Available online at www.sciencedirect.com

SciVerse ScienceDirect

ADVANCES IN Mathematics

Advances in Mathematics 231 (2012) 2858–2891

www.elsevier.com/locate/aim

Equivariant cobordism for torus actions

Amalendu Krishna

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai, India

Received 30 October 2010; accepted 19 July 2012
Available online 8 September 2012

Communicated by Bertrand Toen

Abstract

We study the equivariant cobordism groups for the action of a split torus T on varieties over a field k of characteristic zero. We show that for T acting on a variety X, there is an isomorphism $\Omega^T_*(X) \otimes_{\Omega^*(BT)} \mathbb{L} \xrightarrow{\cong} \Omega_*(X)$. As applications, we show that for a connected linear algebraic group G acting on a k-variety X, the forgetful map $\Omega^G_*(X) \to \Omega_*(X)$ is surjective with rational coefficients. As a consequence, we describe the rational algebraic cobordism ring of algebraic groups and flag varieties.

We prove a structure theorem for the equivariant cobordism of smooth projective varieties with torus action. Using this, we prove various localization theorems and a form of Bott residue formula for such varieties. As an application, we show that the equivariant cobordism of a smooth variety X with torus action is generated by the invariant cobordism cycles in $\Omega_*(X)$ as $\Omega^*(BT)$-module.

MSC: primary 14C25; secondary 19E15

Keywords: Algebraic cobordism; Group actions

1. Introduction

Let k be a field of characteristic zero and let G be a linear algebraic group over k. The equivariant algebraic cobordism groups for smooth varieties were defined by Deshpande in [8]. They were subsequently developed into a complete theory of equivariant cobordism for all k-schemes in [19]. This theory is based on the analogous construction of the equivariant Chow groups by Totaro [31] and Edidin–Graham [9]. In [19], we established all the basic properties of

E-mail address: amal@math.tifr.res.in.

0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.07.025

the equivariant cobordism which are known for the non-equivariant cobordism theory of Levine and Morel [25]. In this paper, we continue the study of these cobordism groups with special focus on the case when the underlying group is a torus.

It was shown in [19, Theorem 8.7] that with rational coefficients, the equivariant cobordism of a k-variety X with the action of a connected linear algebraic group G is simply the subgroup of invariants inside the equivariant cobordism for the action of a maximal torus of G under the action of the Weyl group. This reduces most of the computations of equivariant cobordism to the case when the underlying group is a torus. Our aim in this paper is to study this special case in more detail and derive some important consequences for the action of arbitrary connected groups. These results are applied in [20] to describe the non-equivariant cobordism rings of principal and flag bundles. The results of this paper are also used in [21] to compute the algebraic cobordism of toric varieties. Some more applications will appear in [17]. We now describe some of the main results of this paper.

In this paper, a scheme will mean a quasi-projective k-scheme and all G-actions will be assumed to be linear. Let T be a split torus and let $S(T)$ denote the cobordism ring $\Omega^*_T(k)$ of the classifying space of T. If X is a scheme with T-action, we show in Theorem 3.4 that the forgetful map from the equivariant cobordism to the ordinary cobordism group of X induces an isomorphism

$$r^T_X : \Omega^T_*(X) \otimes_{S(T)} \mathbb{L} \xrightarrow{\cong} \Omega_*(X),$$

which is a ring isomorphism if X is smooth. This result for the equivariant Chow groups was earlier proven by Brion in [4, Corollary 2.3].

Using Theorem 3.4 and the results of Calmès et al. [6], we give an explicit geometric description of the algebraic cobordism ring of flag varieties with rational coefficients. In particular, we show in Theorem 8.1 that if G is a connected reductive group and B is a Borel subgroup containing a split maximal torus T, then with rational coefficients, there is an $\mathbb{L}$-algebra isomorphism

$$S(T) \otimes_{S(G)} \mathbb{L} \xrightarrow{\cong} \Omega^*(G/B), \tag{1.1}$$

where $S(G) = \Omega^*(BG)$. This can be interpreted as the uncompleted version of the results of [6]. In case of $G = GL_n$, this yields an explicit formula for the ring $\Omega^*(G/B)$ as the quotient of the standard polynomial ring $\mathbb{L}[x_1, \dots, x_n]$ by the ideal generated by the homogeneous symmetric polynomials of strictly positive degree. This latter result for GL_n recovers the theorem of Hornbostel and Kiritchenko [16] by a simpler method. We remark that the result of Hornbostel and Kiritchenko is stronger in the sense that they prove it with the integral coefficients. We also obtain similar description for $\Omega^*(G)$ that generalizes the results of Yagita [34].

As an application of Theorems 3.4 and 8.1, we show that if G is a connected linear algebraic group acting on a scheme X, then the forgetful map

$$r^G_X : \Omega^G_*(X) \to \Omega_*(X)$$

is surjective with rational coefficients. This generalizes the analogous results for the K-groups and Chow groups by Graham [13] and Brion [4] to algebraic cobordism.

In our next result, we give a structure theorem (*cf.* Theorem 4.7) for the equivariant cobordism of smooth projective varieties with torus action. The main point of this theorem is that for such a variety X, the equivariant cobordism of X is very closely related to the non-equivariant cobordism of the fixed point loci in X. The main ingredients in the proof are the self-intersection formula for the equivariant cobordism in Proposition 3.1 and a decomposition theorem of Bialynicki-Birula for such varieties.

As an application of Theorem 4.7, we show that if a split torus T acts on a smooth variety X, then the equivariant cobordism group $\Omega_*^T(X)$ is generated by the T-invariant cobordism cycles in $\Omega_*(X)$ as $S(T)$-module. For equivariant Chow groups, this was earlier proven by Brion in [4, Theorem 2.1]. The result of Brion can also be deduced from corresponding result for cobordism and [19, Proposition 7.2].

In [4], Brion proves the localization formulae for the equivariant Chow groups for torus action on smooth projective varieties. These formulae describe the equivariant Chow groups of a smooth projective variety X with T-action in terms of the equivariant Chow groups of the fixed point locus X^T. Since $\Omega_*^T(X^T)$ is relatively simpler to compute, these formulae yield a way of computing the equivariant Chow groups of X. In the final set of results in this paper, we prove these localization formulae for the equivariant cobordism. Our results generalize all the analogous results of Brion to the case of cobordism. These results are expected to be the foundational step in the computation of the cobordism groups of spherical varieties. Using the above result, we compute the cobordism ring of certain spherical varieties in [17].

2. Recollection of equivariant cobordism

Since we shall be concerned with the study of schemes with group actions and the associated quotient schemes, and since such quotients often require the original scheme to be quasi-projective, we shall assume throughout this paper that all schemes over k are quasi-projective.

In this section, we briefly recall the definition of equivariant algebraic cobordism and some of its main properties from [19]. Since most of the results of [19] will be repeatedly used in this text, we summarize them here for reader's convenience. For the definition and all details about the algebraic cobordism used in this paper, we refer the reader to the work of Levine and Morel [25].

Notations. We shall denote the category of quasi-projective k-schemes by $\mathcal{V}_k$. By a scheme, we shall mean an object of $\mathcal{V}_k$. The category of smooth quasi-projective schemes will be denoted by $\mathcal{V}_k^S$. If G is a linear algebraic group over k, we shall denote the category of quasi-projective k-schemes with a G-action and G-equivariant maps by $\mathcal{V}_G$. The associated category of smooth G-schemes will be denoted by $\mathcal{V}_G^S$. All G-actions in this paper will be assumed to be linear. Recall that this means that all G-schemes are assumed to admit G-equivariant ample line bundles. This assumption is always satisfied for normal schemes (*cf.* [29, Theorem 2.5], [30, 5.7]).

Recall that the Lazard ring $\mathbb{L}$ is a polynomial ring over $\mathbb{Z}$ on infinite but countably many variables and is given by the quotient of the polynomial ring $\mathbb{Z}[A_{ij} | (i, j) \in \mathbb{N}^2]$ by the relations, which uniquely define the universal formal group law $F_{\mathbb{L}}$ of rank one on $\mathbb{L}$. Recall that a cobordism cycle over a k-scheme X is a family $\alpha = [Y \xrightarrow{f} X, L_1, \ldots, L_r]$, where Y is a smooth scheme, the map f is projective, and L_i's are line bundles on Y. Here, one allows the set of line bundles to be empty. The degree of such a cobordism cycle is defined to be $\deg(\alpha) = \dim_k(Y) - r$ and its codimension is defined to be $\dim(X) - \deg(\alpha)$. If $\mathcal{Z}_*(X)$ is the free abelian group generated by the cobordism cycles of the above type with Y irreducible, then $\mathcal{Z}_*(X)$ is graded by the degree of cycles. The algebraic cobordism group of X is defined as

$$\Omega_*(X) = \frac{\mathcal{Z}_*(X)}{\mathcal{R}_*(X)},$$

where $\mathcal{R}_*(X)$ is the graded subgroup generated by relations which are determined by the dimension and the section axioms and the above formal group law. If X is equi-dimensional, we set $\Omega^i(X) = \Omega_{\dim(X)-i}(X)$ and grade $\Omega^*(X)$ by codimension of the cobordism cycles. It was shown by Levine and Pandharipande [26] that the cobordism group $\Omega_*(X)$ can also be defined as the quotient

$$\Omega_*(X) = \frac{\mathcal{Z}'_*(X)}{\mathcal{R}'_*(X)},$$

where $\mathcal{Z}'_*(X)$ is the free abelian group on cobordism cycles $[Y \xrightarrow{f} X]$ with Y smooth and irreducible and f projective. The graded subgroup $\mathcal{R}'_*(X)$ is generated by cycles satisfying the relation of *double point degeneration*.

Let X be a k-scheme of dimension d. For $j \in \mathbb{Z}$, let Z_j be the set of all closed subschemes $Z \subset X$ such that $\dim_k(Z) \le j$ (we assume $\dim(\emptyset) = -\infty$). The set Z_j is then ordered by the inclusion. For $i \ge 0$, we set

$$\Omega_i(Z_j) = \varinjlim_{Z \in Z_j} \Omega_i(Z) \quad \text{and} \quad \Omega_*(Z_j) = \bigoplus_{i \ge 0} \Omega_i(Z_j).$$

It is immediate that $\Omega_*(Z_j)$ is a graded $\mathbb{L}_*$-module and there is a graded $\mathbb{L}_*$-linear map $\Omega_*(Z_j) \to \Omega_*(X)$. We define $F_j\Omega_*(X)$ to be the image of the natural $\mathbb{L}_*$-linear map $\Omega_*(Z_j) \to \Omega_*(X)$. In other words, $F_j\Omega_*(X)$ is the image of all $\Omega_*(W) \to \Omega_*(X)$, where $W \to X$ is a projective map such that $\dim(\mathrm{Image}(W)) \le j$. One checks at once that there is a canonical *niveau filtration*

$$0 = F_{-1}\Omega_*(X) \subseteq F_0\Omega_*(X) \subseteq \cdots \subseteq F_{d-1}\Omega_*(X) \subseteq F_d\Omega_*(X) = \Omega_*(X). \tag{2.1}$$

2.1. Equivariant cobordism

In this text, G will denote a linear algebraic group of dimension g over k. All representations of G will be finite dimensional. Recall form [19] that for any integer $j \ge 0$, a *good pair* (V_j, U_j) corresponding to j for the G-action is a pair consisting of a G-representation V_j and an open subset $U_j \subset V_j$ such that the codimension of the complement is at least j and G acts freely on U_j with quotient U_j/G a quasi-projective scheme. It is known that such good pairs always exist.

Let X be a k-scheme of dimension d with a G-action. For $j \ge 0$, let (V_j, U_j) be an l-dimensional good pair corresponding to j. For $i \in \mathbb{Z}$, if we set

$$\Omega_i^G(X)_j = \frac{\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)}{F_{d+l-g-j}\Omega_{i+l-g}\left(X \stackrel{G}{\times} U_j\right)}, \tag{2.2}$$

then it is known that $\Omega_i^G(X)_j$ is independent of the choice of the good pair (V_j, U_j). Moreover, there is a natural surjective map $\Omega_i^G(X)_{j'} \twoheadrightarrow \Omega_i^G(X)_j$ for $j' \ge j \ge 0$.

Definition 2.1. Let X be a k-scheme of dimension d with a G-action. For any $i \in \mathbb{Z}$, we define the *equivariant algebraic cobordism* of X to be

$$\Omega_i^G(X) = \varprojlim_j \Omega_i^G(X)_j.$$

The reader should note from the above definition that unlike the ordinary cobordism, the equivariant algebraic cobordism $\Omega_i^G(X)$ can be non-zero for any $i \in \mathbb{Z}$. We set

$$\Omega_*^G(X) = \bigoplus_{i \in \mathbb{Z}} \Omega_i^G(X).$$

If X is an equi-dimensional k-scheme with G-action, we let $\Omega_G^i(X) = \Omega_{d-i}^G(X)$ and $\Omega_G^*(X) = \oplus_{i \in \mathbb{Z}} \Omega_G^i(X)$. It is known that if G is trivial, then the G-equivariant cobordism reduces to the ordinary one.

Remark 2.2. It is easy to check from the above definition of the niveau filtration that if X is a smooth and irreducible k-scheme of dimension d, then $F_j\Omega_i(X) = F^{d-j}\Omega^{d-i}(X)$, where $F^\bullet\Omega^*(X)$ is the coniveau filtration used in [8]. Furthermore, one also checks in this case that if G acts on X, then

$$\Omega_G^i(X) = \varprojlim_j \frac{\Omega^i\left(X \overset{G}{\times} U_j\right)}{F^j\Omega^i\left(X \overset{G}{\times} U_j\right)}, \tag{2.3}$$

where (V_j, U_j) is a good pair corresponding to any $j \geq 0$. Thus the above definition of the equivariant cobordism coincides with that of [8] for smooth schemes.

For equi-dimensional schemes, we shall write the (equivariant) cobordism groups cohomologically. The G-equivariant cobordism group $\Omega^*(k)$ of the ground field k is denoted by $\Omega^*(BG)$ and is called the cobordism of the *classifying space* of G. We shall often write it as $S(G)$.

The following important result shows that if we suitably choose a sequence of good pairs $\{(V_j, U_j)\}_{j \geq 0}$, then the above equivariant cobordism group can be computed without taking quotients by the niveau filtration. This is often very helpful in computing the equivariant cobordism groups.

Theorem 2.3 (*Cf.* [19, *Theorem 6.1*]). *Let $\{(V_j, U_j)\}_{j \geq 0}$ be a sequence of l_j-dimensional good pairs such that*

(i) $V_{j+1} = V_j \oplus W_j$ *as representations of* G *with* $\dim(W_j) > 0$ *and*
(ii) $U_j \oplus W_j \subset U_{j+1}$ *as G-invariant open subsets.*

Then for any scheme X as above and any $i \in \mathbb{Z}$,

$$\Omega_i^G(X) \xrightarrow{\cong} \varprojlim_j \Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right).$$

Moreover, such a sequence $\{(V_j, U_j)\}_{j \geq 0}$ of good pairs always exists.

2.2. Change of groups

If $H \subset G$ is a closed subgroup of dimension h, then any l_j-dimensional good pair (V_j, U_j) for G-action is also a good pair for the induced H-action. Moreover, for any $X \in \mathcal{V}_G$ of dimension d, $X \overset{H}{\times} U_j \to X \overset{G}{\times} U_j$ is an étale locally trivial G/H-fibration and hence a smooth map (*cf.* [2, Theorem 6.8]) of relative dimension $g - h$. Taking the inverse limit of corresponding pull-back maps on the cobordism groups, this induces the restriction map

$$r^G_{H,X} : \Omega^G_*(X) \to \Omega^H_*(X). \tag{2.4}$$

Taking $H = \{1\}$, we get the *forgetful* map

$$r^G_X : \Omega^G_*(X) \to \Omega_*(X) \tag{2.5}$$

from the equivariant to the non-equivariant cobordism. Since $r^G_{H,X}$ is obtained as a pull-back under the smooth map, it commutes with any projective push-forward and smooth pull-back (*cf.* Theorem 2.5).

