Journal of Algebra 322 (2009) 2118–2130

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the negative K-theory of schemes in finite characteristic

Amalendu Krishna

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai, 400005, India

ARTICLE INFO

Article history:
Received 4 December 2008
Available online 9 June 2009
Communicated by Steven Dale Cutkosky

Keywords:
K-theory
Singular varieties
Topological cyclic homology

ABSTRACT

We show that if X is a d-dimensional scheme of finite type over an infinite perfect field k of characteristic $p > 0$, then $K_i(X) = 0$ and X is K_i-regular for $i < -d - 2$ whenever the resolution of singularities holds over k. This proves the K-dimension conjecture of Weibel [C. Weibel, K-theory and analytic isomorphisms, Invent. Math. 61 (1980) 177–197, 2.9] (except for $-d-1 \leqslant i \leqslant -d-2$) in all characteristics, assuming the resolution of singularities.

1. Introduction

It is by now well known that the negative K-theory of singular schemes is non-zero in general and bears a significant information about the nature of the singularity of the scheme. Hence it is a very interesting question to know how much of the negative K-theory of a singular scheme can survive. A beautiful answer was given in terms of the following very general conjecture of Weibel.

Conjecture 1.1. *(See Weibel [23].) Let X be a Noetherian scheme of dimension d. Then $K_i(X) = 0$ for $i < -d$ and X is K_{-d}-regular.*

This conjecture was proved recently by Cortinas, Haesemeyer, Schlichting and Weibel [4, Theorem 6.2] for schemes of finite type over a field of characteristic zero. If X is a scheme of finite type over a field of positive characteristic, the above conjecture was proved by Weibel [24] provided the dimension of X is at most two. Our aim in this paper is to prove the conjecture for a d-dimensional scheme in the positive characteristic (except for $i = -d-1, -d-2$) assuming the resolution of singularities. A *variety* X in this paper will mean a scheme of finite type over a ground field k.

E-mail address: amal@math.tifr.res.in.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.05.038

Definition 1.2 *(Resolution of singularities).* We will say that the resolution of singularities holds over k if given any equidimensional scheme X of finite type over k, there exists a sequence of monoidal transformations

$$X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$$

such that the following hold:

(i) the reduced subscheme X_r^{red} is smooth over k,
(ii) the center D_i of the monoidal transformation $X_{i+1} \to X_i$ is smooth and connected and nowhere dense in X_i.

The resolution of singularities holds over fields of characteristic zero by the work of Hironaka [11, Theorem 1*]. For fields of positive characteristics, this problem is still not known though widely expected to be true. Recently, Hironaka [12] has outlined a complete program to solve this problem and work on this program is in progress. We will assume throughout this paper that our ground field k is an infinite perfect field of characteristic $p > 0$ and the resolution of singularities holds over k.

For a variety X over a field k, let $X[T_1, \cdots, T_r]$ denote the polynomial extension $X \otimes_k k[T_1, \cdots, T_r]$ over X. Recall that a variety X is said to be K_i-regular if the natural map $K_i(X) \to K_i(X[T_1, \cdots, T_r])$ is an isomorphism for all $r \geqslant 1$, where $K_i(X)$ is the ith stable homotopy group of the non-connective spectrum $K(X)$ of perfect complexes on X. We shall often write the polynomial extension $X[T_1, \cdots, T_r]$ in short as $X[T]$ in this paper, where r can be any natural number. It is known from a result of Vorst [22, Corollary 2.1] that a scheme which is K_i-regular, is also K_j-regular for $j \leqslant i$. For an abelian group A, let $A\{p\}$ denote the p-primary torsion subgroup of A. We now state the main result of this paper.

Theorem 1.3. *Assume that the resolution of singularities holds over k and let X be a variety of dimension d over k. Then*

(i) $K_i(X, \mathbb{Z}/n) = 0$ *for* $i < -d-1$ *and for all* $n \geqslant 1$.
(ii) $K_{-d-2}(X[T])$ *is a divisible group and* $K_{-d-2}(X[T]) = K_{-d-2}(X[T])\{p\}$.
(iii) $K_i(X) = 0$ *and X is K_i-regular for* $i < -d-2$.

We now briefly describe the organization of this paper. We briefly review our main objects, the topological cyclic homology and the cyclotomic trace map from the K-theory to the topological cyclic homology in the next section. In Section 3, we prove the *cdh*-descent for the homotopy fiber of the cyclotomic trace map using a characteristic p variant of Haesemeyer's criterion [9]. This is used in the next section to prove some vanishing results for the homotopy groups of this fiber. In Section 5, we compare the Nisnevich and the *cdh*-cohomology of the sheaf of the first negative topological cyclic homology using Witt vectors, and then prove some vanishing and homotopy invariance for the homotopy groups of $K(-, \mathbb{Z}/p^n)$. The main theorem is finally proved in the last section by combining the vanishing of the negative K-theory with finite coefficients and with the rational coefficients.

To end this introduction, we mention that a few days after this paper appeared on the math arXiv (*cf.* arXiv:0811.0302v1), Geisser and Hesselholt posted the joint paper [7], which also proves the main result of this paper (including the case of $i = -d-1, -d-2$), except the K-regularity part of our theorem.

2. *K*-theory and topological cyclic homology

In this section, we briefly recall the topological cyclic homology of rings and schemes and the cyclotomic trace map from the K-theory to the topological cyclic homology. We will show in the next section that the homotopy fiber of this trace map satisfies the descent for the *cdh*-topology.

