (Balakrishnan 1981)

$$W(n, t) = (\lambda t)^n/(1+\lambda t)^{n+1}. \tag{26}$$

This is not a CTRW (or renewal process), and we cannot write (17) connecting $W(n, t)$ and $\pi(n, t)$ in this case. (Roughly speaking, the pausing time density ψ may itself be n-dependent in such cases.) The explicit solution for $P(m, t\,|\,0)$ now reads

$$P(m, t\,|\,0) = (1+2\lambda t)^{-1/2}\,(\lambda t)^{|m|}\,[1+\lambda t+(1+2\lambda t)^{1/2}]^{-|m|}. \tag{27}$$

The method of images then yields the result

$$Q(m, t\,|\,0) = |\,m\,|\,t^{-1}\,P(m, t\,|\,0) \tag{28}$$

where P is given by (27). Indeed, one can show that if $W(n, t) \propto x^n(t)$, with no further dependence on n, then property (23) is valid *only* for the functional form $x(t) = \lambda t/(1+\lambda t)$, which is equivalent to (26) on taking into account the normalization of $W(n, t)$. The classification of *all* random walks satisfying relation (23) between P and Q will be dealt with elsewhere.

4. The escape time distribution

4.1 *General formula for* $Q(\pm m, t\,|\,0)$

We now turn to the problem of the escape of the random walker out of the region $(-m, m)$, starting from the origin at $t = 0$. This is equivalent to considering first passage through *either* $-m$ *or* $+m$, and involves the solution to a random walk on the set $\{-m, \ldots, +m\}$ with absorbing barriers at both ends (eg. see Montroll and Scher 1973). Let $Q(\pm m, t\,|\,0)$ denote the desired first passage time distribution, and $\mathscr{P}_{\pm m}(m', t\,|\,0)$ the conditional probability for the random walk referred to. Then (Montroll and West 1979)

$$Q(\pm m, t\,|\,0) = -\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m'=-(m-1)}^{(m-1)} \mathscr{P}_{\pm m}(m', t\,|\,0),\ (m \geqslant 1), \tag{29}$$

for the sum is just the probability that the random walker has survived without absorption at either of the barriers till time t. $\mathscr{P}_{\pm m}$ may be found once again by the method of images. As there are two barriers, the number of images of the interval is infinite, and we have

$$\mathscr{P}_{\pm m}(m', t \mid 0) = \sum_{n=-\infty}^{\infty} [P(m' + 4nm, t \mid 0) - P(-m' - 4nm - 2m, t \mid 0)]. \tag{30}$$

Using the symmetry properties of P, this may be simplified to yield

$$Q(\pm m, t \mid 0) = -\frac{\mathrm{d}}{\mathrm{d}t} \sum_{m'=-(m-1)}^{(m-1)} \sum_{n=-\infty}^{\infty} (-1)^n P(m'+2nm, t \mid 0). \tag{31}$$

This is the formula desired. Its form may be compared with that of (5) for first passage from the point 0 to the point m on an infinite chain (a problem with a single absorbing barrier).

4.2 *Calculation of* $Q(\pm m, t \mid 0)$ *for a* CTRW

For an arbitrary CTRW, $\tilde{P}$ is given by (15). Inserting this in the Laplace transform of (31) and carrying out the summations involved, we get (after a considerable amount of algebra) the very simple answer

$$\tilde{Q}(\pm m, u \mid 0) = (\tilde{\psi}_0/\tilde{\psi}) \operatorname{sech}(m\xi). \tag{32}$$

Here $\sec h\, \xi = \tilde{\psi}$, as already defined (equation (14)). This is the result required. As the right side of (32) tends to unity as $u \to 0$, the distribution $Q(\pm m, t \mid 0)$ is also normalised to unity.