The equivariant cobordism for the action of a group G is related with the equivariant cobordism for the action of the various subgroups of G by the following.

Proposition 2.4 (*Morita Isomorphism*). *Let $H \subset G$ be a closed subgroup and let $X \in \mathcal{V}_H$. Then there is a canonical isomorphism*

$$\Omega^G_*\left(G \overset{H}{\times} X\right) \xrightarrow{\cong} \Omega^H_*(X). \tag{2.6}$$

2.3. Fundamental class of cobordism cycles

Let $X \in \mathcal{V}_G$ and let $Y \xrightarrow{f} X$ be a morphism in $\mathcal{V}_G$ such that Y is smooth of dimension d and f is projective. For any $j \geq 0$ and any l-dimensional good pair (V_j, U_j), $[Y_G \xrightarrow{f_G} X_G]$ is an ordinary cobordism cycle of dimension $d + l - g$ by [19, Lemma 5.1] and hence defines an element $\alpha_j \in \Omega^G_d(X)_j$. Moreover, it is evident that the image of $\alpha_{j'}$ is α_j for $j' \geq j$. Hence we get a unique element $\alpha \in \Omega^G_d(X)$, called the *G-equivariant fundamental class* of the cobordism cycle $[Y \xrightarrow{f} X]$. We also see from this more generally that if $[Y \xrightarrow{f} X, L_1, \ldots, L_r]$ is as above with each L_i a G-equivariant line bundle on Y, then this defines a unique class in $\Omega^G_{d-r}(X)$. It is interesting question to ask under what conditions on the group G, the equivariant cobordism group $\Omega^G_*(X)$ is generated by the fundamental classes of G-equivariant cobordism cycles on X. We shall show in this text that this is indeed true for a torus action on smooth varieties.

2.4. Basic properties

The following result summarizes the basic properties of the equivariant cobordism.

Theorem 2.5 (*Cf. [19, Theorem 5.2],[14, Theorem 20]*). *The equivariant algebraic cobordism satisfies the following properties.*

(i) *Functoriality: The assignment $X \mapsto \Omega_*(X)$ is covariant for projective maps and contravariant for smooth maps in $\mathcal{V}_G$. It is also contravariant for l.c.i. morphisms in $\mathcal{V}_G$. Moreover, for a fibre diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

in $\mathcal{V}_G$ with f projective and g smooth, one has $g^* \circ f_* = {f'}_* \circ {g'}^* : \Omega^G_*(X) \to \Omega^G_*(Y')$.

(ii) *Localization: For a G-scheme X and a closed G-invariant subscheme $Z \subset X$ with complement U, there is an exact sequence*

$$\Omega_*^G(Z) \to \Omega_*^G(X) \to \Omega_*^G(U) \to 0.$$

(iii) *Homotopy: If $f : E \to X$ is a G-equivariant vector bundle, then $f^* : \Omega_*^G(X) \xrightarrow{\cong} \Omega_*^G(E)$.*

(iv) *Chern classes: For any G-equivariant vector bundle $E \xrightarrow{f} X$ of rank r, there are equivariant Chern class operators $c_m^G(E) : \Omega_*^G(X) \to \Omega_{*-m}^G(X)$ for $0 \le m \le r$ with $c_0^G(E) = 1$. These Chern classes have same functoriality properties as in the non-equivariant case. Moreover, they satisfy the Whitney sum formula.*

(v) *Free action: If G acts freely on X with quotient Y, then $\Omega_*^G(X) \xrightarrow{\cong} \Omega_*(Y)$.*

(vi) *Exterior product: There is a natural product map*

$$\Omega_i^G(X) \otimes_{\mathbb{Z}} \Omega_{i'}^G(X') \to \Omega_{i+i'}^G(X \times X').$$

In particular, $\Omega_*^G(k)$ is a graded algebra and $\Omega_*^G(X)$ is a graded $\Omega_*^G(k)$-module for every $X \in \mathcal{V}_G$. For X smooth, the pull-back via the diagonal $X \hookrightarrow X \times X$ turns $\Omega_G^*(X)$ into an $S(G)$-algebra.

(vii) *Projection formula: For a projective map $f : X' \to X$ in $\mathcal{V}_G^S$, one has for $x \in \Omega_*^G(X)$ and $x' \in \Omega_*^G(X')$, the formula: $f_*\left(x' \cdot f^*(x)\right) = f_*(x') \cdot x$.*

2.5. Formal group law

We recall from [19] that the first Chern class of the tensor product of two equivariant line bundles satisfies the formal group law of the ordinary cobordism. That is, for $L_1, L_2 \in \mathrm{Pic}^G(X)$, one has

$$c_1^G(L_1 \otimes L_2) = c_1^G(L_1) + c_1^G(L_2) + c_1^G(L_1)c_1^G(L_2) \sum_{i,j \ge 1} a_{i,j} \left(c_1^G(L_1)\right)^{i-1} \left(c_1^G(L_2)\right)^{j-1}, \tag{2.7}$$

where $F(u, v) = u + v + uv \sum_{i,j\ge1} a_{i,j} u^{i-1} v^{j-1}$, $a_{i,j} \in \mathbb{L}_{1-i-j}$ is the universal formal group law on $\mathbb{L}$. We shall often write $c_1^G(L_1 \otimes L_2) = c_1^G(L_1) +_F c_1^G(L_2)$.

If X is smooth, the commutative sub-$\mathbb{L}$-algebra (under composition) of $\mathrm{End}_{\mathbb{L}}\left(\Omega_G^*(X)\right)$ generated by the Chern classes of the vector bundles is canonically identified with a sub-$\mathbb{L}$-algebra of the cobordism ring $\Omega_G^*(X)$ via the identification $c_i^G(E) \mapsto \widetilde{c_i^G(E)} = c_i^G(E)\left([X \xrightarrow{id} X]\right)$. We shall denote this image also by $c_i^G(E)$ or, by c_i^G if the underlying vector bundle is understood. The formal group law of the algebraic cobordism then gives a map of pointed sets

$$\mathrm{Pic}^G(X) \to \Omega_G^1(X), \qquad L \mapsto c_1^G(L) \tag{2.8}$$

such that $c_1^G(L_1 \otimes L_1) = c_1^G(L_1) +_F c_1^G(L_2)$. It is known that even though $c_1^G(L)$ is not nilpotent in $\Omega_G^*(X)$ for $L \in \mathrm{Pic}^G(X)$ (unlike in the ordinary case), $c_1^G(L_1) +_F c_1^G(L_2)$ is a well defined element of $\Omega_G^1(X)$. In this paper, we shall view the (equivariant) Chern classes as elements of the (equivariant) cobordism ring of a smooth variety in the above sense. In the rest of this paper, the sum $x +_F y$ for $x, y \in \Omega_G^1(X)$ will denote the element $F(x, y) \in \Omega_G^1(X)$, the addition according to the formal group law.

2.6. Cobordism ring of classifying spaces

Let R be a Noetherian ring and let $A = \oplus_{j\in\mathbb{Z}} A_j$ be a $\mathbb{Z}$-graded R-algebra with $R \subset A_0$. Recall that the *graded power series ring* $S^{(n)} = \oplus_{i\in\mathbb{Z}} S_i$ is a graded ring such that S_i is the set of formal power series of the form $f(\mathbf{t}) = \sum_{m(\mathbf{t})\in\mathcal{C}} a_{m(\mathbf{t})} m(\mathbf{t})$ such that $a_{m(\mathbf{t})}$ is a homogeneous element of A of degree $|a_{m(\mathbf{t})}|$ and $|a_{m(\mathbf{t})}| + |m(\mathbf{t})| = i$. Here, $\mathcal{C}$ is the set of all monomials in $\mathbf{t} = (t_1, \dots, t_n)$ and $|m(\mathbf{t})| = i_1 + \dots + i_n$ if $m(\mathbf{t}) = t_1^{i_1} \cdots t_n^{i_n}$. We call $|m(\mathbf{t})|$ to be the degree of the monomial $m(\mathbf{t})$.

We shall often write the above graded power series ring as $A[[\mathbf{t}]]_{\mathrm{gr}}$ to distinguish it from the usual formal power series ring. Notice that if A is only non-negatively graded, then $S^{(n)}$ is nothing but the standard polynomial ring $A[t_1, \dots, t_n]$ over A. It is also easy to see that $S^{(n)}$ is indeed a graded ring which is a subring of the formal power series ring $A[[t_1, \dots, t_n]]$. The following result summarizes some basic properties of these rings. The proof is straightforward and is left as an exercise.

Lemma 2.6. (i) *There are inclusions of rings* $A[t_1, \dots, t_n] \subset S^{(n)} \subset A[[t_1, \dots, t_n]]$, *where the first is an inclusion of graded rings.*

(ii) *These inclusions are analytic isomorphisms with respect to the* $\mathbf{t}$*-adic topology. In particular, the induced maps of the associated graded rings*

$$A[t_1, \dots, t_n] \to \mathrm{Gr}_{(\mathbf{t})} S^n \to \mathrm{Gr}_{(\mathbf{t})} A[[t_1, \dots, t_n]]$$

are isomorphisms.

(iii) $S^{(n-1)}[[t_i]]_{\mathrm{gr}} \xrightarrow{\cong} S^{(n)}$.

(iv) $\frac{S^{(n)}}{(t_{i_1}, \dots, t_{i_r})} \xrightarrow{\cong} S^{(n-r)}$ *for any* $n \ge r \ge 1$, *where* $S^{(0)} = A$.

(v) *The sequence* $\{t_1, \dots, t_n\}$ *is a regular sequence in* $S^{(n)}$.

(vi) *If* $A = R[x_1, x_2, \dots]$ *is a polynomial ring with* $|x_i| < 0$ *and* $\lim_{i\to\infty} |x_i| = -\infty$, *then* $S^{(n)} \xrightarrow{\cong} \varprojlim_i R[x_1, \dots, x_i][[\mathbf{t}]]_{\mathrm{gr}}$.

Since we shall mostly be dealing with the graded power series ring in this text, we make the convention of writing $A[[\mathbf{t}]]_{\mathrm{gr}}$ as $A[[\mathbf{t}]]$, while the standard formal power series ring will be written as $\widehat{A[[\mathbf{t}]]}$.

It is known [19, Proposition 6.7] that if T is a split torus of rank n and if $\{\chi_1, \dots, \chi_n\}$ is a chosen basis of the character group $\widehat{T}$, then there is a canonical isomorphism of graded rings

$$\mathbb{L}[[t_1, \dots, t_n]] \xrightarrow{\cong} \Omega^*(BT), \qquad t_i \mapsto c_1^T(L_{\chi_i}). \tag{2.9}$$

Here, L_χ is the T-equivariant line bundle on $\mathrm{Spec}(k)$ corresponding to the character χ of T. One also has isomorphisms

$$\Omega^*(BGL_n) \xrightarrow{\cong} \mathbb{L}[[\gamma_1, \dots, \gamma_n]] \quad \text{and} \quad \Omega^*(BSL_n) \xrightarrow{\cong} \mathbb{L}[[\gamma_2, \dots, \gamma_n]] \tag{2.10}$$

of graded $\mathbb{L}$-algebras, where γ_i's are the elementary symmetric polynomials in $t_1, \dots, t_n$ that occur in $\Omega^*(BT)$.

We finally recall the following result of [19] that will be useful for us.

Theorem 2.7 (*Cf. [19, Theorem 8.6]*)**.** *Let G be a connected linear algebraic group and let L be a Levi subgroup of G with a split maximal torus T. Let W denote the Weyl group of L with* *respect to T. Then for any $X \in \mathcal{V}_G$, the natural map*

$$\Omega^G_*(X) \to \left(\Omega^T_*(X)\right)^W \tag{2.11}$$

is an isomorphism.

3. The forgetful map

In this section, we study the forgetful map

$$r^G_X : \Omega^G_*(X) \to \Omega_*(X)$$

of (2.5) from the equivariant to the non-equivariant cobordism when G is a split torus. It was shown by Brion in [4, Corollary 2.3] (see also [18, Corollary 1.4]) that the natural map $CH^G_*(X) \otimes_{S(G)} \mathbb{Z} \to CH_*(X)$ is an isomorphism. Our aim in this section is to prove an analogous result for the algebraic cobordism. We do this by using a technique which appears to be simpler than the one Brion used for studying the Chow groups. For the rest of this paper, we shall denote the cobordism ring $S(T) = \Omega^*_T(k)$ of the classifying space of a split torus T simply by S. Let $I_T \subset S$ be the augmentation ideal so that $S/I_T \xrightarrow{\cong} \mathbb{L}$. We first prove the following very useful self-intersection formula for the equivariant cobordism.

Proposition 3.1 (*Self-Intersection Formula*)**.** *Let G be a linear algebraic group and let $Y \xrightarrow{f} X$ be a regular G-equivariant embedding in $\mathcal{V}_G$ of pure codimension d and let $N_{Y/X}$ denote the equivariant normal bundle of Y inside X. Then one has for every $y \in \Omega^G_*(Y)$, $f^* \circ f_*(y) = c^G_d(N_{Y/X})(y)$.*

Proof. First we prove this for the non-equivariant algebraic cobordism. But this is a direct consequence of the construction of the refined pull-back map and the excess intersection formula for algebraic cobordism in [25, Section 6]. One simply has to use Lemma 6.6.2 and Theorem 6.6.9 of [25] and follow exactly the same argument as in the proof of the self-intersection formula for Chow groups in [12, Theorem 6.2].

To prove the equivariant version, let $j \geq 0$ and let (V_j, U_j) be a good pair. Then $Y_G \xhookrightarrow{\overline{f}} X_G$ is a regular closed embedding. Writing the normal bundle $N_{Y/X}$ simply by N, we see that N_G is the normal bundle of Y_G inside X_G and $c^G_d(N)$ is induced by $c_d(N_G)$. The non-equivariant self-intersection formula yields $\overline{f}^* \circ \overline{f}_* = c_d(N_G)(-)$ on $\Omega_*(Y_G)$. Moreover, the maps $\overline{f}^*$ and $\overline{f}_*$ descend to a compatible system of maps (*cf.* [19, Theorem 5.2])

$$\Omega^G_*(Y)_j \xrightarrow{\overline{f}_*} \Omega^G_*(X)_{j+d}, \qquad \Omega^G_*(X)_{j+d} \xrightarrow{\overline{f}^*} \Omega^G_{*-d}(Y)_{j+d},$$

whose composite with the natural surjection $\Omega^G_{*-d}(Y)_{j+d} \twoheadrightarrow \Omega^G_{*-d}(Y)_j$ (*cf.* [19, Lemma 4.3]) gives a map of inverse systems $\{\Omega^G_*(Y)_j\} \xrightarrow{\overline{f}^* \circ \overline{f}_*} \{\Omega^G_{*-d}(Y)_j\}$.