Let p be a fixed prime number. Let A be a commutative ring which is a Noetherian k-algebra, where k is a perfect field of characteristic p. Recall from [5] that the topological Hochschild spectrum $T(A)$ is a symmetric $\mathbb{S}$-spectrum, where $\mathbb{S}$ is the circle group. Let $C_r \subset \mathbb{S}$ be the cyclic subgroup of order r. Then one defines

$$TR^n(A; p) = F\left(\mathbb{S}/C^{p^{n-1}}, T(A)\right)^{\mathbb{S}}$$

to be the fixed point spectrum of the function spectrum $F(\mathbb{S}/C^{p^{n-1}}, T(A))$. There are the frobenius and the restriction maps of spectra

$$F, R : TR^n(A; p) \to TR^{n-1}(A; p).$$

The spectrum TC^n is defined as the homotopy equalizer of the maps F and R. That is,

$$TC^n(A; p) = eq\left(TR^n(A; p) \xrightarrow{R,F} TR^n(A; p)\right),$$

and the topological cyclic homology spectrum $TC(A; p)$ is defined as the homotopy limit

$$TC(A; p) = \operatorname{holim} \mathrm{TC}^n(A; p).$$

One similarly defines

$$TR(A; p) = \operatorname*{holim}_{R} \mathrm{TR}^n(A; p),$$

$$TF(A; p) = \operatorname*{holim}_{F} \mathrm{TR}^n(A; p).$$

It was shown by Geisser and Hesselholt in [5] that the topological Hochschild and cyclic homology satisfy descent for a Cartesian diagram of rings and they were able to define these homology spectra for a Noetherian scheme X using the Thomason's construction of the hypercohomology spectrum [18, 1.33]. Recently, Blumberg and Mandell [1] have made a significant progress in the study of the topological Hochschild and cyclic homology of schemes. They globally define the spectrum $T(X)$ for a Noetherian scheme X as the topological Hochschild homology spectrum of the spectral category $D(\mathit{Perf}/X)$ which is the Thomason's derived category of perfect complexes on X [19]. They then define the topological cyclic homology $TC(X)$ spectrum in exactly the same way as above. They show that their definition of these spectra coincides with the above definition for affine schemes. They also prove the localization and the Zariski descent properties of the topological Hochschild and cyclic homology of schemes. We refer to [1] for more details. In this paper, the topological Hochschild and cyclic homology of schemes will be considered in the sense of [1, Definitions 3.3, 3.7]. For any symmetric spectrum E and for $n \geqslant 1$, let E/p^n denote the smash product of E with a mod p^n Moore spectrum Σ^∞/p^n.

Let $K(X)$ denote the Thomason's non-connective spectrum of the perfect complexes on X. For a ring A, there is a cyclotomic trace map [2] of non-connective spectra

$$K/p^n(A) \xrightarrow{tr} TC/p^n(A; p).$$

Since K-theory satisfies Zariski descent by [19] and so does the topological cyclic homology by [1], taking the induced map on the Zariski hypercohomology spectra gives for any Noetherian scheme X, the cyclotomic trace map of spectra

$$K/p^n(X) \xrightarrow{tr} TC/p^n(X; p). \tag{2.1}$$

Let $L^n(X)$ denote the homotopy fiber of the trace map in (2.1). If Sch/k denotes the category of varieties over k, then one gets a presheaf of homotopy fibrations of spectra on Sch/k

$$L^n \to K/p^n \xrightarrow{tr} TC/p^n(-;p). \tag{2.2}$$

3. *cdh*-Descent for L^n

We remind the reader that our ground field k in this paper is an infinite perfect field of characteristic $p > 0$ which admits the resolution of singularities. We recall from [4] that a presheaf of spectra $\mathcal{E}$ on the category Sch/k satisfies the *Mayer–Vietoris property* for a Cartesian square of schemes

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & X \end{tikzcd} \tag{3.1}$$

if applying $\mathcal{E}$ to this square results in a homotopy Cartesian square of spectra. We say that $\mathcal{E}$ satisfies the Mayer–Vietoris property for a class of squares provided it satisfies this property for each square in that class. One says that the presheaf of spectra $\mathcal{E}$ is *invariant under infinitesimal extension* if for any affine scheme X and a closed subscheme Y of X defined by a sheaf of nilpotent ideals $\mathcal{I}$, the spectrum $\mathcal{E}(X,Y)$ is contractible, where the latter is the homotopy fiber of the map $\mathcal{E}(X) \to \mathcal{E}(Y)$. One says that $\mathcal{E}$ satisfies the *excision property* if for any morphism of affine schemes $f : Y \to X$ and a sheaf of ideals $\mathcal{I}$ on X such that $\mathcal{I} \cong f_* f^*(\mathcal{I})$, the spectrum $\mathcal{E}(X,Y,\mathcal{I})$ is contractible, where $\mathcal{E}(X,Y,\mathcal{I})$ is defined as the homotopy fiber of the map $\mathcal{E}(X,\mathcal{I}) \to \mathcal{E}(Y,\mathcal{I})$. An elementary Nisnevich square is a Cartesian square of schemes as above such that $Y \to X$ is an open embedding, $X' \to X$ is étale and $(X' - Y') \to (X - Y)$ is an isomorphism. Then one says that $\mathcal{E}$ satisfies *Nisnevich descent* if it satisfies the Mayer–Vietoris property for all elementary Nisnevich squares.

We next recall from [4] (see also [21]) that a *cd*-structure on a small category $\mathcal{C}$ is a class $\mathcal{P}$ of commutative squares in $\mathcal{C}$ that is closed under isomorphisms. Any such *cd*-structure defines a topology on $\mathcal{C}$. As k admits the resolution of singularities, the *combined cd-structure* on the category Sch/k consists of all elementary Nisnevich squares and all abstract blow-ups, where an abstract blow-up is a Cartesian square as in (3.1) such that $Y \to X$ is a closed embedding, $X' \to X$ is proper and the induced map $(X' - Y')_{\mathrm{red}} \to (X - Y)_{\mathrm{red}}$ is an isomorphism. The topology generated by the combined *cd*-structure is called the *cdh*-topology. Let Sm/k denote the category of smooth varieties over k. Since the resolution of singularities holds over k, the restriction of the *cd*-structure to the category Sm/k where abstract blow-ups are replaced by the smooth blow-ups is complete, bounded and regular (*cf.* [21, Section 4]). The topology generated by this *cd*-structure on Sm/k is called the *scdh*-topology. This is just the restriction of the *cdh*-topology on the subcategory Sm/k. In this paper, we shall consider the local injective model structure on the category of presheaves of spectra on Sch/k as described in [4].