A comparison of (32) with (16) for the characteristic function $\tilde{Q}(m, u \mid 0)$ of the first passage time distribution shows that (setting $\psi_0 = \psi$)

$$\tilde{Q}(\pm m, u \mid 0) = 2\tilde{Q}(m, u \mid 0) / [1 + \tilde{Q}^2(m, u \mid 0)]. \tag{33}$$

The structure of this result suggests the following interesting connection between the escape time distribution and the distribution of the time of first passage to *either* end of the region of interest, *i.e.*, $\pm m$, in the *absence* of the other barrier. Owing to the symmetry of the problem, we have already seen that $Q(m, t \mid 0) = Q(-m, t \mid 0)$. For brevity, let us write $Q_\pm(t)$ for $Q(\pm m, t \mid 0)$, $Q_+(t) = Q(m, t \mid 0)$, $Q_-(t) = Q(-m, t \mid 0)$. Then (33) can be recast as

$$\left.\begin{aligned} \tilde{Q}_\pm(u) &= \tilde{Q}_+/(1 + \tilde{Q}_-\tilde{Q}_+) + \tilde{Q}_- / (1 + \tilde{Q}_+\tilde{Q}_-) \\ &\equiv \tilde{Q}_{\text{right}}(u) + \tilde{Q}_{\text{left}}(u), \end{aligned}\right\} \tag{34}$$

where $Q_{\text{right}}(t)$ is the probability per unit time of absorption at $+m$ *in the presence* of the other absorbing barrier at $-m$, with a similar interpretation for $Q_{\text{left}}(t)$. We have solved the problem under consideration for *biased* random walks as well (Khantha and Balakrishnan 1983a), and equation (34) continues to hold good in that case. These matters will be elaborated upon in the paper referred to above.

4.3 *Mean escape time*

As in the case of the mean first passage time (equation (20)), the mean time of escape from the region $(-m, m)$, starting from the origin, is

$$\left.\begin{aligned}\langle t(\pm m)\rangle &= \int_0^{\infty} \mathrm{d}t\, Q(\pm m, t\,|\,0)\, \mathrm{d}t \\ &= -[\partial \tilde{Q}(\pm m, u\,|\,0)/\partial u]_{u=0}.\end{aligned}\right\} \tag{35}$$

For a CTRW, $\tilde{Q}(\pm m, u\,|\,0)$ is given by (32). It turns out that the derivative required in (35) is finite (see below) when the mean residence time τ at a site is finite —that is, when the Laplace transform of $\psi(t)$ has the small u expansion

$$\tilde{\psi}(u) \simeq 1 - u\tau + (\text{higher orders in } u). \tag{36}$$

In all such cases, we have (taking $\psi_0 = \psi$ for simplicity)

$$\langle t(\pm m)\rangle = m^2 \tau. \tag{37}$$

The Markov case $\psi(t) = \lambda \exp(-\lambda t)$ thus yields $\langle t(\pm m)\rangle = m^2/\lambda$, as is known (Seshadri and West 1982).

It is interesting to examine $\langle t(\pm m)\rangle$ when the pausing time distribution has a long tail (does not fall off like an exponential, or a finite sum of exponentials) (see e.g. Shlesinger 1973). Such distributions are necessary to explain anomalies in charge transport phenomena in amorphous solids (Scher and Lax 1973; Tunaley 1976; Montroll and West 1979). In these cases, $\tilde{\psi}(u)$ has in general a small u expansion of the form

$$\tilde{\psi}(u) \simeq 1 - ru^{\alpha} + su^{\beta} + \text{higher orders}, \tag{38}$$

where $0 < \alpha < 1$ and $\beta > \alpha$. The mean residence time is evidently infinite in all these instances. Using the asymptotic expansion (38), we find (recalling that $\xi = \operatorname{sech}^{-1}\tilde{\psi}$) the expansion

$$\exp(-\xi) = 1 - (2r)^{1/2} u^{\alpha/2} + O(u^{\gamma}), \tag{39}$$

where $\gamma = \operatorname{Min}(\alpha, \beta - \alpha/2)$. This leads to

$$\tilde{Q}(\pm m, u\,|\,0) \simeq 1 - m^2 ru^{\alpha} + \text{higher orders in } u, \tag{40}$$

and hence (remembering that $\alpha < 1$)

$$\langle t\,(\pm m)\rangle \to \infty. \tag{41}$$

This is the reason (*i.e.* a divergent mean time for escape out of a *bounded* region) why we stated in § 1 that such pausing time distributions do not lead to a true long range diffusion of the random walker. This is reinforced by the fact (Khantha and Balakrishnan 1983a) that the foregoing conclusions are not altered by the inclusion of a bias in the random walk.