The equivariant Chern class $c^G_d(N)$ is induced by taking the inverse limit of maps $c^G_{d,j}(N_G) : \{\Omega^G_*(Y)_j\} \to \{\Omega^G_{*-d}(Y)_j\}$. Since $\overline{f}^* \circ \overline{f}_* = c^G_{d,j}(N_G)(-)$ for each $j \geq 0$ by the non-equivariant case proven above, we conclude that $f^* \circ f_* = c^G_d(N)(-)$ on the equivariant cobordism. □

Corollary 3.2. *Let G be a linear algebraic group acting on a k-variety X and let $p : L \to X$ be a G-equivariant line bundle. Let $f : X \to L$ be the zero-section embedding. Then $f^* \circ p^* = \mathrm{Id}$ and $f^* \circ f_* = c^G_1(L)$.*

Proof. Since p is an equivariant line bundle, the homotopy invariance property of Theorem 2.5 implies the first isomorphism. The second isomorphism is a direct consequence of Proposition 3.1. $\square$

Lemma 3.3. *Let G be a connected linear algebraic group acting on a k-variety X. Then all irreducible components of X are G-invariant and the map*

$$\bigoplus_{1 \le i \le r} \Omega_*^G(X_i) \to \Omega_*^G(X)$$

is surjective, where $\{X_1, \ldots, X_r\}$ is the set of irreducible components of X.

Proof. Since G is connected and hence irreducible, it is clear that all components of X are G-invariant. We now prove the surjectivity assertion. By an induction on the number of irreducible components, it suffices to consider the case when $X = X_1 \cup X_2$, where each X_i is a G-invariant closed (not necessarily irreducible) subscheme. Set $Y = X_1 \cap X_2$. It is then clear that Y is a G-invariant closed subscheme of X. By Theorem 2.5 (ii), we get a commutative diagram

$$\begin{tikzcd} \Omega_*^G(Y) \arrow[r] \arrow[d] & \Omega_*^G(X_1) \arrow[r] \arrow[d] & \Omega_*^G(X_1 - Y) \arrow[r] \arrow[d, equal] & 0 \\ \Omega_*^G(X_2) \arrow[r] & \Omega_*^G(X) \arrow[r] & \Omega_*^G(X - X_2) \arrow[r] & 0 \end{tikzcd} \tag{3.1}$$

with exact rows. It follows from the diagram chase that the map $\Omega_*^G(X_1) \oplus \Omega_*^G(X_2) \to \Omega_*^G(X)$ is surjective. $\square$

Theorem 3.4. *Let T be a split torus acting on a k-variety X. Then the forgetful map r_X^T induces an isomorphism*

$$\bar{r}_X^T : \Omega_*^T(X) \otimes_S \mathbb{L} \xrightarrow{\cong} \Omega_*(X).$$

If X is smooth, this is an $\mathbb{L}$-algebra isomorphism.

Proof. We have already seen that r_X^T is an $\mathbb{L}$-algebra homomorphism if X is smooth. So we only need to show that it descends to the desired isomorphism for any $X \in \mathcal{V}_T$.

Let n be the rank of T and let $\{\chi_1, \ldots, \chi_n\}$ be a chosen basis of the character group M of T. We denote the coordinates of the affine space $\mathbb{A}_k^n$ by $x = (x_1, \ldots, x_n)$ and define a linear action of T on $\mathbb{A}^n$ by

$$t \cdot x = y = (y_1, \ldots, y_n), \quad \text{where } y_i = \chi_i(t)x_i \ \forall i. \tag{3.2}$$

Let $H_i \subset \mathbb{A}^n$ be the hyperplane $\{x_i = 0\}$ for $1 \le i \le n$. Then each H_i is T-invariant and

$$T \cong \mathbb{A}^n - \left(\bigcup_{i=1}^{n} H_i \right) = \mathbb{A}^n - H \quad \text{(say)}. \tag{3.3}$$

Note that since $\{\chi_1, \ldots, \chi_n\}$ is a basis and since the action of $\chi_i(t)$ is multiplication by t_i if $t = (t_1, \ldots, t_n)$ is a coordinate of T, we see that under the identification in (3.3), the restriction of the above action of T on $\mathbb{A}^n$ to the open subset T is by simply the left (and hence right) multiplication.

The localization sequence of Theorem 2.5 (ii) gives an exact sequence

$$\Omega^T_*(X \times H) \to \Omega^T_*(X \times \mathbb{A}^n) \to \Omega^T_*(X \times T) \to 0, \tag{3.4}$$

where T acts on $X \times \mathbb{A}^n$ via the diagonal action. Using Proposition 2.4 and Lemma 3.3, this reduces to an exact sequence

$$\bigoplus_{1 \le i \le n} \Omega^T_*(X \times H_i) \xrightarrow{\sum j^i_*} \Omega^T_*(X \times \mathbb{A}^n) \to \Omega_*(X) \to 0, \tag{3.5}$$

where $j^i : H_i \to \mathbb{A}^n$ is the inclusion. By the equivariant homotopy invariance, we can identify $\Omega^T_*(X \times H_i)$ by $\Omega^T_*(X)$ via the projection

$$\begin{tikzcd} H_i \arrow[r, "j^i"] \arrow[dr, "p_i"'] & \mathbb{A}^n \arrow[d, "p"] \\ & \mathrm{Spec}(k) \end{tikzcd} \tag{3.6}$$

and similarly identify $\Omega^T_*(X \times \mathbb{A}^n)$ by $\Omega^T_*(X)$ via p^*. Since T acts on $\mathbb{A}^n$ coordinate-wise, we can also equivariantly identify $\mathbb{A}^n$ as $H_i \times \mathbb{A}^1$ and then by the homotopy invariance, we have isomorphisms $\Omega^T_*(X \times \mathbb{A}^n) \cong \Omega^T_*(X \times \mathbb{A}^1 \times H_i) \cong \Omega^T_*(X \times \mathbb{A}^1)$.

Under the identifications $\Omega^T_*(X) \underset{p_i^*}{\xrightarrow{\cong}} \Omega^T_*(X \times H_i)$ and $\Omega^T_*(X \times \mathbb{A}^1) \cong \Omega^T_*(X \times \mathbb{A}^n)$, the map $\Omega^T_*(X \times H_i) \xrightarrow{j^i_*} \Omega^T_*(X \times \mathbb{A}^n)$ is the composite

$$\Omega^T_*(X) \xrightarrow{j_*} \Omega^T_*(X \times \mathbb{A}^1) \xrightarrow{j^*} \Omega^T_*(X), \tag{3.7}$$

where $j : \mathrm{Spec}(k) \to \mathbb{A}^1$ is the zero-section embedding. In particular, we can apply Corollary 3.2 to identify the map j^i_* as the map $\Omega^T_*(X) \xrightarrow{c^T_1(L_{\chi_i})} \Omega^T_*(X)$. Hence, the first map of (3.5) is identified as the map

$$\begin{aligned} &\left(\Omega^T_*(X)\right)^{\oplus n} \xrightarrow{\sum c^T_1(L_{\chi_i})} \Omega^T_*(X) \\ &(a_1, \ldots, a_n) \mapsto \sum_{i=1}^{n} c^T_1(L_{\chi_i}) \cdot a_i. \end{aligned} \tag{3.8}$$

Since each L_{χ_i} is a T-equivariant line bundle on $\mathrm{Spec}(k)$, the above map is same as

$$S^{\oplus n} \otimes_S \Omega^T_*(X) \xrightarrow{\Phi \otimes id} S \otimes_S \Omega^T_*(X),$$

where $S = \Omega^*_T(k) = \mathbb{L}[[x_1, \ldots, x_n]]$ and $\Phi : S^{\oplus n} \to S$ is given as in (3.8).

Now, the map $c^T_1(L_{\chi_i})$ on S is given by the multiplication by x_i for $1 \le i \le n$. In particular, we see that $[S^{\oplus n} \xrightarrow{\Phi} S]$ is the complex $(K_1 \xrightarrow{\Phi} K_0)$, where

$$K_\bullet = K_\bullet(S \xrightarrow{x_i} S) = (K_n \to \cdots \to K_1 \xrightarrow{\Phi} K_0)$$

is the Koszul complex associated to the sequence $(x_1, \ldots, x_n)$ in S. Hence, by comparing this with (3.5) and (3.8), we conclude that (3.5) is the exact sequence

$$K_1 \otimes_S \Omega^T_*(X) \to K_0 \otimes_S \Omega^T_*(X) \to \Omega_*(X) \to 0. \tag{3.9}$$

On the other hand, $(x_1, \ldots, x_n)$ is a regular sequence in S by Lemma 2.6. This in turn implies using [27, Theorem 16.5] that $K^\bullet$ is a resolution of $\mathbb{L}$ as S-module. In particular, the sequence

$$K_1 \otimes_S \Omega^T_*(X) \to K_0 \otimes_S \Omega^T_*(X) \to \mathbb{L} \otimes_S \Omega^T_*(X) \to 0$$

is exact. The proof of the theorem follows by comparing this exact sequence with (3.9). □

Remark 3.5. We expect the assertion of Theorem 3.4 to hold for all linear algebraic groups with rational coefficients. Although we are unable to prove this generalization, we shall show a weaker version later in this paper. We shall also show the isomorphism $\Omega^G_*(X) \otimes_{S(G)} \mathbb{L} \xrightarrow{\cong} \Omega_*(X)$ when X is the flag variety G/B (*cf.* Theorem 8.1).

4. Structure theorems for torus action

In this section, we prove some structure theorems for the equivariant cobordism of smooth projective varieties with torus action. To state our first result of this kind, recall that for homomorphisms $A_i \xrightarrow{f_i} B$, $i = 1, 2$ of abelian groups, $A_1 \underset{B}{\times} A_2$ denotes the fibre product $\{(a_1, a_2) | f_1(a_1) = f_2(a_2)\}$.

Proposition 4.1. *Let T be a split torus and let $Y \hookrightarrow X$ be a T-equivariant closed embedding of smooth varieties of codimension $d \geq 0$. Assume that $c^G_d(N_{Y/X})$ is a non-zero divisor in the cobordism ring $\Omega^*_T(Y)$. Let $Y \overset{i}{\hookrightarrow} X$ and $U \overset{j}{\hookrightarrow} X$ be the inclusion maps, where U is the complement of Y in X. Then:*

(i) *The localization sequence*

$$0 \to \Omega^*_T(Y) \xrightarrow{i_*} \Omega^*_T(X) \xrightarrow{j^*} \Omega^*_T(U) \to 0$$

is exact.

(ii) *The restriction ring homomorphisms*

$$\Omega^*_T(X) \xrightarrow{(i^*, j^*)} \Omega^*_T(Y) \times \Omega^*_T(U)$$

give an isomorphism of rings

$$\Omega^*_T(X) \xrightarrow{\cong} \Omega^*_T(Y) \underset{\widetilde{\Omega^*_T(Y)}}{\times} \Omega^*_T(U),$$

where $\widetilde{\Omega^*_T(Y)} = \Omega^*_T(Y) / \left(c^T_d\left(N_{Y/X}\right)\right)$, and the maps

$$\Omega^*_T(Y) \to \widetilde{\Omega^*_T(Y)}, \qquad \Omega^*_T(U) \to \widetilde{\Omega^*_T(Y)}$$

are respectively, the natural surjection and the map

$$\Omega^*_T(U) = \frac{\Omega^*_T(X)}{i_*\left(\Omega^*_T(Y)\right)} \xrightarrow{i^*} \frac{\Omega^*_T(Y)}{c^T_d\left(N_{Y/X}\right)} = \widetilde{\Omega^*_T(Y)},$$

which is well-defined by Proposition 3.1.

Proof. Since all the statements are obvious for $d = 0$, we assume that $d \geq 1$. In view of Theorem 2.5 (ii), we only need to show that i_* is injective to prove the first assertion. But this follows from Proposition 3.1 and the assumption that $c^T_d(N_{Y/X})$ is a non-zero divisor in the ring $\Omega^*_T(Y)$. Since i^* and j^* are ring homomorphisms, the proof of the second part follows directly from the first part and [32, Lemma 4.4]. $\square$

4.1. *The motivic cobordism theory*

Before we prove our other results of this section, we recall the theory of motivic algebraic cobordism $MGL_{*,*}$ introduced by Voevodsky in [33]. This is a bi-graded ring cohomology theory in the category of smooth schemes over k. Levine has recently shown in [22] that $MGL_{*,*}$ extends uniquely to a bi-graded oriented Borel–Moore homology theory $MGL'_{*,*}$ on the category of all schemes over k. This homology theory has exterior products, homotopy invariance, localization exact sequence and Mayer–Vietoris among other properties (*cf.* [22, Section 3]). Moreover, the universality of Levine–Morel cobordism theory implies that there is a unique map

$$\vartheta : \Omega_* \to MGL'_{2*,*}$$

of oriented Borel–Moore homology theories. Our motivation for studying the motivic cobordism theory in this text comes from the following result of Levine.

Theorem 4.2 (*[23]*). *For any $X \in \mathcal{V}_k$, the map ϑ_X is an isomorphism.*

Proposition 4.3. *Let X be a k-scheme with a filtration by closed subschemes*

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X \tag{4.1}$$

and maps $\phi_m : U_m = (X_m \setminus X_{m-1}) \to Z_m$ for $0 \leq m \leq n$ which are all vector bundles. Assume moreover that each Z_m is smooth and projective. Let MGL' be the Motivic Borel–Moore cobordism theory. Then there is a canonical isomorphism

$$\bigoplus_{m=0}^{n} MGL'_{*,*}(Z_m) \xrightarrow{\cong} MGL'_{*,*}(X).$$

Proof. We prove it by induction on n. For $n = 0$, the map $X = X_0 \xrightarrow{\phi_0} Z_0$ is a vector bundle over a smooth scheme and hence the lemma follows from the homotopy invariance of the MGL-theory. We now assume by induction that $1 \leq m \leq n$ and

$$\bigoplus_{j=0}^{m-1} MGL'_{*,*}(Z_j) \xrightarrow{\cong} MGL'_{*,*}(X_{m-1}). \tag{4.2}$$

Let $i_{m-1} : X_{m-1} \hookrightarrow X_m$ and $j_m : U_m = (X_m \setminus X_{m-1}) \hookrightarrow X_m$ be the closed and open embeddings. We consider the localization exact sequence for MGL' (*cf.* [22, p. 35]).

$$\cdots \to MGL'_{*,*}(X_{m-1}) \xrightarrow{i_{(m-1)*}} MGL'_{*,*}(X_m) \xrightarrow{j^*_m} MGL'_{*,*}(U_m) \to \cdots. \tag{4.3}$$

Using (4.2), it suffices now to construct a canonical splitting of the smooth pull-back j^*_m in order to prove the proposition.

Let $V_m \subset U_m \times Z_m$ be the graph of the projection $U_m \xrightarrow{\phi_m} Z_m$ and let $\overline{V}_m$ denote the closure of V_m in $X_m \times Z_m$. Let $Y_m \to \overline{V}_m$ be a resolution of singularities. Since V_m is smooth, we see that $V_m \overset{\overline{j}_m}{\hookrightarrow} Y_m$ as an open subset. We consider the composite maps

$$\begin{aligned} &p_m : V_m \hookrightarrow U_m \times Z_m \to U_m, \qquad q_m : V_m \hookrightarrow U_m \times Z_m \to Z_m \quad \text{and} \\ &\overline{p}_m : Y_m \to X_m \times Z_m \to X_m, \qquad \overline{q}_m : Y_m \to X_m \times Z_m \to Z_m. \end{aligned}$$

Note that $\overline{p}_m$ is a projective morphism since Z_m is projective. The map q_m is smooth and p_m is an isomorphism.