For the local injective model structure on the category of presheaves of spectra on Sch/k with a given Grothendieck topology $\mathcal{C}$, a fibrant replacement of a presheaf of spectra $\mathcal{E}$ is a trivial cofibration $\mathcal{E} \to \mathcal{E}'$ where $\mathcal{E}'$ is fibrant. We shall write such a fibrant replacement as $\mathbb{H}_{\mathcal{C}}(-,\mathcal{E})$. We shall say that $\mathcal{E}$ satisfies the *cdh*-descent if it satisfies the Mayer–Vietoris property for all elementary Nisnevich squares and all abstract blow-ups. By [4, Theorem 3.4], this is equivalent to the assertion that the map $\mathcal{E} \to \mathbb{H}_{cdh}(-,\mathcal{E})$ is a global weak equivalence in the sense that $\mathcal{E}(X) \to \mathbb{H}_{cdh}(X,\mathcal{E})$ is a weak equivalence for all $X \in Sch/k$. Let a denote the natural morphism from the *cdh*-site to the Zariski site on the category Sch/k. For any Zariski sheaf $\mathcal{F}$, let $a_{cdh}\mathcal{F}$ denote the *cdh*-sheafification of $\mathcal{F}$.

Theorem 3.1. *Let $\mathcal{E}$ be a presheaf of spectra on Sch/k such that $\mathcal{E}$ satisfies excision, is invariant under infinitesimal extension, satisfies Nisnevich descent and satisfies the Mayer–Vietoris property for every blow-up* *along a regular closed embedding. Then $\mathcal{E}$ satisfies the cdh-descent. In particular, there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H^p\big(X_{cdh}, a_{cdh}\pi_q(\mathcal{E})\big) \quad \Rightarrow \quad \pi_{q-p}\mathcal{E}(X),$$

where the differentials of the spectral sequence are $d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$.

Proof. The proof of this theorem is very similar to the proof of the analogous theorem in [4, Theorem 3.12] when k has characteristic zero. We only give the brief sketch. As shown above, it suffices to show that the map

$$\mathcal{E}(X) \to \mathbb{H}_{cdh}(X, \mathcal{E}) \tag{3.2}$$

is a weak equivalence for all varieties X over k. Since the *scdh*-topology on Sm/k is generated by elementary Nisnevich squares and smooth blow-ups and since the closed embeddings of smooth varieties are regular embeddings, we see that $\mathcal{E}$ satisfies the *scdh*-descent in Sm/k.

Now assume X is singular. As explained in [4], the argument goes as in the proof of Theorem 6.4 in [9]. The excision, invariance under infinitesimal extension and Nisnevich descent together imply that $\mathcal{E}$ satisfies the Mayer–Vietoris property for closed covers and for finite abstract blow-ups. Now if X is a hypersurface in a smooth scheme, we can follow the proof of Theorem 6.1 in [9] to conclude that (3.2) holds for X since the resolution of singularities holds over k, which is also perfect and infinite. If X is a complete intersection inside a smooth k-scheme, then we can use the hypersurface case, the Mayer–Vietoris for the closed covers and an induction on the embedding dimension of X to conclude (3.2) for X. The general case follows from this as shown in [9, Theorem 6.4]. The spectral sequence now follows from [9, Theorem 2.8] since the *cdh*-cohomological dimension of X is bounded by its Krull dimension by [17, Theorem 12.5]. $\square$

Corollary 3.2. *The presheaf of spectra L^n (cf. (2.2)) satisfies the cdh-descent.*

Proof. We need to show that L^n satisfies all the conditions of Theorem 3.1. We have the homotopy fibration of presheaves of spectra

$$L^n \to K/p^n \to TC/p^n(-; p).$$

The fact that L^n satisfies excision was proved by Geisser–Hesselholt [6, Theorem 1]. The invariance of L^n under infinitesimal extension was proved by McCarthy [16, Main Theorem]. Next we show that L^n satisfies Nisnevich descent. K/p^n satisfies Nisnevich descent by [19, Theorem 10.8]. $TC/p^n(-; p)$ satisfies Nisnevich descent by [5, Corollary 3.3.4] and by the agreement of the definition of the topological cyclic homology as given in [5] with that of [1] since the topological cyclic homology of [1] satisfies the Zariski descent (see the discussion in Remark 3.3.5 of [5]). We now consider the following commutative diagram of spectra for a given variety X.

$$\begin{array}{ccccc}
L^n(X) & \longrightarrow & K/p^n(X) & \longrightarrow & TC/p^n(X; p) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}_{Nis}(X, L^n) & \longrightarrow & \mathbb{H}_{Nis}(X, K/p^n) & \longrightarrow & \mathbb{H}_{Nis}(X, TC/p^n(-; p))
\end{array} \tag{3.3}$$

Since the top row in the above diagram is a homotopy fibration and the bottom row is a fibrant replacement of the top row, the bottom row is also a homotopy fibration (*cf.* [18, 1.35], see also [4, Section 5]). Now, since the middle and the right vertical maps are weak equivalences, we see that the left vertical map is also a weak equivalence. This verifies the Nisnevich descent for L^n. Finally, L^n satisfies the Mayer–Vietoris property for the blow-up along regular closed embeddings by [1, Theorem 1.4]. We conclude from Theorem 3.1 that L^n satisfies *cdh*-descent. □

Corollary 3.3. *Let $\mathcal{KH}$ denote the presheaf of homotopy invariant K-theory on Sch/k (cf. [25]). Then $\mathcal{KH}$ satisfies cdh-descent on Sch/k.*

Proof. The presheaf of spectra $\mathcal{KH}$ satisfies excision by [25, Corollary 2.2], it satisfies invariance under infinitesimal extension by [25, Theorem 2.3]. The Nisnevich descent for $\mathcal{KH}$ follows from the analogous property of K-theory [19, Theorem 10.8] and the spectral sequence [25, Theorem 1.3]

$$E^1_{p,q} = N^p K_q(X) \quad \Rightarrow \quad KH_{p+q}(X).$$

It satisfies the Meyer–Vietoris property for blow-up under regular closed embeddings by [9, Theorem 3.6]. The corollary now follows from Theorem 3.1. □

4. Vanishing and homotopy invariance for L^n

Following [4], we let $\widetilde{C}_j\mathcal{E}$ denote the homotopy cofiber of the natural map $\mathcal{E} \to \mathcal{E}(- \times \mathbb{A}^j)$ for any presheaf of spectra $\mathcal{E}$ on Sch/k. Note that $\mathcal{E}(- \times \mathbb{A}^j)$ is a canonical direct sum of $\mathcal{E}$ and $\widetilde{C}_j\mathcal{E}$ and hence the functor $\widetilde{C}_j$ preserves the homotopy fibration sequences. In particular, we get a presheaf of fibration sequences

$$\widetilde{C}_j L^n \to \widetilde{C}_j K/p^n \to \widetilde{C}_j TC/p^n(-;p). \tag{4.1}$$

Furthermore, since L^n and $L^n(- \times \mathbb{A}^j)$ satisfy *cdh*-descent by Corollary 3.2, we see that $\widetilde{C}_j L^n$ also satisfies *cdh*-descent. For a presheaf of spectra E on Sch/k, let E_i (or $\pi_i(E)$) denote the presheaf of ith stable homotopy groups of E.

