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On phases and length of curves in a cyclic quantum evolution

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Abstract. The concept of a curve traced by a state vector in the Hilbert space is introduced into the general context of quantum evolutions and its length defined. Three important curves are identified and their relation to the dynamical phase, the geometric phase and the total phase are studied. These phases are reformulated in terms of the dynamical curve, the geometric curve and the natural curve. For any arbitrary cyclic evolution of a quantum system, it is shown that the dynamical phase, the geometric phase and their sums and/or differences can be expressed as the integral of the contracted length of some suitably-defined curves. With this, the phases of the quantum mechanical wave function attain new meaning. Also, new inequalities concerning the phases are presented.

Keywords. Geometric phase; distance function; length of the curves.

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1. Introduction

The phase change of the quantum mechanical wave function is of vital importance as it gives information about the dynamical changes in the system as well as the geometry of the path of the evolution. This paper aims at giving a new meaning towards all phase changes in general. When we are dealing with the cyclic evolution of a physical system, the possible natural questions that concern us are: (i) what is the phase factor that is associated with the final state (ii) how much distance the quantum state has travelled during the period of evolution in the projective Hilbert space $\mathscr{P}$ as measured by the Fubini–Study metric and (iii) what is the total length of the curve traced by the state vector (or other curves traced by the phase-transformed vectors)? The answer to (i) is well-known: the total phase factor contains two parts. One is the usual dynamical phase that gives the information about the duration of the evolution of the system and the second phase factor is the geometric phase (Berry phase [1, 2]) that gives the information about the geometry of the path of the evolution in $\mathscr{P}$ [3, 4]. The answer to the question (ii), though not widely-studied, has drawn some attention in connection with the introduction of geometric structures into quantum mechanics [5–14] and in relating the geometric phase to the geometric distance function [11–13]. The total distance travelled by the quantum states along a given curve $\hat{C}$ in $\mathscr{P}$, as measured by the Fubini–Study metric [7–13] is the time integral of the uncertainty in the energy of the system. Anandan and Aharonov [7] have remarked that it is also a geometric quantity analogous to the geometric phase, in the sense that it does not depend on the particular Hamiltonian used to move the quantum system along a given curve $\hat{C}$ in $\mathscr{P}$. In fact these are not just analogies but are related quantities. Recently, the author [11] has shown that the geometric phase