We consider the diagram

$$\begin{tikzcd} MGL'_{*,*}(Z_m) \arrow[r, "\overline{q}_m^*"] \arrow[d, "\phi_m^*"', "\cong"] & MGL'_{*,*}(Y_m) \arrow[d, "\overline{p}_{m*}"] \\ MGL'_{*,*}(U_m) & MGL'_{*,*}(X_m). \arrow[l, "j_m^*"] \end{tikzcd} \tag{4.4}$$

Since each Z_m is smooth by our assumption, the map ϕ_m^* is an isomorphism by the homotopy invariance of the MGL-theory. The map $\overline{q}_m^*$ is the pull-back $MGL'_{*,*}(Z_m) \cong MGL^{*,*}(Z_m) \to MGL^{*,*}(Y_m) \cong MGL'_{*,*}(Y_m)$ between Voevodsky's motivic cobordism of smooth varieties. It suffices to show that this diagram commutes. For, the map $\overline{p}_{m*} \circ \overline{q}_m^* \circ \phi_m^{*\,-1}$ will then give the desired splitting of the map j_m^*.

We now consider the following commutative diagram.

$$\begin{tikzcd} X_m & U_m \arrow[l, "j_m"'] & \\ Y_m \arrow[u, "\overline{p}_m"] \arrow[dr, "\overline{q}_m"'] & V_m \arrow[l, "\overline{j}_m"] \arrow[u, "p_m"'] \arrow[d, "q_m"] & U_m \arrow[l, "{(\phi_m, id)}"] \arrow[ul, "id"'] \arrow[dl, "\phi_m"] \\ & Z_m. & \end{tikzcd}$$

Since the top left square is Cartesian and j_m is an open immersion, we have $j_m^* \circ \overline{p}_{m*} = p_{m*} \circ \overline{j}_m^*$ by the functoriality property of the pull-back of MGL'-theory with respect to an open immersion. Now, using the fact that (ϕ_m, id) is an isomorphism, we get

$$\begin{aligned} j_m^* \circ \overline{p}_{m*} \circ \overline{q}_m^* &= p_{m*} \circ \overline{j}_m^* \circ \overline{q}_m^* = p_{m*} \circ q_m^* \\ &= p_{m*} \circ (\phi_m, id)_* \circ (\phi_m, id)^* \circ q_m^* = id_* \circ \phi_m^* \\ &= \phi_m^*. \end{aligned}$$

This proves the commutativity of (4.4) and hence the proposition. $\square$

Lemma 4.4. *Let $Y \xrightarrow{f} X$ be a morphism of smooth k-varieties of relative codimension d. Then the map $f^* : \Omega_p(X) \to \Omega_{p-d}(Y)$ preserves the niveau filtration (cf. (2.1)).*

Proof. We need to show that $f^*\left(F_q\Omega_p(X)\right) \subset F_{q-d}\Omega_{p-d}(Y)$. We can factor f as $Y \xrightarrow{i} U \xrightarrow{j} \mathbb{P}^n_X \xrightarrow{\pi} X$, where i and j are respectively closed and open immersions, and π is the usual projection. One has then $f^* = i^* \circ j^* \circ \pi^*$ (*cf.* [25, 5.1.3]). Since π is the smooth projection and j is an open immersion, it is clear from the definition of the cobordism (*cf.* [19, Lemma 3.3]) that $j^* \circ \pi^* \left(F_q \Omega_p(X)\right) \subset F_{q+n}\Omega_{p+n}(U)$. Since $d = \operatorname{codim}_U(Y) - n$, we need to prove the lemma when f is a closed immersion.

So let $\alpha = [W \xrightarrow{g} X]$ be a cobordism cycle where W is an irreducible and smooth variety and g is projective. By [24, Lemma 7.1], we can assume that g is transverse to f. In particular, $W' = W \times_X Y$ is smooth and there is a Cartesian square

$$\begin{tikzcd} W' \arrow[r, "f'"] \arrow[d, "g'"'] & W \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} \tag{4.5}$$

in $\mathcal{V}_k^S$, where the horizontal arrows are closed immersions and the vertical arrows are projective. We then have

$$f^*(\alpha) = f^* \circ g_*(\mathrm{Id}_W) = g'_* \circ f'^*(\mathrm{Id}_W) = g'_*(\mathrm{Id}_{W'}),$$

where the second equality follows from the transversality condition (*cf.* [25, 5.1.3]). Since f and f' have same codimension, the lemma follows. □

4.2. *Equivariant cobordism of filtrable varieties*

We recall from [4, Section 3] that a k-variety X with an action of a split torus T is called *filtrable* if the fixed point locus X^T is smooth and projective, and there is a numbering $X^T = \coprod_{m=0}^n Z_m$ of the connected components of the fixed point locus, a filtration of X by T-invariant closed subschemes

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X \tag{4.6}$$

and maps $\phi_m : U_m = (X_m \setminus X_{m-1}) \to Z_m$ for $0 \le m \le n$ which are all T-equivariant vector bundles. The following celebrated theorem of Bialynicki-Birula [1] (generalized to the case of non-algebraically closed fields by Hesselink [15]) will be crucial for understanding the equivariant cobordism of smooth projective varieties.

Theorem 4.5 (*Bialynicki-Birula, Hesselink*)**.** *Let X be a smooth projective variety with an action of T. Then X is filtrable.*

This is roughly proven by choosing a generic one parameter subgroup $\lambda \subset T$ such that $X^\lambda = X^T$ and taking the various strata of the filtration to be the locally closed T-invariant subsets of the form

$$X_+(Z, \lambda) = \{x \in X \mid \lim_{t \to 0} \lambda(t) \cdot x \in Z\},$$

where Z is a connected component of X^T. Note that the above limit exists since X is projective. The fibre of the equivariant vector bundle $X_+(Z, \lambda) \to Z$ is the positive eigenspace $T_z^+ X$ for the T-action on the tangent space of X at any point $z \in Z$.

Lemma 4.6. *Let X be a filtrable variety with an action of a split torus T as above. Then for every $0 \le m \le n$, there is a canonical split exact sequence*

$$0 \to \Omega_*^T(X_{m-1}) \xrightarrow{i_{(m-1)*}} \Omega_*^T(X_m) \xrightarrow{j_m^*} \Omega_*^T(U_m) \to 0. \tag{4.7}$$

Proof. Let r be the rank of T and for each $j \ge 1$, let (V_j, U_j) be the good pair for the T-action corresponding to j as chosen in the proof of Lemma 6.1. Notice in particular that $U_j/T \cong \left(\mathbb{P}_k^{j-1}\right)^r$. For any scheme Y with T-action, we set $Y^j = Y \overset{T}{\times} U_j$ for every j. Given the T-equivariant filtration as in (4.6), it is easy to see that for each j, there is an associated system of filtrations

$$\emptyset = (X_{-1})^j \subset (X_0)^j \subset \cdots \subset (X_n)^j = X^j \tag{4.8}$$

and maps $\phi_m : (U_m)^j = (X_m)^j \setminus (X_{m-1})^j \to (Z_m)^j$ for $0 \le m \le n$ which are all vector bundles. Observe also that as T acts trivially on each Z_m, we have that $(Z_m)^j \cong Z_m \times \left(U_j/T\right) \cong Z_m \times \left(\mathbb{P}_k^{j-1}\right)^r$. Since Z_m is smooth and projective, this in turn implies that $(Z_m)^j$ is a smooth projective variety. We conclude that the filtration (4.8) of X^j satisfies all the conditions of Proposition 4.3. In particular, we get split exact sequences

$$0 \to MGL'_{*,*}\left((X_{m-1})^j\right) \to MGL'_{*,*}\left((X_m)^j\right) \to MGL'_{*,*}\left((U_m)^j\right) \to 0.$$

Applying Theorem 4.2, we get for each $0 \le m \le n$, $i \in \mathbb{Z}$ and $j \ge 1$, the canonical split exact sequence

$$0 \to \Omega_i\left((X_{m-1})^j\right) \xrightarrow{i_{(m-1)*}} \Omega_i\left((X_m)^j\right) \xrightarrow{j_m^*} \Omega_i\left((U_m)^j\right) \to 0.$$

By [19, Theorem 3.5], this sequence remains exact at each level of the niveau filtration. We now claim that the inverse $(j_m^*)^{-1}$ also preserves this filtration. By the construction of the inverse of j_m^* in diagram (4.4), we only need to show that $\overline{p}_{m*}$ and $\overline{q}_m^*$ preserve the niveau filtration. This holds for $\overline{p}_{m*}$ by [19, Lemma 3.3] and so is the case for $\overline{q}_m^*$ by Lemma 4.4. This proves the claim. We conclude that there are canonical split exact sequences

$$0 \to \Omega_i\,(X_{m-1})_j \xrightarrow{i_{(m-1)*}} \Omega_i\,(X_m)_j \xrightarrow{j_m^*} \Omega_i\,(U_m)_j \to 0.$$

Taking the inverse limit over j, we get the desired split exact sequence (4.7). □

The following is the main result about the equivariant cobordism for the torus action on smooth filtrable varieties.

Theorem 4.7. *Let T be a split torus of rank r acting on a filtrable variety X and let $i : X^T = \coprod_{m=0}^n Z_m \hookrightarrow X$ be the inclusion of the fixed point locus. Then there is a canonical isomorphism*

$$\bigoplus_{m=0}^{n} \Omega_*^T(Z_m) \xrightarrow{\cong} \Omega_*^T(X)$$

of S-modules. In particular, there is a canonical isomorphism

$$\Omega_*^T(X) \xrightarrow{\cong} \Omega_*(X)[[t_1, \ldots, t_r]] \tag{4.9}$$

of S-modules.

Proof. By inducting on n, it follows from Lemma 4.6 that there are canonical isomorphisms

$$\bigoplus_{m=0}^{n} \Omega_*^T(Z_m) \xrightarrow{\cong} \Omega_*^T(X) \quad \text{and} \quad \bigoplus_{m=0}^{n} \Omega_*(Z_m) \xrightarrow{\cong} \Omega_*(X)$$

of S-modules. The second assertion now follows from these two isomorphisms and Lemma 6.1 since T acts trivially on X^T. $\square$

Recall from Section 2.3 that if $Y \to X$ is a T-equivariant projective morphism of k-schemes with T-action such that Y is smooth, then $[Y \to X]$ defines unique elements in $\Omega_*^T(X)$ and $\Omega_*(X)$, called the fundamental classes of $[Y \to X]$.

Corollary 4.8. *Let T be a split torus of rank r acting on a filtrable variety X such that X^T is the finite set of smooth closed points $\{x_0, \dots, x_n\}$. For $0 \le m \le n$, let $f_m : \widetilde{X}_m \to X_m$ be a T-equivariant resolution of singularities and let $\widetilde{x}_m$ be the fundamental class of the T-invariant cobordism cycle $[\widetilde{X}_m \to X]$ in $\Omega_*^T(X)$. Then $\Omega_*^T(X)$ is a free S-module with a basis $\{\widetilde{x}_0, \dots, \widetilde{x}_n\}$. In particular, $\Omega_*(X)$ is a free $\mathbb{L}$-module with a basis $\{\widetilde{x}_0, \dots, \widetilde{x}_n\}$.*

Proof. It follows by inductively applying Lemma 4.6 that $\{\widetilde{x}_0, \dots, \widetilde{x}_n\}$ spans $\Omega_*^T(X)$ as S-module. We show its linear independence by the induction on the length of the filtration. There is nothing to prove for $n = 0$ and so we assume that $n \ge 1$. Let $i_{(n-1)} : X_{n-1} \to X_n$ be the inclusion map. Notice that for $0 \le m \le n-1$, $\widetilde{x}_m$ defines a unique class $\widetilde{x}'_m$ in $\Omega_*^T(X_{n-1})$ and $\widetilde{x}_m = i_{(n-1)*}(\widetilde{x}'_m)$.

Since U_n is an affine space, it follows from Lemma 4.6 that $\sum_{m=0}^{n} a_m \widetilde{x}_m = 0$ implies that $a_n j_n^*(\widetilde{x}_n) = 0$. Since $j_n^*(\widetilde{x}_n) = 1$, we conclude that $a_n = 0$. It follows again from Lemma 4.6 that $\sum_{m=0}^{n-1} a_m \widetilde{x}'_m = 0$ in $\Omega_*^T(X_{n-1})$ and hence $a_m = 0$ for each m by induction. This proves the first assertion of the corollary. The second assertion follows from the first and Theorem 3.4. $\square$

4.3. *Case of toric varieties*

Let $X = X(\Delta)$ be a smooth projective toric variety associated to a fan Δ in $N = \mathrm{Hom}(\mathbb{G}_m, T)$. Then the fixed point locus X^T is the disjoint union of the closed orbits O_σ, where σ runs over the set of r-dimensional cones in Δ. Since X is projective and any orbit is affine, we see that σ is a maximal cone if and only if O_σ is a closed point and hence a fixed point. In other words, $X^T = \coprod_{\sigma \in \Delta_{\max}} O_\sigma = \{x_0, \cdots, x_n\}$. Note also that all these closed points are in fact k-rational. For any $\sigma \in \Delta$, let $V(\sigma)$ denote the closure of the orbit O_σ in X.

It follows from Theorem 4.7 that $\Omega_T^*(X)$ is a free S-module of rank $|\Delta_{\max}| = n+1$. Moreover, it follows from Corollary 4.8 that a basis of this free module is given by $\{\widetilde{x}_0, \dots, \widetilde{x}_n\}$, where $\widetilde{x}_m$ is the fundamental class of the T-invariant cobordism cycle $[X_m \to X]$.

If we now fix an ordering $\{\sigma_0, \dots, \sigma_n\}$ of $\Delta_{\max}$ and let $\tau_m \subset \sigma_m$ be the cone which is the intersection of σ_m with all those σ_j such that $j \ge m$ and which intersect σ_m in dimension $r-1$, then we can choose the ordering of $\Delta_{\max}$ such that

$$\tau_i \subset \sigma_j \quad \text{only if } i \le j.$$

In this case, X_m is same as $V(\tau_{n-m})$ for $0 \le m \le n$, which is itself a smooth toric variety. Let $[V(\tau_i)]$ denote the fundamental class of $[V(\tau_i) \to X]$. We conclude the following.

Corollary 4.9. *Let $X = X(\Delta)$ be as above.*

(i) *$\Omega^*_T(X)$ is free S-module with basis $\{[V(\tau_0)], \dots, [V(\tau_n)]\}$.*
(ii) *$\Omega^*(X)$ is free $\mathbb{L}$-module with basis $\{[V(\tau_0)], \dots, [V(\tau_n)]\}$.*
(iii) *There is an $MGL^{*,*}(k)$-linear isomorphism $MGL^{*,*}(X) \cong (MGL^{*,*}(k))^{n+1}$.*

Proof. We have already shown (i) above and (ii) follows from (i) and Theorem 3.4. The last part follows from Proposition 4.3. □

4.4. *Case of flag varieties*

Let G be a connected reductive group with a split maximal torus T and let B be a Borel subgroup of G containing T. Let $X = G/B$ be the associated flag variety of the left cosets of B in G. Then the left multiplication of G on itself induces a natural G-action on X. In particular, X is a smooth projective T-variety and hence filtrable. The Bruhat decomposition of G implies that $X^T = \coprod_{w \in W} wB$, where W is the Weyl group of G with respect to T. Notice here that the coset wB makes sense even though w is not an element of G. We can choose an ordering $\{w_0, \dots, w_n\}$ of the elements of W such that X_m is the union of the Schubert varieties X_{w_i} with $i \le m$. For each $w \in W$, there is canonical B-equivariant resolution of singularities $f_w : \widetilde{X}_w \to X_w$ which is defined by inducting on the length of w with X_0 being the closed point associated to the identity element of W. The varieties $\widetilde{X}_w$ are called the *Bott–Samelson* varieties associated to X. We refer to [5, Subsection 2.2] for a very nice exposition of these facts. We shall call the fundamental classes $[\widetilde{X}_w \to X]$ in $\Omega^*_T(X)$ as the Bott–Samelson cobordism classes.