Lemma 4.1. *For a d-dimensional variety X over k, one has $L^n_i(X) = 0 = \pi_i \widetilde{C}_j L^n(X)$ for all $j \geqslant 0$ and $i < -d-2$.*

Proof. Using Theorem 3.1 and Corollary 3.2 and [17, Theorem 12.5], it suffices to show that $a_{cdh}\pi_i(L^n)$ and $a_{cdh}\pi_i(\widetilde{C}_j L^n)$ are zero for $i < -2$. The presheaf of fibration sequences (2.2) gives the long exact sequence of presheaves of homotopy groups on Sch/k

$$\cdots \to L^n_i \to K/p^n{}_i \to TC/p^n{}_i(-;p) \to L^n_{i-1} \to \cdots.$$

Since the sheafification is an exact functor, we get the corresponding long exact sequence of *cdh*-sheaves

$$\cdots \to a_{cdh} L^n_i \to a_{cdh} K/p^n{}_i \to a_{cdh} TC/p^n{}_i(-;p) \to a_{cdh} L^n_{i-1} \to \cdots. \tag{4.2}$$

We similarly get a long exact sequence of *cdh*-sheaves

$$\begin{aligned} \cdots &\to a_{cdh}\pi_i\big(\widetilde{C}_j L^n\big) \to a_{cdh}\pi_i\big(\widetilde{C}_j K/p^n\big) \to a_{cdh}\pi_i\big(\widetilde{C}_j TC/p^n(-;p)\big) \\ &\to a_{cdh}\pi_{i-1}\big(\widetilde{C}_j L^n\big) \to \cdots. \end{aligned} \tag{4.3}$$

Since the smooth schemes have no non-zero negative K-theory, we have $a_{cdh} K/p^n{}_i = 0$ for $i < 0$ and hence there are isomorphisms

$$a_{cdh}TC/p^n{}_i(-;p) \xrightarrow{\cong} a_{cdh}L^n_{i-1} \quad \text{and}$$
$$a_{cdh}\pi_i\big(\widetilde{C}_jTC/p^n(-;p)\big) \xrightarrow{\cong} a_{cdh}\pi_{i-1}\big(\widetilde{C}_jL^n\big) \quad \text{for } i<0. \tag{4.4}$$

Thus it suffices to show that the left terms of both the isomorphisms vanish for $i<-1$. For this, it suffices to show that $TC_i(A;p,\mathbb{Z}/p^n)=0$ for $i<-1$ for any commutative Noetherian k-algebra A. One knows from a result of Hesselholt (*cf.* [10], see also [5, Section 3]) that $TC_i(A;p)=0$ for $i<-1$ and the same conclusion then holds with finite coefficients by the exact sequence

$$TC_i(A;p) \to TC_i\big(A;p,\mathbb{Z}/p^n\big) \to TC_{i-1}(A;p). \qquad \square$$

Lemma 4.2. *Let X be a k-variety of dimension d. Then there are natural isomorphisms*

$$H^d\big(X_{\mathcal{C}}, a_{\mathcal{C}}\pi_{-1}\big(TC/p^n(-;p)\big)\big) \xrightarrow{\cong} \mathbb{H}^{d+1}_{\mathcal{C}}\big(X, TC/p^n(-;p)\big), \tag{4.5}$$
$$H^d\big(X_{\mathcal{C}}, a_{\mathcal{C}}\pi_{-1}\big(\widetilde{C}_jTC/p^n(-;p)\big)\big) \xrightarrow{\cong} \mathbb{H}^{d+1}_{\mathcal{C}}\big(X, \widetilde{C}_jTC/p^n(-;p)\big) \quad \textit{and}$$
$$\mathbb{H}^i_{\mathcal{C}}\big(X, TC/p^n(-;p)\big) = 0 = \mathbb{H}^i_{\mathcal{C}}\big(X, \widetilde{C}_jTC/p^n(-;p)\big) \tag{4.6}$$

for $i \geqslant d+2$, where $\mathcal{C}$ is any of Zariski, Nisnevich and cdh sites. The above results also hold for the presheaf of spectra $TC(-;p)$.

Proof. Since the Zariski, Nisnevich, or the *cdh*-cohomological dimension of X is bounded by d, one has a strongly convergent spectral sequence

$$E_2^{s,t} = H^s\big(X_{\mathcal{C}}, a_{\mathcal{C}}\pi_t\big(TC/p^n(-;p)\big)\big) \quad \Rightarrow \quad \mathbb{H}^{s-t}_{\mathcal{C}}\big(X, TC/p^n(-;p)\big) \tag{4.7}$$

and the similar spectral sequence holds for the homotopy groups of the $\widetilde{C}_j$ functors.

Since $TC_i(A;p)=0$ for any ring A and for any $i<-1$ as mentioned above, we conclude that for $s<d$, one has $-d-1+s<-1$ and hence

$$H^s\big(X_{\mathcal{C}}, a_{\mathcal{C}}\pi_{-d-1+s}\big(TC/p^n(-;p)\big)\big) = 0 = H^s\big(X_{\mathcal{C}}, a_{\mathcal{C}}\pi_{-d-1+s}\big(\widetilde{C}_jTC/p^n(-;p)\big)\big)$$

and also $E_2^{d-2,-2}=0$. Hence the above spectral sequence degenerates enough to give the desired isomorphisms in (4.5). The spectral sequence (4.7) and the vanishing of $TC_{\leqslant -2}(A;p)$ also prove (4.6) at once. The same proof works for $TC(-;p)$ as well. $\square$

Lemma 4.3. *Let X be as in Lemma* 4.2*. Then the natural maps*

$$H^d\big(X_{cdh}, a_{cdh}\pi_{-2}\big(L^n\big)\big) \to \mathbb{H}^{d+2}_{cdh}\big(X, L^n\big),$$
$$H^d\big(X_{cdh}, a_{cdh}\pi_{-2}\big(\widetilde{C}_jL^n\big)\big) \to \mathbb{H}^{d+2}_{cdh}\big(X, \widetilde{C}_jL^n\big)$$

are isomorphisms.