4.4 *Temporally fractal random walks*

When the pausing time distribution $\psi(t)$ has no finite first moment, the mean time between the jumps of the random walker is infinite, and there is no finite time scale in the problem. This is a necessary condition (but of course not a sufficient one) for a self-similar clustering or fractal behaviour (Mandelbrot 1977) of the epochs at which jumps occur, as described by the distribution $W(n, t)$. A class of such processes that is within the purview of CTRW's is provided by the (normalized) pausing time distribution (Shlesinger and Hughes 1981)

$$\psi(t) = \frac{\lambda(1-a)}{a} \sum_{k=1}^{\infty} (a\,b)^k \exp\,(-\,\lambda\, b^k\, t), \tag{42}$$

where $0 < a, b < 1$, and λ^{-1} is a positive constant with the dimensions of time. This expression is an infinite superposition of exponentials in which the jump rate $\lambda\, b^k$ occurs with a probability proportional to a^k. The mean residence time is

$$\tau = \int_0^{\infty} t\,\psi(t)\,\mathrm{d}t = \frac{(1-a)}{\lambda\, a} \sum_{k=1}^{\infty} (a/b)^k, \tag{43}$$

so that τ is infinite if $a \geqslant b$. If this is so, there is no finite time scale in the problem. The long-time decay of such a 'frozen' process is governed by a power law when $a > b$, *i.e.*

$$\psi(t) \simeq O\,(t^{-1-H}) \tag{44}$$

where H is a positive number to be identified shortly. Equivalently, the Laplace transform of $\psi(t)$ is not analytic at $u = 0$, and can be shown (Shlesinger and Hughes 1981) to have the small u behaviour

$$\tilde{\psi}(u) \simeq 1 + u^H\, K(u) + O(u), \tag{45}$$

where $K(u)$ is a periodic function of $\ln\,(u/\lambda)$ that does not seriously affect the behaviour of $\tilde{\psi}$ as $u \to 0$. The leading power H is given by

$$H = \ln a/\ln b \tag{46}$$

so that $0 < H < 1$ (since $0 < b < a < 1$). The exponent H can be viewed (in an average sense) as the fractal or Hausdorff-Besicovitch dimension characterising the CTRW. How does one probe this quantity?

The mean escape time from a bounded region, say $(-m, m)$ provides a direct answer. It is immediately evident from what has been deduced earlier that for the CTRW specified by (42) with $b < a$,

$$\tilde{Q}(\pm m, u \mid 0) \simeq 1 + m^2 u^H K(u) + O(u), \tag{47}$$

with H given by (46). Hence $\langle t(\pm m)\rangle \to \infty$ in this case. As

$$\langle t(\pm m)\rangle = \lim_{T\to\infty} \int_0^T t\, Q(\pm m, t \mid 0)\, dt, \tag{48}$$

the divergence of $\langle t(\pm m)\rangle$ with the time of observation T goes like T^{1-H} for very large times T. This therefore yields a convenient index for the estimation of the fractal dimensionality H of the CTRW. The introduction of a uniform bias in the random walk does not affect this result, as stated earlier.

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Appendix

The function $\tilde{P}(m, u \mid 0)$ (eqns. (7), (12)) involves the sum

$$S_1(x) = \sum_{n=0}^{\infty} x^n p_n(m), \quad (|x| < 1),$$

where x is a function of u, and (see (8))

$$p_n(m) = \begin{cases} \binom{n}{\frac{n-m}{2}} 2^{-n} & \text{if } n = m \bmod 2,\ n \geqslant |m|; \\ 0 & \text{otherwise.} \end{cases}$$