It is known that these Bott–Samelson classes are not invariants of the associated Schubert variety X_w and they depend on a reduced decomposition of $w \in W$ in terms of the simple roots (*cf.* [3]). We fix a choice $\{\widetilde{X}_w | w \in W\}$ of the Bott–Samelson varieties. As an immediate consequence of Corollary 4.8, we obtain the following. It was earlier shown in [16, Proposition 3.1] that the ordinary cobordism group $\Omega^*(X)$ is a free $\mathbb{L}$-module. We shall use the following to compute the cobordism ring of G/B later in this paper.

Corollary 4.10. *Let $X = G/B$ be as above. Then $\Omega^*_T(X)$ is a free S-module with a basis given by the Bott–Samelson classes. The similar conclusion holds for $\Omega^*(X)$.*

As another consequence of Theorem 4.7, we obtain the following generalization of Brion's result [4, Theorem 2.1] for smooth schemes.

Theorem 4.11. *Let T be a split torus acting on a smooth k-variety X. Then the S-module $\Omega^*_T(X)$ is generated by the fundamental classes of the T-invariant cobordism cycles in $\Omega^*(X)$.*

Proof. Since we are in characteristic zero, we can use the canonical resolution of singularities to get a T-equivariant open embedding $j : X \hookrightarrow Y$, where Y is a smooth and projective T-variety. Since the restriction map $\Omega^*_T(Y) \xrightarrow{j^*} \Omega^*_T(X)$ is S-linear and surjective by Theorem 2.5(ii), it suffices to prove the theorem when X is projective. In this case, the assertion is proved exactly like the proofs of Theorems 4.5 and 4.7 using an induction on the length of the filtration. □

Remark 4.12. In [4, Theorem 2.1], Brion also describes the relations which explicitly describe the T-equivariant Chow groups in terms of the invariant cycles. It will be interesting to know what are the analogous relations for cobordism. We shall come back to this problem in a subsequent work.

Example 4.13. As an illustration of the above structure theorems, we deduce the formula for the equivariant cobordism ring of the projective line where $\mathbb{G}_m$ acts with weight χ. We can write $\mathbb{P}^1_k = \mathbb{A}^1_0 \cup \mathbb{A}^1_\infty$ as union of $\mathbb{G}_m$-invariant affine lines where the first (resp. second) affine line is the complement of ∞ (resp. 0). We get pull-back maps $\Omega^*_{\mathbb{G}_m}(\mathbb{P}^1) \xrightarrow{i_0^*} \Omega^*_{\mathbb{G}_m}(\mathbb{A}^1_0) \xrightarrow{\cong} \Omega^*_{\mathbb{G}_m}(\{0\}) \cong S(\mathbb{G}_m)$. We similarly have the pull-back map i^*_∞. It follows from Lemma 4.6 that there is a short exact sequence of ring homomorphisms

$$0 \longrightarrow \Omega^*_{\mathbb{G}_m}(\mathbb{P}^1) \xrightarrow{(i_0^*, i_\infty^*)} \Omega^*_{\mathbb{G}_m}(\mathbb{A}^1_0) \times \Omega^*_{\mathbb{G}_m}(\mathbb{A}^1_\infty) \xrightarrow{j_0^* - j_\infty^*} \Omega^*_{\mathbb{G}_m}(\mathbb{G}_m) \longrightarrow 0.$$

Identifying the last term with $\Omega^*(k) \cong \mathbb{L}$ and $S(\mathbb{G}_m)$ with $\mathbb{L}[\![t]\!]$, we have a short exact sequence of ring homomorphisms

$$0 \longrightarrow \Omega^*_{\mathbb{G}_m}(\mathbb{P}^1) \xrightarrow{(i_0^*, i_\infty^*)} \mathbb{L}[[t]] \times \mathbb{L}[[t]] \xrightarrow{j_0^* - j_\infty^*} \mathbb{L} \longrightarrow 0. \tag{4.10}$$

It follows from this exact sequence and Lemma 5.4 that there is a ring isomorphism

$$\frac{\mathbb{L}[\![x, y]\!]}{(xy)} \xrightarrow{\cong} \Omega^*_{\mathbb{G}_m}(\mathbb{P}^1) \tag{4.11}$$

where the images of the variables are the fundamental classes of 0 and ∞. This gives the Stanley–Reisner presentation for the equivariant cobordism of the projective line.

5. Localization for torus action: first steps

The localization theorems are the most powerful tools in the study of the equivariant cohomology of smooth varieties with torus action. They provide simple formulae to describe the relation between the equivariant cobordism ring of a smooth projective variety and that of the fixed point locus for the action of a torus. The localization theorems for the equivariant Chow groups and their important applications are considered in [4]. Our goal in the rest of this paper is to prove such results for the equivariant cobordism. These results will be used in [17] to compute the equivariant and the non-equivariant cobordism rings of certain spherical varieties. More applications of these results appear in [21]. Below, we prove some algebraic results that we need for the localization theorems.

Let R be a commutative Noetherian ring and let $A = \bigoplus_{j \in \mathbb{Z}} A_j$ be a commutative graded R-algebra with $R \subset A_0$. Let $S^{(n)} = \bigoplus_{i \in \mathbb{Z}} S_i$ be the graded power series ring $A[\![\mathbf{t}]\!] := A[\![t_1, \ldots, t_n]\!]$ (*cf.* Section 2). Recall that S_i is the set of formal power series of the form $f(\mathbf{t}) = \sum_{m(\mathbf{t}) \in \mathcal{C}} a_{m(\mathbf{t})} m(\mathbf{t})$ such that $a_{m(\mathbf{t})}$ is a homogeneous element in A and $|a_{m(\mathbf{t})}| + |m(\mathbf{t})| = i$. Here, $\mathcal{C}$ is the set of all monomials in $\mathbf{t} = (t_1, \ldots, t_n)$ and $|m(\mathbf{t})| = i_1 + \cdots + i_n$ if $m(\mathbf{t}) = t_1^{i_1} \cdots t_n^{i_n}$. Notice that $S^{(n-1)}[\![t_i]\!] \cong S^{(n)}$ and $S^{(n)}/(t_i) \cong S^{(n-1)}$.

Recall from [25] that a formal (commutative) group law over A is a power series $F(u, v) \in A[\![u, v]\!]$ such that

(i) $F(u, 0) = F(0, u) = u \in A[\![u]\!]$
(ii) $F(u, v) = F(v, u)$
(iii) $F(u, F(v, w)) = F(F(u, v), w)$
(iv) there exists (unique) $\rho(u) \in A[\![u]\!]$ such that $F(u, \rho(u)) = 0$.

We write $F(u, v)$ as $u +_F v$. We shall denote $\rho(u)$ by $[-1]_F u$. Inductively, we have $[n]_F u = [n-1]_F u +_F u$ if $n \geq 1$ and $[n]_F u = [-n]_F \rho(u)$ if $n \leq 0$. The sum $\sum_{i=1}^{m} [n_i]_F u_i$ will mean $[n_1]_F u_1 +_F \cdots +_F [n_m]_F u_m$ for $n_i \in \mathbb{Z}$. It is known that such a power series is of the form

$$F(u, v) = (u + v) + uv \left(\sum_{i,j \geq 1} a_{i,j} u^{i-1} v^{j-1} \right) \in A[\![u, v]\!], \quad \text{where } a_{i,j} \in A_{1-i-j}. \quad (5.1)$$

Notice that $F(u, v)$ is a homogeneous element of degree one in the graded power series ring $A[\![u, v]\!]$ if u, v are homogeneous elements of degree one. It follows from the above conditions that $\rho(u) = -u + u^2 \sum_{j \geq 0} b_j u^j$. We conclude that for a set $\{u_1, \ldots, u_m\}$ of homogeneous elements of degree one, one has

$$\sum_{i=1}^{m} [n_i]_F u_i = \sum_{i=1}^{m} n_i u_i + \sum_{|m(\mathbf{u})| \geq 2} a_{m(\mathbf{u})} m(\mathbf{u}) \quad (5.2)$$

is also homogeneous and of degree one. For the rest of this text, our ring A will be a graded $\mathbb{L}$-algebra and $F(u, v)$ will be the formal group law on A induced from that of the universal formal group law $F_{\mathbb{L}}$ on $\mathbb{L}$. We recall the following inverse function theorem for a power series ring.

Lemma 5.1. *Let A be any commutative ring and let $\{f_1, \ldots, f_n\}$ be a set of power series in the formal power series ring $\widehat{A[\![\mathbf{t}]\!]}$ such that $((\frac{\partial f_i}{\partial t_j}))(0) \in GL_n(A)$. Then the A-algebra homomorphism*

$$\phi : \widehat{A[\![\mathbf{t}]\!]} \to \widehat{A[\![\mathbf{t}]\!]}, \qquad \phi(t_j) = f_j$$

is an isomorphism.

Proof. *Cf.* [11, Exercise 7.25]. □

Lemma 5.2. *Let A and $S^{(n)}$ be as above such that no non-zero element of R is a zero divisor in A. Let $f = \sum_{i=1}^{n} [m_i]_F t_i$ be a non-zero homogeneous element of degree one in $S^{(n)}$. Let $g(f) = f^r + \alpha_{r-1} f^{r-1} + \cdots + \alpha_1 f + \alpha_0 \in S^{(n)}$ be such that each α_j is homogeneous of degree $r - j$ in A and is nilpotent. Then $g(f)$ is a non-zero divisor in $S^{(n)}$.*

Proof. Since $S^{(n)}$ is a subring of the formal power series ring $\widehat{A[\![\mathbf{t}]\!]}$, it suffices to show that $g(f)$ is a non-zero divisor in $\widehat{A[\![\mathbf{t}]\!]}$. We can thus assume that $S^{(n)}$ is the formal power series ring.

We prove the lemma by induction on n. We first assume that $n = 1$, in which case $S^{(1)} = \widehat{A[\![t]\!]}$ and $f = [m]_F t = mt + t^2 f'(t)$ with $m \neq 0$ by (5.2).

It follows from our assumption that every non-zero element of R is a non-zero divisor in $S^{(1)}$. Since the map $S^{(1)} \to S^{(1)}[m^{-1}]$ is then an injective map of rings, it suffices to show that $g(f)$ is not a zero divisor in $S^{(1)}[m^{-1}]$. That is, we can assume that $m = 1$. It follows then from Lemma 5.1 that there is an A-automorphism of $S^{(1)}$ which takes f to t. Thus we can assume that $f = t$ and $g(f) = g(t) = t^r + \alpha_{r-1} t^{r-1} + \cdots + \alpha_1 t + \alpha_0$.

Now, let $g'(t) = \sum_{j=0}^{\infty} a_j t^j$ be a power series in S. Then,

$$g(t)g'(t) = 0 \Rightarrow \sum_{j=0}^{\infty} \left(a_{j-r} + a_{j-r+1}\alpha_{r-1} + \cdots + a_{j-1}\alpha_1 + a_j\alpha_0\right) t^j = 0$$

$$\Rightarrow a_j = -\sum_{i=0}^{r-1} a_{j+r-i}\alpha_i \quad \forall\, j \geq 0. \tag{$*$}$$

Applying $(*)$ recursively, we see that each a_j can be expressed as an A-linear combination of monomials in α_i's of arbitrarily large degree in A. Since each α_i is of positive degree and nilpotent, we must have $a_j = 0$ for $j \geq 0$.

To prove the general case, suppose the lemma is proven when the number of variables is strictly less than n with $n \geq 2$. Suppose $g'(\mathbf{t}) \in S$ is such that $g(f)g'(\mathbf{t}) = 0$. If $g'(\mathbf{t})$ is not zero, we remove those terms $a_{m(\mathbf{t})}m(\mathbf{t})$ from its expression for which $a_{m(\mathbf{t})} = 0$. In particular, we get that none of the coefficients of $g'(\mathbf{t})$ is zero. We show that this leads to a contradiction.

Set $m_0(\mathbf{t}) = t_1 t_2 \cdots t_n$ and write $g'(\mathbf{t}) = (m_0(\mathbf{t}))^p\, h(\mathbf{t})$ such that $h(\mathbf{t})$ is not divisible by $m_0(\mathbf{t})$. Then, there is a term $a_{m'(\mathbf{t})}m'(\mathbf{t})$ of $h(\mathbf{t})$ which is not divisible by at least one t_i. By permutation, we can assume it is not divisible by t_n. If $m_i = 0$ for $1 \leq i \leq n-1$, then f is of the form $f = [m_n]_F t_n = m_n t_n + t_n^2 \sum_{j\geq 0} b_j t_n^j$ with $m_n \neq 0$ by (5.2). Hence, we are in the situation of the above one variable case with A replaced by $A[[t_1, \ldots, t_{n-1}]]$. So we assume that $m_i \neq 0$ for some $i \neq n$.

Since $m_0(\mathbf{t})$ is a non-zero divisor in S, we must have $g(f)h(\mathbf{t}) = 0$. Let $\overline{h(\mathbf{t})}$ be the image of a power series $h(\mathbf{t})$ under the quotient map $S^{(n)} \twoheadrightarrow S^{(n-1)}$. Then we get $\overline{g(f)}\ \overline{h(\mathbf{t})} = 0$ in $S^{(n-1)}$. By our choice, the term $a_{m'(\mathbf{t})}m'(\mathbf{t})$ still survives in $S^{(n-1)}$. On the other hand, as $\overline{g(t)}$ is of the same form as $g(t)$, the induction implies that $\overline{h(\mathbf{t})}$ must be zero. In particular, we must have $a_{m'(\mathbf{t})} = 0$. Since the coefficients of $h(\mathbf{t})$ are same as those of $g'(\mathbf{t})$, we arrive at a contradiction. $\square$

The following is a variant of Lemma 5.2 and is proved in the similar way.

Lemma 5.3. *Let A and $S^{(n)}$ be as above such that no non-zero element of R is a zero divisor in A. Let $f = \sum_{i=1}^{n}[m_i]_F t_i$ be a non-zero homogeneous element of degree one in $S^{(n)}$. Let $v \in A$ be a homogeneous element of degree one which is nilpotent. Then $F(f, v)$ is a non-zero divisor in $S^{(n)}$.*

Proof. The proof is exactly along the same lines as that of Lemma 5.2. We give the main steps. We can assume as before that $S^{(n)}$ is the formal power series ring. It follows from (5.1) that $F(f, v)$ is of the form $f + v\left(1 + fF'(f, v)\right)$, where $1 + fF'(f, v)$ is an invertible element. In particular, we can write $F(f, v) = uf + v$, where $u = \left(1 - fF' + f^2F'^2 - \cdots\right)$ is an invertible element in $S^{(n)}$. As before, we prove by induction on n. In case of $n = 1$, we can write $S^{(1)} = A[[t]]$ and $f = mt + t^2 f'(t)$. We can further assume that $m = 1$. In particular, we get $uf = t + t^2 f''(t)$. It follows from Lemma 5.1 that there an A-automorphism of $S^{(1)}$ which takes uf to t. This reduces to showing that $t + v$ is a non-zero divisor in $S^{(1)}$, which is shown in Lemma 5.2.