Proof. Since the smooth schemes have no negative K-theory, we have $a_{cdh}K/p^n{}_i=0$ for $i<0$. We have also seen above that $a_{cdh}TC/p^n{}_i(-;p)=0$ for $i<-1$, and the same vanishing holds for the homotopy groups of the $\widetilde{C}_j$-functors. We conclude from the exact sequences (4.2) and (4.3) that $a_{cdh}L^n_i = 0 = a_{cdh}\pi_i(\widetilde{C}_jL^n)$ for $i<-2$. The spectral sequence (4.7) now implies the lemma. $\square$

5. Vanishing and homotopy invariance for K/p^n

In this section, we prove the vanishing results for some negative homotopy groups of K/p^n and $\widetilde{C}_j K/p^n$ using the results of the previous section. Recall from Section 2 that $TC^\bullet = \{TC^n\}$ is an inverse system of presheaves of spectra on Sch/k. For a k-variety X, let $W\mathcal{O}_X$ denote the sheaf of Witt vectors on X (*cf.* [13]). The sheaf $W\mathcal{O}$ is a sheaf of rings on the category Sch/k.

Lemma 5.1. *Let $X = \mathrm{Spec}(A)$ be a scheme of dimension d, where A is a commutative Noetherian k-algebra. Then one has*

$$H^d_{cont}\big(X_{\mathcal{C}}, a_{\mathcal{C}} TC^\bullet_{-1}(-;p)\big) = 0 = H^d_{cont}\big(X_{\mathcal{C}}, a_{\mathcal{C}}\pi_{-1}\big(\widetilde{C}_j TC^\bullet_{-1}(-;p)\big)\big)$$

whenever $\mathcal{C}$ is any of the Zariski, Nisnevich and the cdh sites, where the terms are the continuous cohomology in the sense of Jannsen [14].

Proof. The continuous cohomology are related to the ordinary cohomology via the exact sequence

$$\begin{aligned} 0 \to \varprojlim{}^1 H^{s-1}\big(X_{\mathcal{C}}, a_{\mathcal{C}} TC^n_t(-;p)\big) &\to H^s_{cont}\big(X_{\mathcal{C}}, a_{\mathcal{C}} TC^\bullet_t(-;p)\big) \\ &\to \varprojlim H^s\big(X_{\mathcal{C}}, a_{\mathcal{C}} TC^n_t(-;p)\big) \to 0. \end{aligned} \tag{5.1}$$

Thus it suffices to prove that the inverse system $\{H^d(X_{\mathcal{C}}, a_{\mathcal{C}} TC^n_{-1}(-;p))\}$ is zero and $\{H^{d-1}(X_{\mathcal{C}}, a_{\mathcal{C}} TC^n_{-1}(-;p))\}$ satisfies the Mittag–Leffler condition, and same for the cohomology of the sheaves $a_{\mathcal{C}}\pi_{-1}(\widetilde{C}_j TC^n(-;p))$.

The exact sequences of sheaves on Sch/k

$$0 \to \mathcal{O} \to W_n\mathcal{O} \to W_{n-1}\mathcal{O} \to 0, \tag{5.2}$$

$$W_n\mathcal{O} \xrightarrow{1-F} W_n\mathcal{O} \to TC^n_{-1}(-;p) \to 0 \tag{5.3}$$

give us an exact sequence of sheaves

$$a_{\mathcal{C}} TC^1_{-1}(-;p) \to a_{\mathcal{C}} TC^n_{-1}(-;p) \to a_{\mathcal{C}} TC^{n-1}_{-1}(-;p) \to 0. \tag{5.4}$$

We claim that $H^d(X_{\mathcal{C}}, a_{\mathcal{C}} TC^n_{-1}(-;p)) = 0$ for all $n \geqslant 1$. It suffices to show using (5.3) and the right exactness of H^d that $H^d(X_{\mathcal{C}}, a_{\mathcal{C}} W_n\mathcal{O}_X) = 0$ for all $n \geqslant 1$. For $n = 1$, this is proved in [4, Theorem 6.1]. We remark here that Theorem 6.1 of [4] is proved when the base field is of characteristic zero. However exactly the same argument works even if k is of positive characteristic as long as the resolution of singularities holds over k, which we have assumed throughout. Only extra ingredients needed are the results of [20] and the formal function theorem which are characteristic free. We refer to [20] for details of the proof. For $n \geqslant 2$, the claim follows from the exact sequence (5.2), the right exactness of H^d and an induction on n.

Now the exact sequence (5.4) gives the cohomology exact sequence

$$H^{d-1}\big(X_{\mathcal{C}}, a_{\mathcal{C}} TC^n_{-1}(-;p)\big) \to H^{d-1}\big(X_{\mathcal{C}}, a_{\mathcal{C}} TC^{n-1}_{-1}(-;p)\big) \to F \to 0,$$

where the last term F is a quotient of $H^d(X_{\mathcal{C}}, a_{\mathcal{C}} TC^1_{-1}(-;p))$ and hence is zero by the above claim. We conclude that the inverse system $\{H^d(X_{\mathcal{C}}, a_{\mathcal{C}} TC^n_{-1}(-;p))\}$ is zero and $\{H^{d-1}(X_{\mathcal{C}}, a_{\mathcal{C}} TC^n_{-1}(-;p))\}$ satisfies the Mittag–Leffler condition.