After a change of variables,

$$S_1(x) = \sum_{k=0}^{\infty} \binom{2k + |m|}{k} \left(\frac{x}{2}\right)^{2k+|m|},$$

which reduces after some manipulation to

$$S_1(x) = (x/2)^{|m|}\ {}_2F_1\left(\frac{|m|+1}{2},\ \frac{|m|}{2}+1;\ |m|+1;\ x^2\right)$$

$$= (x/2)^{|m|}\,(1-x^2)^{-1/2}\ F\left(\frac{|m|}{2},\ \frac{|m|+1}{2};\ |m|+1;\ x^2\right).$$

Call $x^2 = 4z\,(1-z)$, and use a transformation property of the hypergeometric function that relates the above to a function with argument z. It is then possible to identify S_1 to be

$$S_1(x) = (1-x^2)^{-1/2}\left(\frac{1-(1-x^2)^{1/2}}{x}\right)^{|m|}.$$

Similarly, the characteristic function of the first passage time distribution, $\tilde{Q}\,(m, u|0)$, involves the sum (see (9), (18))

$$S_2(x) = \sum_{n=0}^{\infty} (1/n)\, x^n\, p_n(m), \quad (|x| < 1).$$

Proceeding as before, we find

$$S_2(x) = (x/2)^{|m|}\ {}_2F_1\left(\frac{|m|+1}{2},\ \frac{|m|}{2};\ |m|+1;\ x^2\right),$$

which, by an inspection of the earlier result, is just

$$S_2(x) = \left(\frac{1-(1-x^2)^{1/2}}{x}\right)^{|m|}.$$

References

Balakrishnan V 1980 in Proc. Meeting on Spin-Glass Alloys, University of Roorkee, Roorkee, India (unpublished)

Balakrishnan V 1981 *Pramana* **17** 55

Balakrishnan V and Venkataraman G 1981 *Pramana* **16** 109

Chandrasekhar S 1943 *Rev. Mod. Phys.* **15** 1

Cox D R 1967 *Renewal theory* (London: Methuen)

Darling D A and Siegert A J F 1953 *Ann. Math. Stat.* **24** 624

Feller W 1966 *An introduction to probability theory and its applications* (New York: Wiley) Vols. 1 and 2

Goel N S and Richter-Dyn N 1974 *Stochastic models in biology* (New York: Academic)

Hänggi P and Talkner P 1981 *Z. Phys.* **B45** 79

Hughes B D, Shlesinger M F and Montroll E W 1981 *Proc. Natl. Acad. Sci.* (*USA*) **78** 3287

Hughes B D, Montroll E W and Shlesinger M F 1982 *J. Stat. Phys.* **28** 111

Kehr K W and Haus J W 1978 *Physica* **A93** 412

Khantha M and Balakrishnan V 1983 *Pramana* (Accepted)
Khantha M and Balakrishnan V 1983a manuscript in preparation
Lax M and Scher H 1977 *Phys. Rev. Lett.* **39** 781
Mandelbrot B B 1977 *Fractals: form, chance and dimension* (San. Francisco: Freeman)
Montroll E W 1969 *J. Math. Phys.* **10** 753
Montroll E W and Weiss G H 1965 *J. Math. Phys.* **6** 167
Montroll E W and Scher H 1973 *J. Stat. Phys.* **9** 101
Montroll E W and West B J 1979 in *Fluctuation phenomena* (eds.) E W Montroll and J L Lebowitz (Amsterdam: North-Holland)
Polya G 1921 *Math. Ann.* **84** 149
Pontryagin L, Andronov A and Witt A 1933 *Zh. Eskp. Theor. Fiz.* **3** 172
Scher H and Lax M 1973 *Phys. Rev.* **B7** 4491
Seshadri V and West B J 1982 *Proc. Natl. Acad. Sci.* (*USA*) **79** 4501
Shlesinger M F 1973 *J. Stat. Phys.* **10** 421
Shlesinger M F and Hughes B D 1981 *Physica* **A109** 597
Siegert A J F 1951 *Phys. Rev.* **81** 617
Stratonovich R L 1963 *Topics in the theory of random noise* (New York: Gorden-Breach) Vol. 1
Tunaley J K E 1976 *J. Stat. Phys.* **15** 149
Weiss G H 1966 *Adv. Chem. Phys.* **13** 1
Weiss G H 1981 *J. Stat. Phys.* **24** 587