Now suppose the result holds for all $m \leq n-1$ with $n \geq 2$. If $m_i = 0$ for $1 \leq i \leq n-1$, then $f = [m_n]_F t_n$ with $m_n \neq 0$ and hence we are in the situation of one variable case. So we can assume that $m_i \neq 0$ for some $i \neq n$. Since the image $\overline{f}$ of f under the natural surjection

$S^{(n)} \twoheadrightarrow S^{(n-1)}$ is of the same form as that of f and $\overline{f} \neq 0$, we argue as in the proof of Lemma 5.2 to conclude that $F(f, v)$ is a non-zero divisor in $S^{(n)}$. □

Lemma 5.4. *Let T be split torus of rank n and let $\{\chi_1, \ldots, \chi_s\}$ be a set of characters of T such that for $j \neq j'$, the set $\{\chi_j, \chi_{j'}\}$ is a part of a basis of $\widehat{T}$. Let $\gamma_j = \left(c_1^T(L_{\chi_j})\right)^{d_j}$ with $d_j \geq 0$. Then*

$$(\gamma_1 \cdots \gamma_s) = \bigcap_{j=1}^{s} (\gamma_j)$$

as ideals in $S(T)$.

Proof. By ignoring those $j \geq 1$ such that $d_j = 0$, we can assume that $d_j \geq 1$ for all j. Using a simple induction, it suffices to show that for $j \neq j'$, the relation $\gamma_j | q\gamma_{j'}$ implies that $\gamma_j | q$. So we can assume that $s = 2$. By assumption, we can extend $\{\chi_1, \chi_2\}$ to a basis $\{\chi_1, \ldots, \chi_n\}$ of $\widehat{T}$. In particular, we can identify $S(T)$ with the graded power series ring $\mathbb{L}[[t_1, \ldots, t_n]]$ such that $t_j = c_1^T(L_{\chi_j})$ (*cf.* (2.9)). We can thus write $\gamma_j = t_j^{d_j}$ for $j = 1, 2$.

We now write $S(T)$ as $A[[t_1]]$ where $A = \mathbb{L}[[t_2, \ldots, t_n]]$ (*cf.* Lemma 2.6) and suppose that $p\gamma_2 = q\gamma_1$ in $S(T)$. We can write $p = s\gamma_1 + r$, where r is a polynomial in t_1 with coefficients in A of degree less than d_1.

We now claim that $q - s\gamma_2 = 0$, which will finish the proof of the lemma. Set $g(t_1) = q - s\gamma_2 = \sum_{j \geq 0} a_j t_1^j$. This yields $g(t_1)\gamma_1 = \sum_{j=0}^{\infty} a_j t_1^{d_1+j}$. Since $g(t_1)\gamma_1 = r\gamma_2$ is of degree less than d_1 in t_1, we see that $a_j = 0$ for all $j \geq 0$. □

6. Chern classes of equivariant bundles

We now apply the above algebraic results to deduce some consequences for the Chern classes of the equivariant vector bundles on smooth varieties with a torus action. We need the following description of the equivariant cobordism for the trivial action of a torus which follows essentially from the definitions. Recall that for a split torus T of rank n, $S = \mathbb{L}[[t_1, \ldots, t_n]]$ denotes the cobordism ring of the classifying space of T.

Lemma 6.1. *Let T be a split torus of rank n acting trivially on a smooth variety X of dimension d and let $\{\chi_1, \ldots, \chi_n\}$ be a chosen basis of $\widehat{T}$. Then the assignment $t_i \mapsto c_1^T(L_{\chi_i})$ induces an isomorphism of graded rings*

$$\Omega^*(X)[[t_1, \ldots, t_n]] \xrightarrow{\cong} \Omega_T^*(X). \tag{6.1}$$

Proof. This is a direct application of the projective bundle formula in the non-equivariant cobordism and is similar to the calculation of $\Omega_T^*(k)$ (*cf.* [19, Example 6.6]). We give the sketch.

The chosen basis $\{\chi_1, \ldots, \chi_n\}$ of $\widehat{T}$ equivalently yields a decomposition $T = T_1 \times \cdots \times T_n$ with each T_i isomorphic to $\mathbb{G}_m$ such that χ_i is a generator of $\widehat{T_i}$. Let L_χ be the one-dimensional representation of T, where T acts via χ. For any $j \geq 1$, we take the good pair (V_j, U_j) such that $V_j = \prod_{i=1}^n L_{\chi_i}^{\oplus j}$, $U_j = \prod_{i=1}^n \left(L_{\chi_i}^{\oplus j} \setminus \{0\}\right)$ and T acts on V_j by $(t_1, \ldots, t_n)(x_1, \ldots, x_n) = (\chi_1(t_1)(x_1), \ldots, \chi_n(t_n)(x_n))$. Since T acts trivially on X, it is easy to see that $X \overset{T}{\times} U_j \cong X \times (X_1 \times \cdots \times X_n)$ with each X_i isomorphic to $\mathbb{P}_k^{j-1}$. Moreover, the T-line bundle L_{χ_i} gives the line bundle $L_{\chi_i} \overset{T_i}{\times} \left(L_{\chi_i}^{\oplus j} \setminus \{0\}\right) \to X_i$ which is $\mathcal{O}(\pm 1)$. Letting ζ_i be the first Chern class of this line bundle, it follows from the projective bundle formula for the non-equivariant cobordism and Theorem 2.3 that

$$\Omega^i_T(X) = \prod_{p_1,\dots,p_n \geq 0} \Omega^{i-\left(\sum\limits_{i=1}^{n} p_i\right)}(X)\zeta_1^{p_1}\cdots\zeta_n^{p_n},$$

which is isomorphic to the set of formal power series in $\{\zeta_1, \dots, \zeta_n\}$ of degree i with coefficients in $\Omega^*(X)$. The desired isomorphism of (6.1) is easily deduced from this. $\square$

Let T be a split torus of rank n and let $\{\chi_1, \dots, \chi_n\}$ be a chosen basis of the character group $M = \widehat{T}$. We have then seen that there is a graded ring isomorphism $S = \Omega^*_T(k) \cong \mathbb{L}[[t_1, \dots, t_n]]$, where $t_j = c_1^T(L_{\chi_j})$ for $1 \leq j \leq n$. It follows from (2.7) and the formula $c_1^T(L_1 \otimes L_2) = c_1^T(L_1) +_F c_1^T(L_2)$ that $c_1^T(L_{\chi_j^m}) = mt_j + t_j^2 \sum_{i \geq 0} a_i t_j^i$. In particular, we have for a character $\chi = \prod_{j=1}^n \chi_j^{m_j}$,

$$c_1^T(L_\chi) = \sum_{j=1}^n [m_j]_F t_j = \sum_{j=1}^n m_j t_j + \sum_{|m(\mathbf{t})| \geq 2} a_{m(\mathbf{t})} m(\mathbf{t}). \tag{6.2}$$

Let $S(T)[M^{-1}]$ denote the ring obtained by inverting all non-zero linear forms $\sum_{j=1}^n m_j t_j$ in $S(T)$. Then $S(T)[M^{-1}]$ is also a graded ring and letting $f = \sum_{j=1}^n m_j t_j$ in (6.2), we can write

$$c_1^T(L_\chi) = f\left(1 + f^{-1} \sum_{|m(\mathbf{t})| \geq 2} a_{m(\mathbf{t})} m(\mathbf{t})\right) \tag{6.3}$$

in $S(T)[M^{-1}]$. Notice then that the term inside the parenthesis is a homogeneous element of degree zero in $S(T)[M^{-1}]$ which is invertible. We conclude in particular that $c_1^T(L_\chi)$ is an invertible element of $S(T)[M^{-1}]$. For a smooth k-scheme X with an action of a split torus T, we shall write $\Omega^*_T(X) \otimes_S S[M^{-1}]$ as $\Omega^*_T(X)[M^{-1}]$.

Lemma 6.2 (*Splitting Principle*)**.** *Let T be a split torus of rank n acting trivially on a smooth scheme X and let $\{E_1, \dots, E_s\}$ be a finite collection of vector bundles on X. Then there is a smooth morphism $p : Y \to X$ which is a composition of affine and projective bundle morphisms such that*

(i) *$p^*(E_i)$ is a direct sum of line bundles for $1 \leq i \leq s$,*
(ii) *the pull-back map $p^* : \Omega^*_T(X) \to \Omega^*_T(Y)$ is split injective, where T acts trivially on Y, and*
(iii) *the map $\frac{\Omega^*_T(X)}{(c_i^T(E))} \to \frac{\Omega^*_T(Y)}{(p^*(c_i^T(E)))}$ is split injective for any T-equivariant vector bundle E on X and any $i \geq 0$.*

Proof. If $E = E_1$ is a single vector bundle on X, then it is shown in [28, Lemma 3.24] that there are maps $X_2 \xrightarrow{p_2} X_1 \xrightarrow{p_1} X$ such that p_1 is the natural projection $X_1 = \mathbb{P}(E) \to X$ and p_2 is an affine bundle map. Furthermore, $(p_1 \circ p_2)^*(E) = E' \oplus E''$, where E' is a line bundle. By repeating this process finitely many times, we see that there is a smooth map $p : Y \to X$ which is composition of projective and affine bundle morphisms and such that the condition (i) of the lemma holds on Y.

Letting T act trivially on all the intermediate schemes mapping to X, we see that all the intermediate maps are T-equivariant. Since an affine bundle keeps the (equivariant) cobordism rings invariant, we only need to show (ii) and (iii) when $p : Y \to X$ is a projective bundle morphism.

So let $Y = \mathbb{P}(V) \xrightarrow{p} X$ be a such a projective bundle of relative dimension r and let $\zeta \in \Omega^1(Y)$ be the first Chern class of the tautological bundle $\mathcal{O}_Y(-1)$ on Y. Then $\zeta = c_1^T\left(\mathcal{O}_Y(-1) \otimes L_{\chi_0}\right)$, where χ_0 is the trivial character of T. It also follows from Lemma 6.1 that p^* is the natural map of the graded power series rings $p^* : \Omega^*(X)[[t_1, \ldots, t_n]] \to \Omega^*(Y)[[t_1, \ldots, t_n]]$. Moreover, the projective bundle formula for the ordinary cobordism and the projection formula for the equivariant cobordism (*cf.* Theorem 2.5(vii)) imply that

$$p_*\left(\zeta^r \cdot p^*(x)\right) = x \cdot p_*(\zeta^r) = x \tag{6.4}$$

for any $x \in \Omega_T^*(X)$. This immediately implies the condition (ii) of the lemma.

To prove the final assertion, we first deduce from the standard properties of the Chern classes that $p^*\left(c_i^T(E)\right) = c_i^T\left(p^*(E)\right)$ for any T-equivariant vector bundle E on X. Setting $x = c_i^T(E) \in \Omega_T^*(X)$ and $y = c_i^T\left(p^*(E)\right)$, we conclude that p^* induces the map

$$\overline{p^*} : \frac{\Omega_T^*(X)}{(x)} \to \frac{\Omega_T^*(Y)}{(y)}.$$

Let $q_* : \Omega_T^*(Y) \to \Omega_T^*(X)$ be the map $q(z) = p_*\left(\zeta^r \cdot z\right)$. Then q_* is clearly $\Omega_T^*(X)$-linear. Moreover, it follows from the projection formula and (6.4) that q_* induces maps

$$\frac{\Omega_T^*(X)}{(x)} \xrightarrow{\overline{p^*}} \frac{\Omega_T^*(Y)}{(y)} \xrightarrow{\overline{q_*}} \frac{\Omega_T^*(X)}{(x)}$$

such that the composite is identity. This completes the proof of the lemma. □

Notation. All results in the rest of this paper will be proven with the rational coefficients. In order to simplify our notations, an abelian group A from now on will actually mean the $\mathbb{Q}$-vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$, and an inverse limit of abelian groups will mean the limit of the associated $\mathbb{Q}$-vector spaces. In particular, all cohomology groups will be considered with the rational coefficients and $\Omega_i^G(X) = \varprojlim_j \Omega_i^G(X)_j$, where each $\Omega_i^G(X)_j$ at jth level is the non-equivariant cobordism group with rational coefficients.

Lemma 6.3. *Let T be a split torus of rank n acting trivially on a smooth variety X and let E be a T-equivariant vector bundle of rank d on X which is a direct sum of line bundles. Assume that in the eigenspace decomposition of E with respect to T, the submodule corresponding to the trivial character is zero. Then $c_d^T(E)$ is a non-zero divisor in $\Omega_T^*(X)$ and is invertible in $\Omega_T^*[M^{-1}]$.*

Proof. To prove the first assertion, it suffices to show by Lemma 6.1 that $c_d^T(E)$ is a non-zero divisor in $\Omega^*(X)[[t_1, \ldots, t_n]]$. As in the proof of Lemma 5.2, it suffices to show that $c_d^T(E)$ is a non-zero divisor in the formal power series ring $\widehat{\Omega^*(X)[[\mathbf{t}]]}$, where $\mathbf{t} = (t_1, \ldots, t_n)$.

Let $q : X \to \mathrm{Spec}(k)$ be the structure map. Since T acts on X trivially, E has a unique direct sum decomposition

$$E = \bigoplus_{i=1}^{m} E_i \otimes q^*(L_{\chi_i}),$$

where each E_i is an ordinary vector bundle on X and L_{χ_i} is the line bundle in $\mathrm{Pic}_T(k)$ corresponding to a character χ_i of T. Since $\mathrm{rank}(E) = d$, the Whitney sum formula (*cf.* Theorem 2.5, [25, Proposition 4.1.15]) yields

$$c_d^T(E) = \prod_{i=1}^{m} c_{d_i}^T \left(E_i \otimes q^* \left(L_{\chi_i}\right)\right).$$

We can thus assume that $E = E_\chi \otimes q^*(L_\chi)$, where χ is not a trivial character by our assumption. Since $\mathrm{Pic}(BT) \cong \mathbb{Q}^n$ and since χ is not trivial, we can extend χ to a basis of $\mathbb{Q}^n$. We can now use (6.2) and Lemma 5.1 to assume that $c_1^T(L_\chi) = t_1$.

We can also write $E = \oplus_{i=1}^s L_i$, where each L_i is a line bundle on X. The Whitney sum formula again implies that

$$\begin{aligned} c_d^T\left(E \otimes L_\chi\right) &= c_d^T\left(\bigoplus_{i=1}^{s}\left(L_i \otimes L_\chi\right)\right) \\ &= \prod_{i=1}^{s} c_1^T\left(L_i \otimes L_\chi\right) \\ &= \prod_{i=1}^{s} F\left(t_1, c_1(L_i)\right), \end{aligned} \tag{6.5}$$

where $F(u, v)$ is the universal formal group law on $\mathbb{L}$. Since $c_1(L) \in \Omega^1(X)$ and since $\Omega^{>\dim(X)}(X) = 0$, we see that $c_1(L_i)$ is nilpotent in $\Omega^*(X)$. In fact, this implies that $c_j(E)$ is nilpotent in $\Omega^*(X)$ for all vector bundles E on X and for all $j \geq 1$. We now apply Lemma 5.3 with $R = \mathbb{Q}$ and $A = \Omega^*(X)$ to conclude that $c_d^T\left(E \otimes L_\chi\right)$ is a non-zero divisor in $\widehat{\Omega^*(X)[\![\mathbf{t}]\!]}$.