To prove the above conclusion for the cohomology of the sheaves $a_{\mathcal{C}}\pi_{-1}(\widetilde{C}_j TC^n(-;p))$, we use exactly the same steps as above to reduce to showing that $H^d(X_{\mathcal{C}}, a_{\mathcal{C}}\pi_{-1}(\widetilde{C}_j TC^n(-;p))) = 0$ for all

$n \geqslant 1$. For $n = 1$, we use (5.3) to see that $a_{\mathcal{C}}\pi_{-1}(\widetilde{C}_j TC^1(-;p))$ is of the form $a_{\mathcal{C}} TC^1_{-1}(-;p) \otimes_{\mathbb{F}_p} V$, for some $\mathbb{F}_p$-vector space V, and hence

$$H^d\big(X_{\mathcal{C}}, a_{\mathcal{C}}\pi_{-1}\big(\widetilde{C}_j TC^1(-;p)\big)\big) = H^d\big(X_{\mathcal{C}}, a_{\mathcal{C}} TC^1_{-1}(-;p)\big) \otimes_{\mathbb{F}_p} V = 0.$$

The general case now follows from induction as above. □

Lemma 5.2. *Let X be as in Lemma* 5.1. *Then one has*

$$H^d\big(X_{cdh}, a_{cdh}\pi_{-1}\big(TC/p^n(-;p)\big)\big) = 0 = H^d\big(X_{cdh}, a_{cdh}\pi_{-1}\big(\widetilde{C}_j TC/p^n(-;p)\big)\big).$$

Proof. As the fibrant replacement preserves the homotopy fibration, we have an exact sequence

$$\mathbb{H}^{d+1}_{cdh}\big(X, TC(-;p)\big) \to \mathbb{H}^{d+1}_{cdh}\big(X, TC/p^n(-;p)\big) \to \mathbb{H}^{d+2}_{cdh}\big(X, TC(-;p)\big).$$

Combining this with Lemma 4.2, we are reduced to proving that

$$\mathbb{H}^{d+1}_{cdh}\big(X, TC(-;p)\big) = 0 = \mathbb{H}^{d+1}_{cdh}\big(X, \widetilde{C}_j TC(-;p)\big). \tag{5.5}$$

Recall from Section 2 that $TC(-;p) = \operatorname{holim} TC^n(-;p)$. Now the Milnor exact sequence

$$0 \to \varprojlim{}^1 TC^n_{i+1}(-;p) \to TC_i(-;p) \to \varprojlim TC^n_i(-;p) \to 0$$

and the corresponding exact sequence for the inverse system $\mathbb{H}_{cdh}(-;TC^n(-;p))$ imply that the map $TC(-;p) \to \operatorname{holim} \mathbb{H}_{cdh}(-;TC^n(-;p))$ is a weak equivalence in the *cdh* topology and hence the map

$$\mathbb{H}_{cdh}\big(-;TC(-;p)\big) \to \operatorname{holim} \mathbb{H}_{cdh}\big(-;TC^n(-;p)\big)$$

is a weak equivalence. Thus we get a convergent spectral sequence

$$E_2^{s,t} = H^s_{cont}\big(X_{cdh}, a_{cdh} TC^{\bullet}_t(-;p)\big) \quad \Rightarrow \quad \mathbb{H}^{s-t}_{cdh}\big(X; TC(-;p)\big)$$

by [5, Proposition 3.1.2]. Using this spectral sequence, the exact sequence (5.1), the vanishing of the sheaves $a_{\mathcal{C}} TC^n_i(-,p) = 0$ for $i < -1$ and [17, Theorem 12.5], we get the following exact sequence:

$$\begin{aligned} \varprojlim{}^1 H^d\big(X_{cdh}, a_{cdh} TC^n_0(-;p)\big) &\to \mathbb{H}^{d+1}_{cdh}\big(X; TC(-;p)\big) \\ &\to H^d_{cont}\big(X_{cdh}, a_{cdh} TC^n_{-1}(-;p)\big) \to 0. \end{aligned} \tag{5.6}$$

One gets the similar exact sequence representing $\mathbb{H}^{d+1}_{cdh}(X; \widetilde{C}_j TC(-;p))$. The right term of (5.6) vanishes by Lemma 5.1. To compute the left term of this exact sequence, we can first assume that X is reduced [17, Lemma 12.1]. In this case, we note from [5, Theorem 4.2.2] that

$$a_{cdh}\pi_0\big(\widetilde{C}_j TC^n_0(-;p)\big) = a_{cdh} TC^n_0(-;p) = \mathbb{Z}/p^n \oplus a_{cdh}\big(R^1\epsilon_*\big(\mathcal{O}^{\times}/p^n\big)\big), \tag{5.7}$$

where ϵ is the natural morphism from the étale to the Zariski site. Since the first term is finite and since the map $R^1\epsilon_*(\mathcal{O}^{\times}/p^n) \to R^1\epsilon_*(\mathcal{O}^{\times}/p^{n-1})$ is surjective [5, page 19], we see from [17, Theorem 12.5] that the inverse system $\{H^d(X_{cdh}, a_{cdh} TC^n_0(-;p))\}$ satisfies the Mittag–Leffler condition and hence the left term of (5.6) is zero. In particular, $\mathbb{H}^{d+1}_{cdh}(X; TC(-;p))$ and $\mathbb{H}^{d+1}_{cdh}(X; \widetilde{C}_j TC(-;p))$ are zero, proving (5.5). □

Proposition 5.3. *Let X be a k-variety of dimension d. Then the natural maps*

$$H^d\big(X_{Nis}, a_{Nis}\pi_{-1}\big(TC/p^n(-;p)\big)\big) \to H^d\big(X_{cdh}, a_{cdh}\pi_{-1}\big(TC/p^n(-;p)\big)\big),$$
$$H^d\big(X_{Nis}, a_{Nis}\pi_{-1}\big(\widetilde{C}_j TC/p^n(-;p)\big)\big) \to H^d\big(X_{cdh}, a_{cdh}\pi_{-1}\big(\widetilde{C}_j TC/p^n(-;p)\big)\big)$$

are surjective.