To prove the invertibility of $c_d^T(E)$ in $\Omega_T^*[M^{-1}]$, we can again use the above reductions to assume that $E = L \otimes L_\chi$ where χ is not a trivial character of T. In this case, $F(f, v)$ is of the form $f + v\left(1 + fF'(f, v)\right)$, where $1 + fF'(f, v)$ is an invertible element of degree zero. In particular, we can write $F(f, v) = uf + v$, where $u = \left(1 - fF' + f^2F'^2 - \cdots\right)$ is an invertible element of degree zero in $S^{(n)}$. In particular, v is nilpotent. We have seen in (6.3) that f is invertible in $S[M^{-1}]$. Since u is invertible in $\Omega_T^*(X)$ and v is nilpotent, it follows that $uf + v$ is invertible in $\Omega_T^*[M^{-1}]$. □

Lemma 6.4. *Let T be a split torus of rank n acting trivially on a smooth variety X. Let χ be a non-trivial character of T and let $E = \bigoplus \left(E_q \otimes L_{\chi^q}\right)$, where each E_q is either zero or a direct sum of line bundles on X. Assume that $d = \mathrm{rank}(E) \geq 1$. Then $c_d^T(E)$ is of the form*

$$c_d^T(E) = u\left(x^d + \gamma_{d-1}x^{d-1} + \cdots + \gamma_1 x + \gamma_0\right),$$

with $x = c_1^T\left(L_\chi\right)$ such that

(i) *u is an invertible and homogeneous element of degree zero in $\Omega_T^*(X)$, and*
(ii) *$\gamma_i = u_i\gamma_i'$, where u_i is an invertible and homogeneous element of degree zero in $\Omega_T^*(X)$ and γ_i' is a product of the first Chern classes of line bundles on X.*

In particular, each γ_i is a nilpotent homogeneous element of $\Omega_T^*(X)$.

Proof. By Lemma 6.1, $\Omega^*_T(X)$ is a graded power series ring $\Omega^*(X)[\![t_1, \ldots, t_n]\!]$. Setting $E_q = \oplus_{i=1}^{d_q} L_{q,i}$, $v_{q,i} = c_1(L_{q,i})$ and $x_q = c_1^T\left(L_{\chi^q}\right)$, we can write

$$c^T_{d_q}\left(E_q \otimes L_{\chi^q}\right) = \prod_{i=1}^{d_q} F(x_q, v_{q,i}) = \prod_{i=1}^{s} u_{q,i}(x_q + v_{q,i}), \tag{6.6}$$

where $u_{q,i}$ is an invertible homogeneous element of degree zero in $\Omega^*_T(X)$. Using the formula $x_q = [q]_F x = qx + x^2 \sum_{i\geq 0} a_i x^i$, we can write $x_q = qx(1 + xg_q(x))$, where $1 + xg_q(x)$ is a homogeneous element of degree zero in $S(T)$ which is invertible. Setting $u_q = q(1 + xg_q(x))$, we obtain

$$c^T_{d_q}\left(E_q \otimes L_{\chi^q}\right) = \prod_{i=1}^{d_q} u_{q,i}(u_q x + v_{q,i}). \tag{6.7}$$

The desired form of $c^T_d(E)$ follows immediately from this and the Whitney sum formula. □

Corollary 6.5. *Let T be a split torus of rank n acting trivially on a smooth variety X and let E be a T-equivariant vector bundle of rank d on X. Assume that in the eigenspace decomposition of E with respect to T, the submodule corresponding to the trivial character is zero. Then $c^T_d(E)$ is a non-zero divisor in $\Omega^*_T(X)$ and is invertible in $\Omega^*_T[M^{-1}]$.*

Proof. This follows immediately from Lemmas 6.2 and 6.3. □

7. Localization theorems

We now prove our following localization theorems for the equivariant cobordism of smooth filtrable varieties with torus action.

Theorem 7.1. *Let X be a smooth and filtrable variety with the action of a split torus T of rank n. Let $i : X^T \hookrightarrow X$ be the inclusion of the fixed point locus. Then the S-algebra map*

$$i^* : \Omega^*_T(X) \to \Omega^*_T(X^T)$$

is injective and is an isomorphism over $S[M^{-1}]$.

Proof. Consider the filtration of X as in (4.6). It follows from the description of this filtration that the tangent space $T_x(Z_0)$ is the weight zero subspace $T_x(X)_0$ of the tangent space $T_x(X)$. Moreover, the bundle $X_0 \to Z_0$ corresponds to the positive weight space $T_x(X)_+$ (*cf.* [4, Theorem 3.1]). It follows from this that in the eigenspace decomposition of the T-equivariant vector bundle $\phi_0 : X_0 \to Z_0$, the submodule corresponding to the trivial character is zero. We conclude from Corollary 6.5 that the top Chern class of the normal bundle of Z_0 in X a non-zero divisor in $\Omega^*_T(Z_0)$. Since $X_0 \xrightarrow{\phi_0} Z_0$ is a T-equivariant vector bundle, it follows from the homotopy invariance that $c_0 := c^T_{d_0}\left(N_{X_0/X}\right)$ is a non-zero divisor in $\Omega^*_T(X_0)$. In particular, letting $U_0 = X \setminus X_0$, we conclude from Proposition 4.1 that the sequence

$$0 \to \Omega^*_T(X_0) \xrightarrow{i_*} \Omega^*_T(X) \xrightarrow{j^*} \Omega^*_T(U_0) \to 0 \tag{7.1}$$

is exact and there is a ring isomorphism

$$\Omega^*_T(X) \xrightarrow{\cong} \Omega^*_T(X_0) \underset{\widetilde{\Omega^*_T(X_0)}}{\times} \Omega^*_T(U_0), \tag{7.2}$$

where $\widetilde{\Omega^*_T(X_0)} = \Omega^*_T(X_0)/(c_0)$. In other words, there is a short exact sequence

$$0 \to \Omega^*_T(X) \xrightarrow{(i^*,j^*)} \Omega^*_T(X_0) \times \Omega^*_T(U_0) \to \frac{\Omega^*_T(X_0)}{(c_0)} \to 0. \tag{7.3}$$

Furthermore, c_0 is invertible in $\Omega^*_T(X_0)[M^{-1}]$ by Corollary 6.5. The proof of the theorem follows immediately from (7.3) and an induction argument once we observe that U_0 is itself a smooth and filtrable variety with smaller number of strata than in X. □

Corollary 7.2. *Let X be a smooth and projective variety with the action of a split torus T of rank n. Let $i : X^T \hookrightarrow X$ be the inclusion of the fixed point locus. Then the S-algebra map*

$$i^* : \Omega^*_T(X) \to \Omega^*_T(X^T)$$

is injective and is an isomorphism over $S[M^{-1}]$.

Proof. This follows immediately from Theorems 4.5 and 7.1. □

Corollary 7.3 (*Localization Formula*)**.** *Let X be a smooth and projective variety with an action of a split torus T of rank n. Then the push-forward map*

$$\Omega^T_*(X^T) \xrightarrow{i^T_*} \Omega^T_*(X)$$

is an isomorphism of S-modules over $S[M^{-1}]$.

In particular, one has for any $\alpha \in \Omega^*_T(X)[M^{-1}]$,

$$\alpha = \sum_F i_{F*} \frac{i^*_F(\alpha)}{c^T_{d_F}(N_{F/X})},$$

where the sum is over the components F of X^T and d_F is the codimension of F in X.

Proof. This follows immediately from Corollary 7.2 and Proposition 3.1. □

Remark 7.4. The analogous result for the T-equivariant Chow groups was proven by Edidin and Graham [10] and also by Brion [4]. Edidin and Graham prove their result for Chow groups also for singular varieties by a different method.

Let $E_1, \dots, E_s$ be a set of T-equivariant vector bundles on an n-dimensional smooth projective variety X. Let $p(x^1_1, \dots, x^1_s, \dots, x^n_1, \dots, x^n_s)$ be a polynomial of weighted degree n, where x^i_j has weighted degree i. Let $p(E_1, \dots, E_s)$ denote the polynomial in the Chern classes of $E_1, \dots, E_s$ obtained by setting $x^i_j = c_i(E_j)$. Let $q_X : X \to \mathrm{Spec}(k)$ be the structure map. Since X is projective, there is a push-forward map $\deg = q_{X*} : \Omega_0(X) \to \Omega_0(k) \cong \mathbb{Q}$. As an immediate consequence of the above localization formula, we get the following *Bott residue formula* for the algebraic cobordism which computes the degree of the cobordism cycle $\deg\,(p(E_1, \dots, E_s) \cap [X])$ in terms of the restriction of E_i on X^T.

Corollary 7.5 (*Bott Residue for Cobordism*)**.** *Let $E_1, \dots, E_s$ be a set of T-equivariant vector bundles on an n-dimensional smooth projective variety X as above. Then*

$$\deg\,(p(E_1, \dots, E_s) \cap [X]) = \sum_F q_{F*} \left(\frac{p^T(E|_F) \cap [F]_T}{c^T_{d_F}(N_{F/X})} \right).$$

Proof. This is a direct consequence of Corollary 7.3 and can be proved exactly in the same way as the proof of the analogous result for Chow groups in [10, Theorem 3]. We skip the details. □

7.1. Description of the image of i^*

The following result for equivariant Chow groups was proven by Brion in [4, Theorem 3.3].

Theorem 7.6. *Let X be a smooth filtrable variety where a split torus T of rank n acts with finitely many isolated fixed points. Let $i : X^T \to X$ be the inclusion of the fixed point locus. Then the image of $i^* : \Omega_T^*(X) \to \Omega_T^*(X^T)$ is the intersection of the images of the restriction maps*

$$i_{T'}^* : \Omega_T^*(X^{T'}) \to \Omega_T^*(X^T)$$

where T' runs over all subtori of codimension one in T.

Proof. This follows from the results of the previous sections and by following the strategy of Brion. We give the main steps. We prove by induction over the number of strata. If X is the unique stratum, it is a T-equivariant vector bundle over X^T, in which case both the maps i^* and $i_{T'}^*$ are surjective by homotopy invariance. In the general case, let $X_0 \subset X$ be a closed stratum and let U_0 be its complement. Then U_0 is a filtrable smooth variety with smaller number of strata where T acts with isolated fixed points. We have seen in the proof of Theorem 7.1 that there are exact sequences

$$0 \to \Omega_T^*(X_0) \xrightarrow{i_{0*}} \Omega_T^*(X) \xrightarrow{j_0^*} \Omega_T^*(U_0) \to 0 \tag{7.4}$$

$$0 \to \Omega_T^*(X) \xrightarrow{(i_0^*, j_0^*)} \Omega_T^*(X_0) \times \Omega_T^*(U_0) \to \frac{\Omega_T^*(X_0)}{(c_0)} \to 0 \tag{7.5}$$

where $c_0 = c_{d_0}^T\left(N_{X_0/X}\right)$ is the top Chern class of the normal bundle of X_0 in X. Moreover, we have $i_0^* \circ i_{0*} = c_0$.

We identify $\Omega_T^*(X_0)$ with $\Omega_T^*(Z_0)$ which in turn is identified with $\Omega^*(Z_0)[[t_1, \dots, t_n]]$ by Lemma 6.1. In particular, we can evaluate c_0 by identifying its class in $\Omega_T^*(Z_0)$ under the pull-back via the zero-section. Under this identification, we can write $N_{X_0/X}$ as

$$N_{X_0/X} = \bigoplus_{j=1}^{s} E_j$$

where each E_j is obtained by grouping together E_χ and $E_{\chi'}$ when the characters χ and χ' are multiples of a common primitive character of T. In particular, we can write $E_j = \oplus(E_q \otimes L_{\chi_j^q})$, where χ_j is a primitive character. We then have

$$c_0 = \prod_{j=1}^{s} c_{d_j}^T(E_j) = \prod_{j=1}^{s} c_{\chi_j}. \tag{7.6}$$

Using (7.4)–(7.6) and following the argument in [4, Theorem 3.3], we only need to show that

$$(c_0) = \bigcap_{j=1}^{s} \left(c_{d_j}^T(E_j)\right) \quad \text{as ideals in } \Omega^*(Z_0)[[t_1, \dots, t_n]]. \tag{7.7}$$

Since all vector bundles on Z_0 are trivial, it follows from Lemma 6.4 that $c^T_{d_j}(E_j) = u_j \left(c^T_1(L_{\chi_j})\right)^{d_j}$ where u_j is invertible in $\Omega^*_T(Z_0) \cong S$. Since $\{\chi_1, \ldots, \chi_s\}$ is pairwise non-proportional and since we are working with the rational coefficients, setting $\gamma_j = \left(c^T_1(L_{\chi_j})\right)^{d_j}$, it follows from Lemma 5.4 that

$$(\gamma_1 \cdots \gamma_s) = \bigcap_{j=1}^{s} (\gamma_j).$$

Since each u_j is a unit in $\Omega^*_T(Z_0)$, the equality of (7.7) now follows. This completes the proof of the theorem. □

Remark 7.7. The analogue of Theorem 7.6 for the Chow groups was proven by Brion without any condition on the nature of the fixed point locus. At this moment, the author is not confident that the result for cobordism might hold in such generality because of the complicated nature of the associated formal group law. However, the above theorem covers most of the situations that occur in practice. In particular, this is applicable for toric varieties, all flag varieties and all symmetric varieties of minimal rank. These are the cases which have been of main interest of enumerative geometry in recent past.

The analogue of the following result for the equivariant Chow groups was proven by Brion in [4, Theorem 3.4].

Theorem 7.8. *Let X be a smooth filtrable variety where a split torus T acts with finitely many fixed points $x_1, \ldots, x_m$ and with finitely many invariant curves. Then the image of*

$$i^* : \Omega^*_T(X) \to \Omega^*_T(X^T)$$

is the set of all $(f_1, \ldots, f_m) \in S^m$ such that $f_i \equiv f_j (\mathrm{mod}\ \chi)$ whenever x_i and x_j lie in an invariant irreducible curve C and the kernel of the T-action on C is the kernel of the character χ.

Proof. This is an easy consequence of Theorem 7.6, as shown by Brion for the equivariant Chow groups. Notice that $\Omega^*_T(X^T)$ is identified with S^m by Lemma 6.1.

Let π be a non-trivial character of T. Then the space $X^{\mathrm{Ker}(\pi)}$ is at most one-dimensional by our assumption, and is smooth.

In particular, it is a disjoint union of points and smooth connected curves. If C is such a curve and contains a unique fixed point x, then $i^*_x : \Omega^*_T(C) \to \Omega^*_T(\{x\}) = S$ is an isomorphism. Otherwise, C must be isomorphic to the projective line with fixed points x and y. It follows from Example 4.13 (see the exact sequence (4.10)) that the image of

$$i^*_C : \Omega^*_T(C) \to \Omega^*_T(C^T)$$

is of the desired form. Since every codimension one subtorus in T is the kernel of a non-trivial character, the main result now follows from Theorem 7.6. □

Remark 7.9. We remark here that the localization theorems and the Bott residue formula for the equivariant Chow groups can be recovered from the above results for the equivariant cobordism and [19, Proposition 7.2].

8. Cobordism ring of flag varieties

Let G be a connected reductive group with a split maximal torus T of rank n and let B be a Borel subgroup of G containing T. Let $X = G/B$ be the associated flag variety of the left cosets of B in G. We have shown in Corollary 4.10 that $\Omega^*_T(X)$ is a free $S(T)$-module on the classes of the Bott–Samelson varieties. The similar conclusion holds for the non-equivariant cobordism of X as well. The Schubert calculus for the non-equivariant algebraic cobordism has been studied recently by Calmès, Petrov and Zainoulline in [6] (see also [16]). Using this, Calmès–Petrov–Zainoulline have obtained a description of the cobordism ring $\Omega^*(X)$.

Theorem 8.1 can be viewed as the uncompleted version of the result of [6] and gives a more geometric expression of $\Omega^*(X)$ in terms of the cobordism ring of the classifying space of the maximal torus T. This formula is a direct analogue of a similar result of Demazure [7] for the Chow ring of X. Theorem 8.1 is used in [20] to describe the algebraic cobordism of flag bundles.

Consider the forgetful map $\Omega^*_G(X) \xrightarrow{r^G_X} \Omega^*(X)$. Using Proposition 2.4, we can identify $\Omega^*_G(X) = \Omega^*_G(G/B)$ with $\Omega^*_B(k) \cong S(T)$. Thus r^G_X is same as the map $S(T) \xrightarrow{r^G_X} \Omega^*(X)$.

Theorem 8.1. *The ring homomorphism r^G_X descends to an isomorphism of $\mathbb{L}$-algebras*

$$c_X : S(T) \otimes_{S(G)} \mathbb{L} \to \Omega^*(X). \tag{8.1}$$

In particular, if $G = GL_n(k)$, then $\Omega^*(G/B)$ is isomorphic to the quotient of the standard polynomial ring $\mathbb{L}[t_1, \dots, t_n]$ by the ideal I generated by the homogeneous symmetric polynomials of strictly positive degree.

Proof. We first prove the isomorphism of c_X. The forgetful map $S(T) = \Omega^*_G(X) \to \Omega^*(X)$ can be geometrically described as follows. Let L_χ be the T-equivariant line bundle on Spec(k) corresponding to the character χ of T. This uniquely defines a line bundle $L_\chi \overset{B}{\times} G \to G/B = X$ on X. We denote this line bundle by $L_{\chi,X}$. The formal group law for the Chern classes in $S(T)$ and $\Omega^*(X)$ implies that the assignment $\chi \mapsto L_{\chi,X}$ induces an $\mathbb{L}$-algebra homomorphism

$$S(T) \to \Omega^*(X), \qquad c^T_1(L_\chi) \mapsto c_1\left(L_{\chi,X}\right) \tag{8.2}$$

and it is easy to see from the definition of r^G_X in (2.5) and the identification $\Omega^*_G(X) \xrightarrow{\cong} S(T)$ that this map descends to the map c_X above.

We identify $S(T)$ with the graded power series ring $\mathbb{L}[[t_1, \dots, t_n]]$ and let I_T be the augmentation ideal $(t_1, \dots, t_n)$. Taking the I_T-adic completions, we get the map $\widehat{S(T)} = \widehat{\mathbb{L}[[\mathbf{t}]]} \xrightarrow{\widehat{c}_X} \widehat{\Omega^*(X)}$.

We claim that c_X is surjective. To prove this, let N denote the cokernel of the map $S(T) \to \Omega^*(X)$ and consider the exact sequence of $S(T)$-modules

$$0 \to M \to \Omega^*(X) \to N \to 0, \tag{8.3}$$

where $M = \text{Image}\,(S(T) \to \Omega^*(X))$. Using the subspace topology on M given by the descending chain of submodules $\{I^n_T \Omega^*(X) \cap M\}_{n \ge 0}$, we get an exact sequence of completions

$$0 \to \widehat{M} \to \widehat{\Omega^*(X)} \to \widehat{N} \to 0 \tag{8.4}$$

by [27, Theorem 8.1].

Since $c_X(t_i) = c_1(L_{t_i,X}) \in \Omega^1(X)$ and since $\Omega^{>\dim(X)}(X) = 0$, we see that $c_X(t_i)$ is nilpotent for all i. In particular, $\Omega^*(X)$ is I_T-adically complete and hence so is N. In particular, the exact sequence (8.4) is same as

$$0 \to \widehat{M} \to \Omega^*(X) \to N \to 0. \tag{8.5}$$

On the other hand, the map $\widehat{S(T)} \to \Omega^*(X)$ is surjective by [6, Corollary 13.6]. Since the I_T-adic topology on M is finer than the subspace topology, we have natural maps $\widehat{S(T)} \to \widehat{M}_{I_S} \to \widehat{M} \to \Omega^*(X) = \widehat{\Omega^*(X)}$. We conclude that the first arrow in (8.5) is surjective and hence $N = 0$, which proves the surjectivity of c_X.

Since $S(T)$ is a subring of $\widehat{S(T)}$, it follows from [6, Corollary 13.7] that the kernel of the map $S(T) \to \Omega^*(X)$ is generated by $(I_T)^W$, where W is the Weyl group of G with respect to T. On the other hand, the commutative diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & I_G \arrow[r] \arrow[d] & S(G) \arrow[r] \arrow[d] & \mathbb{L} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I_T \arrow[r] & S(T) \arrow[r] & \mathbb{L} \arrow[r] & 0, \end{tikzcd}$$

the exactness of the functor of W-invariance on the category of $\mathbb{Q}[W]$-modules and Theorem 2.7 together imply that $I_G = (I_T)^W$. In particular, we get exact sequence

$$I_G \otimes_{S(G)} S(T) \to S(T) \to \Omega^*(X) \to 0$$

which completes the proof of the main part of the theorem.

If $G = GL_n(k)$, then we can choose T to be the set of all diagonal matrices which is clearly split and W is just the symmetric group S_n which acts on $S(T) = \mathbb{L}[\![t_1, \dots, t_n]\!]$ by permuting the variables. In particular, $S(G)$ is the graded power series subring of S generated by the elementary symmetric polynomials. It is easy to see from this that $S \otimes_{S(G)} \mathbb{L}$ has the desired form. □

Remark 8.2. For $G = GL_n(k)$, the cobordism ring of G/B in the above explicit form has been recently computed by Hornbostel and Kiritchenko [16] with integer coefficients. They achieve this using the approach of Schubert calculus.

As an easy consequence of Theorems 2.5, 3.4 and 8.1, one gets the following description of the rational cobordism ring of connected reductive groups.

Corollary 8.3. *For a connected and split linear algebraic group G, the natural ring homomorphism*

$$\mathbb{L} \to \Omega^*(G)$$

is an isomorphism.

Proof. Let G^u be the unipotent radical of G and let $L \subset G$ be a Levi subgroup (which exists because we are in characteristic zero). Since $G \to L$ is principal G^u bundle and since G^u is a split unipotent group (again because of characteristic zero), the homotopy invariance implies that $\Omega^*(L) \xrightarrow{\cong} \Omega^*(G)$. So we can assume that G is a connected and split reductive group.

We choose a split maximal torus T of G of rank n and let $\{\chi_1, \dots, \chi_n\}$ be a basis of the character group of T. By Theorem 3.4, there is an isomorphism of rings $\Omega^*_T(G) \otimes_{S(T)} \mathbb{L} \xrightarrow{\cong} \Omega^*(G)$. On the other hand, it follows from Theorem 2.5(v) that $\Omega^*_T(G) \cong \Omega^*(G/T) \cong$

$\Omega^*(G/B)$. In particular, we get

$$\begin{aligned}\Omega^*(G) &\cong \frac{\Omega^*_T(G)}{(t_1,\dots,t_n)} \cong \frac{\Omega^*(G/B)}{(t_1,\dots,t_n)}\\ &\cong \frac{S(T)\otimes_{S(G)}\mathbb{L}}{(t_1,\dots,t_n)}\\ &\cong \mathbb{L},\end{aligned}$$

where the third isomorphism follows from Theorem 8.1. $\square$

Remark 8.4. For a complex connected Lie group G such that its maximal compact subgroup is simply connected, $\Omega^*(G)$ has been computed with integer coefficients by Yagita [34] using non-equivariant techniques.

It is known (*cf.* [13, Theorem 1.1], [4, Corollary 2.3]) that the forgetful map from the equivariant K_0 (resp. Chow groups) to the ordinary K_0 (resp. Chow groups) is surjective with the rational coefficients. As another application of above results, we obtain a similar result for the cobordism.

Theorem 8.5. *Let G be a connected linear algebraic group acting on a k-scheme X of dimension d. Then the forgetful map $r^G_X : \Omega^G_*(X) \to \Omega_*(X)$ descends to an $\mathbb{L}$-linear map $\Omega^G_*(X)\otimes_{S(G)}\mathbb{L} \to \Omega_*(X)$ and is surjective with rational coefficients.*

Proof. By [19, Proposition 8.2], we can assume that G is reductive with a maximal torus T. There is a finite field extension $k \hookrightarrow l$ such that T_l is split. We then have a commutative diagram

$$\begin{tikzcd} \Omega^{G_l}_*(X_l) \arrow[r, "\pi^G_*"] \arrow[d] & \Omega^G_*(X) \arrow[d] \\ \Omega_*(X_l) \arrow[r, "\pi_*"'] & \Omega_*(X). \end{tikzcd} \tag{8.6}$$

The map $\pi_* \circ \pi^*$ is multiplication by $[l : k]$ by [25, Lemma 2.3.5]. In particular, the bottom horizontal map is surjective with rational coefficients. Hence, we can assume that T is a split torus. Let W denote the Weyl group of G with respect to T. The commutative diagram

$$\begin{tikzcd} I_G \otimes_{S(G)} \Omega^G_*(X) \arrow[r] \arrow[d] & \Omega^G_*(X) \arrow[r] \arrow[d] & \Omega_*(X) \arrow[d] \\ I_T \otimes_{S(T)} \Omega^T_*(X) \arrow[r] & \Omega^T_*(X) \arrow[r] & \Omega_*(X) \end{tikzcd}$$

and Theorem 3.4 imply that r^G_X descends to the map $\Omega^G_*(X)\otimes_{S(G)}\mathbb{L} \xrightarrow{\bar{r}^G_X} \Omega_*(X)$ which is a ring homomorphism if X is smooth.

To show its surjectivity, we use Theorem 2.7 to get

$$\Omega^G_*(X)\otimes_{S(G)}\mathbb{L} \cong \left(\Omega^T_*(X)\right)^W \otimes_{S(G)}\mathbb{L} \cong \left(\Omega^T_*(X)\otimes_{S(G)}\mathbb{L}\right)^W.$$

On the other hand, we have

$$\Omega^T_*(X)\otimes_{S(G)}\mathbb{L} \cong \Omega^T_*(X)\otimes_{S(T)}\left(S(T)\otimes_{S(G)}\mathbb{L}\right) \twoheadrightarrow \Omega^T_*(X)\otimes_{S(T)}\mathbb{L} \cong \Omega_*(X),$$

where the last isomorphism follows from Theorem 3.4. The exactness of the functor of taking the W-invariance on the category of $\mathbb{Q}[W]$-modules implies that $\left(\Omega_*^T(X) \otimes_{S(G)} \mathbb{L}\right)^W \twoheadrightarrow (\Omega_*(X))^W = \Omega_*(X)$. This proves the surjectivity $\Omega_*^G(X) \twoheadrightarrow \Omega_*^G(X) \otimes_{S(G)} \mathbb{L} \twoheadrightarrow \Omega_*(X)$. $\square$

Acknowledgments

Some of the results in Section 7 of this paper were inspired by the discussion with Michel Brion at the Institute Fourier, Université de Grenoble in June, 2010. The author takes this opportunity to thank Brion for invitation and financial support during the visit.

References

[1] A. Bialynicki-Birula, Some theorems on actions of algebraic groups, Ann. of Math. (2) 98 (1973) 480–497.
[2] A. Borel, Linear Algebraic Groups, second ed., in: GTM, vol. 8, Springer-Verlag, 1991.
[3] P. Bressler, S. Evens, Schubert calculus in complex cobordism, Trans. Amer. Math. Soc. 331 (2) (1992) 799–813.
[4] M. Brion, Equivariant Chow groups for torus actions, Transform. Groups 2 (3) (1997) 225–267.
[5] M. Brion, Lectures on the geometry of flag varieties, in: Topics in Cohomological Studies of Algebraic Varieties, in: Trends in Math., Birkhäuser, Basel, 2005, pp. 33–85.
[6] B. Calmès, V. Petrov, K. Zainoulline, Invariants, torsion indices and oriented cohomology of complete flags, 2009. arxiv:math.AG/0905.1341.
[7] M. Demazure, Invariants symmétriques entiers des groupes de Weyl et torsion, Invent. Math. 21 (1973) 53–61.
[8] D. Deshpande, Algebraic cobordism of classifying spaces, 2009. arxiv:math.AG/0907.4437.
[9] D. Edidin, W. Graham, Equivariant intersection theory, Invent. Math. 131 (1998) 595–634.
[10] D. Edidin, W. Graham, Localization in equivariant intersection theory and the Bott residue formula, Amer. J. Math. 120 (3) (1998) 619–636.
[11] D. Eisenbud, Commutative Algebra with a View Towards Algebraic Geometry, in: Graduate Text in Mathematics, vol. 150, Springer, 1994.
[12] W. Fulton, Intersection Theory, second ed., Springer-Verlag, 1998.
[13] W. Graham, The forgetful map in rational K-theory, Pacific J. Math. 236 (1) (2008) 45–55.
[14] J. Heller, J. Malagón-López, Equivariant algebraic cobordism, J. Reine Angew. Math. (2012) http://dx.doi.org/10.1515/crelle-2011-0004.
[15] W. Hesselink, Concentration under actions of algebraic groups, in: Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 33rd Year (Paris), in: Lect. Notes Math., vol. 867, 1980, pp. 55–89.
[16] J. Hornbostel, V. Kiritchenko, Schubert calculus for algebraic cobordism, J. Reine Angew. Math. 656 (2011) 59–85.
[17] V. Kiritchenko, A. Krishna, Equivariant Cobordism of Flag Varieties and of Symmetric Varieties, 2011. arxiv:math.AG/1104.1089.
[18] A. Krishna, Equivariant K-theory and higher Chow groups of smooth varieties, 2009. arxiv:math.AG/0906.3109.
[19] A. Krishna, Equivariant cobordism of schemes, Doc. Math. 17 (2012) 95–134.
[20] A. Krishna, Cobordism of flag bundles, 2010. arxiv:math.AG/1007.1083.
[21] A. Krishna, V. Uma, Cobordism ring of toric varieties, 2010. arxiv:math.AG/1011.0573.
[22] M. Levine, Oriented cohomology, Borel–Moore homology, and algebraic cobordism, Michigan Math. J. 57 (2008) 523–572. Special volume in honor of Melvin Hochster.
[23] M. Levine, Comparison of cobordism theories, J. Algebra 322 (9) (2009) 3291–3317.
[24] M. Levine, F. Morel, Algebraic cobordism I, Preprint, 2002. www.math.uiuc.edu/K-theory/0547.
[25] M. Levine, F. Morel, Algebraic Cobordism, in: Springer Monographs in Mathematics, Springer, Berlin, 2007.
[26] M. Levine, R. Pandharipande, Algebraic cobordism revisited, Invent. Math. 176 (1) (2009) 63–130.
[27] H. Matsumura, Commutative Ring Theory, in: Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, 1997.
[28] I. Panin, Oriented cohomology theories of algebraic varieties, K-Theory 30 (2003) 265–314.
[29] H. Sumihiro, Equivariant completion II, J. Math. Kyoto 15 (1975) 573–605.
[30] R. Thomason, Equivariant algebraic vs. topological K-homology Atiyah–Segal-style, Duke Math. J. 56 (1988) 589–636.
[31] B. Totaro, The Chow ring of a classifying space, in: Algebraic K-Theory, Seattle, WA, 1997, in: Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., 1999, pp. 249–281.

[32] G. Vezzosi, A. Vistoli, Higher algebraic K-theory for actions of diagonalizable groups, Invent. Math. 153 (1) (2003) 1–44.
[33] V. Voevodsky, $\mathbb{A}^1$-homotopy theory, in: Proceedings of the International Congress of Mathematicians, Berlin, 1998, in: Doc. Math., vol. 1, 1998, pp. 579–604.
[34] N. Yagita, Algebraic cobordism of simply connected Lie groups, Math. Proc. Cambridge Philos. Soc. 139 (2) (2005) 243–260.