Proof. Let a^N denote the natural morphism from the *cdh* to the Nisnevich site. Then we have the Leray spectral sequence

$$H^p\big(X_{Nis}, R^q a^N_*\big(a_{cdh}\pi_{-1}\big(TC/p^n(-;p)\big)\big)\big) \quad \Rightarrow \quad H^{p+q}\big(X_{cdh}, a_{cdh}\pi_{-1}\big(TC/p^n(-;p)\big)\big).$$

Using this spectral sequence, it suffices to show that

$$H^p\big(X_{Nis}, R^q a^N_*\big(a_{cdh}\pi_{-1}\big(TC/p^n(-;p)\big)\big)\big) = 0 \quad \text{whenever } q > 0 \text{ and } p \geqslant d-q.$$

So fix $q > 0$ and let $\mathcal{F}^q$ denote the sheaf $R^q a^N_*(a_{cdh}\pi_{-1}(TC/p^n(-;p)))$. Then it suffices to show by [19] (see proof of Lemma E.6, page 429) that $(\mathcal{F}^q)_x$ is zero for any Nisnevich point x whose closure has codimension $\leqslant q$. Since the stalks of $\mathcal{F}^q$ is the *cdh* cohomology of the local rings, it suffices to show that $H^q(X_{cdh}, a_{cdh}\pi_{-1}(TC/p^n(-;p))) = 0$ for X as in Lemma 5.2 and for $q \geqslant \dim(X)$. But this follows directly from Lemma 5.2. The other conclusion of the proposition also follows the same way using Lemma 5.2 again. □

Theorem 5.4. *Let X be a k-variety of dimension d. Then $K_i(X, \mathbb{Z}/p^n) = 0 = \pi_i\widetilde{C}_j K/p^n(X)$ for $j \geqslant 0$ and $i < -d-1$.*

Proof. Using the vanishing of $TC_i(A; p, \mathbb{Z}/p^n) = 0$ for $i < -1$ for any A as in Lemma 5.1, the Zariski and Nisnevich descent for $TC/p^n(-;p)$ as seen in the proof of Corollary 3.2 and the spectral sequence (4.7), we conclude that

$$TC_i\big(X; p, \mathbb{Z}/p^n\big) = 0 = \pi_i\widetilde{C}_j TC/p^n(-;p)(X) \quad \text{for } i < -d-1. \tag{5.8}$$

The homotopy fibration sequence (2.2) gives the long exact sequence of homotopy groups

$$\cdots L^n_i(X) \to K_i\big(X, \mathbb{Z}/p^n\big) \to TC_i\big(X; p, \mathbb{Z}/p^n\big) \to L^n_{i-1}(X) \to \cdots$$

and one has a similar long exact sequence of the homotopy groups of the functors $\widetilde{C}_j$. Lemma 4.1 and (5.8) together now imply that

$$K_i\big(X, \mathbb{Z}/p^n\big) = 0 = \pi_i\widetilde{C}_j K/p^n(X) \quad \text{for } i < -d-2,$$

and there are exact sequences

$$TC_{-d-1}\big(X; p, \mathbb{Z}/p^n\big) \to L^n_{-d-2}(X) \to K_{-d-2}\big(X, \mathbb{Z}/p^n\big) \to 0,$$
$$\pi_{-d-1}\widetilde{C}_j TC/p^n(-;p)(X) \to \pi_{-d-2}\widetilde{C}_j L^n(X) \to \pi_{-d-2}\widetilde{C}_j K/p^n(X) \to 0.$$

Thus we need to show that the first map in both the exact sequences are surjective.

We consider the following commutative diagram.

$$\begin{tikzcd}
H^d(X_{Nis}, a_{Nis}\pi_{-1}(TC/p^n(-;p))) \arrow[r] \arrow[d] & \mathbb{H}^{d+1}_{Nis}(X, TC/p^n(-;p)) \arrow[d] & TC_{-d-1}(X; p, \mathbb{Z}/p^n) \arrow[l] \arrow[dd] \\
H^d(X_{cdh}, a_{cdh}\pi_{-1}(TC/p^n(-;p))) \arrow[r] \arrow[d] & \mathbb{H}^{d+1}_{cdh}(X, TC/p^n(-;p)) \arrow[d] & \\
H^d(X_{cdh}, a_{cdh}\pi_{-2}(L^n)) \arrow[r] & \mathbb{H}^{d+2}_{cdh}(X, L^n) & L^n_{-d-2}(X) \arrow[l]
\end{tikzcd}$$

The left horizontal arrows of all the rows are isomorphisms by Lemmas 4.2 and 4.3. The right horizontal arrow of the top row is an isomorphism by the Nisnevich descent of TC (*cf.* [5,1]). The right horizontal arrow in the bottom row is an isomorphism by Corollary 3.2. The lower vertical arrow on the left column is an isomorphism by (4.4). The upper vertical arrow on the left column is surjective by Proposition 5.3. A diagram chase shows that the long vertical arrow in the extreme right is surjective. The surjectivity of the map $\pi_{-d-1}\widetilde{C}_j TC/p^n(-;p)(X) \to \pi_{-d-2}\widetilde{C}_j L^n(X)$ follows exactly in the same way using Lemmas 4.2, 4.3, Proposition 5.3 and Corollary 3.2. □

6. Vanishing and homotopy invariance for rational K-theory

For any presheaf of spectra $\mathcal{E}$ on Sch/k, let $\mathcal{E}_{\mathbb{Q}}$ denote the direct colimit over the multiplication maps $\mathcal{E} \xrightarrow{n} \mathcal{E}$ by positive integers (*cf.* [19]). Then $\mathcal{E}_{\mathbb{Q}}$ is a presheaf of spectra on Sch/k such that $\pi_i(\mathcal{E}_{\mathbb{Q}}) \cong \pi_i(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$ for $i \in \mathbb{Z}$. Our goal now is to prove the vanishing of the rational K-theory and $K_{\mathbb{Q}}$-regularity in degrees below minus the dimension of a k-variety, where k is an infinite perfect field of positive characteristic as before. Let $\mathcal{K}_{\mathbb{Q}}$ and $\mathcal{HC}_{\mathbb{Q}}$ denote the presheaves of rational K-theory and rational cyclic homology spectra on Sch/k. Let $\widetilde{\mathcal{HN}}_{\mathbb{Q}}$, $\widetilde{\mathcal{HP}}_{\mathbb{Q}}$ and $\widetilde{\mathcal{HC}}_{\mathbb{Q}}$ denote the presheaves of spectra on Sch/k given by $U \mapsto HN(U \otimes \mathbb{Q})$, $U \mapsto HP(U \otimes \mathbb{Q})$ and $U \mapsto HC(U \otimes \mathbb{Q})$, where U is considered as a scheme over $\mathbb{Z}$, and HN, HP and HC respectively are the presheaves of negative cyclic homology, periodic cyclic homology and cyclic homology spectra on the category of schemes over $\mathbb{Z}$. There is a generalized Chern character map (*cf.* [15, Section 8.4])

$$\mathcal{K}_{\mathbb{Q}} \xrightarrow{ch} \widetilde{\mathcal{HN}}_{\mathbb{Q}}.$$

Let $\mathcal{K}^{\inf}_{\mathbb{Q}}$ denote the homotopy fiber of the above map of spectra.

Lemma 6.1. *The presheaf of spectra $\mathcal{K}_{\mathbb{Q}}$ on Sch/k satisfies cdh-descent.*

Proof. Since our schemes are defined over k which is of positive characteristic, we see that the presheaf of spectra $\widetilde{\mathcal{HN}}_{\mathbb{Q}}$ is contractible. Hence using the homotopy fibration sequence

$$\mathcal{K}^{\inf}_{\mathbb{Q}} \to \mathcal{K}_{\mathbb{Q}} \xrightarrow{ch} \widetilde{\mathcal{HN}}_{\mathbb{Q}},$$

it suffices to show that the presheaf of spectra $\mathcal{K}^{\inf}_{\mathbb{Q}}$ satisfies cdh-descent on Sch/k. To this end, it suffices to show that $\mathcal{K}^{\inf}_{\mathbb{Q}}$ satisfies all the conditions of Theorem 3.1. It satisfies excision by [3, Theorem 01] and it is invariant under infinitesimal extension by [8, Main Theorem]. $\mathcal{K}^{\inf}_{\mathbb{Q}}$ satisfies Nisnevich descent by [19, Theorem 10.8] and it satisfies the Mayer–Vietoris property for blow-up under regular closed embeddings by [20, Theorem 2.1]. This completes the proof of the lemma. □

Corollary 6.2. *Let X be a k-variety of dimension d. Then one has $K_i(X) \otimes_{\mathbb{Z}} \mathbb{Q} = 0 = \pi_i(\widetilde{C}_j \mathcal{K}_{\mathbb{Q}})(X)$ for $j \geqslant 0$ and $i < -d$.*

Proof. This follows directly from Lemma 6.1, the spectral sequence of Theorem 3.1 and from [17, Theorem 12.5] since $a_{cdh}\pi_i(\mathcal{K}_{\mathbb{Q}}) = 0$ for $i < 0$. □

Recall from [25] that $\mathcal{KH}$ is the sheaf of spectra of homotopy invariant K-theory on Sch/k. For any $X \in Sch/k$ and $i \in \mathbb{Z}$, $KH_i(X)$ denotes the ith homotopy group of the spectrum $\mathcal{KH}(X)$. We also recall (*cf.* Section 2) that for any spectrum E and any positive integer n, E/n is the smash product of E with a mod n Moore Spectrum Σ^{∞}/n. In particular, there is a fibration sequence [18, A.5]

$$E \xrightarrow{n} E \to E/n.$$

The corresponding long exact homotopy sequence yields for $i \in \mathbb{Z}$, the universal coefficient exact sequence

$$0 \to (\pi_i E)/n \to \pi_i(E/n) \to \operatorname{Tor}^1_{\mathbb{Z}}(\pi_{i-1}E, \mathbb{Z}/n) \to 0. \tag{6.1}$$

Corollary 6.3. *Let X be a k-variety of dimension d and let n be a positive integer prime to p. Then $K_i(X, \mathbb{Z}/n) = 0 = \widetilde{C}_j K_i(X, \mathbb{Z}/n)$ for $j \geqslant 0$ and $i < -d$.*

Proof. We first prove the corollary for the homotopy invariant K-theory. We have seen in Corollary 3.3 that $\mathcal{KH}$ satisfies the cdh-descent. Hence we can compute $KH_i(X)$ and $\widetilde{C}_j KH_i(X)$ using the spectral sequence of Theorem 3.1. Since the cdh-cohomological dimension of X is bounded by d [17, Theorem 12.5], it suffices to show that the sheaves $a_{cdh}(\pi_i(\mathcal{KH}))$ and $a_{cdh}(\pi_i(\widetilde{C}_j\mathcal{KH}))$ vanish for $i < 0$. Since the stalks of a sheaf in the cdh-topology are same as its stalks on the smooth schemes, it is now enough to show that a smooth scheme has no negative homotopy invariant K-theory. But this follows immediately from [25, Proposition 6.10] and the similar fact about the negative algebraic K-theory. This proves the statement of the corollary for the KH-theory. The statement about the KH-theory with finite coefficients now follows immediately from the exact sequence (6.1). To conclude the proof of the corollary, we just have to use the fact that for n prime to p, the natural map $K_i(X, \mathbb{Z}/n) \to KH_i(X, \mathbb{Z}/n)$ is an isomorphism by [25, Proposition 1.6], and the same conclusion holds also for $\widetilde{C}_j K_i(X)$. □

Proof of Theorem 1.3. The first part of Theorem 1.3 follows directly from Theorem 5.4 and Corollary 6.3. For any abelian group A, let ${}_nA$ denote the subgroup of n-torsion elements of A. It follows from Corollary 6.2 that $K_i(X)$ and $\widetilde{C}_j K_i(X)$ are torsion groups for $i < -d$. Thus we only need to show that these groups have no torsion whenever $i < -d-2$ and have only p-primary torsion for $i = -d-2$.

Theorem 5.4 and Corollary 6.3 together imply that $K_i(X, \mathbb{Z}/n) = 0 = \widetilde{C}_j K_i(X, \mathbb{Z}/n)$ for all $n \geqslant 1$ whenever $i < -d-1$. Now we use the exact sequence (6.1) for the algebraic K-theory to see that $K_i(X)$ and $\widetilde{C}_j K_i(X)$ are divisible groups for $i < -d-1$. Since the last term of (6.1) is same as ${}_nK_{i-1}(X)$, we also see that these groups are torsion-free for $i < -d-2$. Finally, the claim that $K_{-d-2}(X)$ and $\widetilde{C}_j K_{-d-2}(X)$ have only p-primary torsion, follows again from Corollary 6.3 and (6.1). This also shows that $K_i(X) = 0$ and X is K_i-regular for $i < -d-2$. □

Acknowledgment

The author would like to thank Chuck Weibel for giving some valuable suggestions for improving the earlier version of this paper.

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