ABSOLUTE CHOW–KÜNNETH DECOMPOSITION FOR RATIONAL HOMOGENEOUS BUNDLES AND FOR LOG HOMOGENEOUS VARIETIES

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Abstract. In this paper, we prove that rational homogeneous bundles and log homogeneous varieties (studied by M. Brion) have an absolute Chow–Künneth decomposition. This strengthens the earlier paper by us on small dimensional varieties with a NEF tangent bundle, and using Hwang-Mok's results, we get the results for new cases in higher dimension as well, for instance upto dimension four.

1. Introduction

Suppose X is a nonsingular projective variety of dimension n defined over the complex numbers. Let $CH^i(X) \otimes \mathbb{Q}$ be the Chow group of codimension i algebraic cycles modulo rational equivalence, with rational coefficients. Jacob Murre [Mu2], [Mu3] has made the following conjecture which leads to a filtration on the rational Chow groups:

Conjecture: The motive $h(X) := (X, \Delta_X)$ of X has a Chow-Künneth decomposition:

$$\Delta_X = \sum_{i=0}^{2n} \pi_i \in CH^n(X \times X) \otimes \mathbb{Q}$$

such that π_i are orthogonal projectors (see §2.2).

In this paper, *absolute* Chow–Künneth decomposition (resply. projectors) is the same as Chow–Künneth decomposition (resply.projectors). We write 'absolute' to emphasize the difference with 'relative' Chow–Künneth projectors which will appear in the paper.

Some examples where this conjecture is verified are: curves, surfaces, a product of a curve and surface [Mu], [Mu3], abelian varieties and abelian schemes [Sh],[De-Mu], unir-uled threefolds [dA-Ml], elliptic modular varieties [Go-Mu], [GHMu2]), universal families over Picard modular surfaces [MWYK] and finite group quotients (maybe singular) of abelian varieties [Ak-Jo], some varieties with a nef tangent bundles [Iy], open moduli spaces of smooth curves [Iy-Ml], universal families over some Shimura surfaces [Mi].

In [Iy], we had looked at varieties which have a nef tangent bundle. Using the structure theorems of Campana and Peternell [Ca-Pe] and Demailly-Peternell-Schneider [DPS], we know that such a variety X admits a finite étale surjective cover $X' \to X$ such that $X' \to A$ is a bundle of smooth Fano varieties over an abelian variety. Furthermore, any

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fibre which is a smooth Fano variety necessarily has a nef tangent bundle. It is an open question [Ca-Pe, p.170] whether such a Fano variety is a rational homogeneous variety. They answered this question positively in dimension at most 3. We showed in [Iy] that whenever the étale cover is a relative cellular variety over A or if it admits a relative Chow–Künneth decomposition, then X' and X have a Chow–Künneth decomposition. In particular, it holds for varieties with a nef tangent bundle of dimension at most 3.

In this paper, we weaken the hypothesis on the cover $X' \to A$ as above and obtain a Chow–Künneth decomposition whenever $X' \to A$ is a rational homogeneous bundle, over an abelian variety A. This strengthens the results in [Iy] and if the open question [Ca-Pe, p.170] is answered positively in higher dimensions then we obtain a Chow–Künneth decomposition for all varieties which have a nef tangent bundle. This question is answered positively in some higher dimensional cases also, see [Hw, section 4] and references therein. Hence we obtain a Chow–Künneth decomposition for new cases as well in higher dimensions, for instance, all varieties upto dimension four.

We state the result and proofs, in a more general situation.

Theorem 1.1. *Suppose S is a smooth projective variety over the complex numbers. Let G be a connected reductive algebraic group and let Z be a rational G homogeneous space over the variety S. Assume that S has a Chow-Künneth decomposition. Then the following hold:*

a) the motive of Z has an absolute Chow–Künneth decomposition.

b) the motive of the bundle $Z \to S$ is expressed as a sum of tensor products of summands of the motive of S with the twisted Tate motive.

One of the main observation in the proof is to note that a rational homogeneous bundle as above is étale locally a relative cellular variety, using the fact that the formal deformations of a rational homogeneous variety are trivial (see Lemma 3.2). Hence we can construct relative Chow–Künneth projectors (in the sense of [De-Mu]) over étale morphisms of S. These projectors lie in the subspace generated by the relative algebraic cells. The corresponding relative cohomology classes patch up since they lie in the subspace generated by the relative analytic cells. Hence the relative orthogonal projectors can be patched up as algebraic cycles to obtain relative projectors, in the rational Chow groups of the associated regular stack. In this case, we show that the relative Chow–Künneth projectors over the regular stack descend to relative Chow–Künneth projectors for $Z \to S$ (see Corollary 3.7). The criterion of Gordon-Hanamura-Murre [GHMu2], for obtaining absolute Chow–Künneth projectors from relative Chow–Künneth projectors can be directly applied, see Proposition 3.8.

A similar proof also holds for a class of log homogeneous varieties studied by M. Brion [Br]. A log homogeneous variety consists of a pair (X, D), where X is a smooth projective variety and D is a normal crossing divisor on X, with the following property. The variety X is said to be log homogeneous with respect to D if the associated logarithmic tangent bundle $\mathcal{T}_X(-D)$ is generated by its global sections. It follows that X is almost homogeneous under the connected automorphism group $G := Aut^0(X,D)$, with boundary D. With notations as above, we show

Theorem 1.2. *Suppose X is log homogeneous with respect to a normal crossing divisor D. Then X has a Chow–Künneth decomposition. Moreover, the motive of X is expressed as a sum of tensor products of the summands of the motive of its Albanese reduction, with the twisted Tate motive.*

See Theorem 4.4.

The proof uses the classification of log homogeneous varieties by Brion [Br]. The fibres of the Albanese morphism are smooth spherical varieties. In this case we check that étale local triviality of the Albanese fibration holds. The proof of Theorem 1.2 relies on the algebraicity of the cohomology of the spherical varieties, similar to Theorem 1.1, and applying the criterion of [GHMu2].

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2. Preliminaries

We work over the field of complex numbers in this paper. We begin by recalling the standard constructions of the category of motives. Since this is fairly discussed in the literature, we give a brief account and refer to [Mu2], [Sc] for details.

2.1. **Category of motives.** The category of nonsingular projective varieties over $\mathbb{C}$ will be denoted by $\mathcal{V}$. For an object X of $\mathcal{V}$, let $CH^i(X)_{\mathbb{Q}} = CH^i(X) \otimes \mathbb{Q}$ denote the rational Chow group of codimension i algebraic cycles modulo rational equivalence. Suppose $X, Y \in Ob(\mathcal{V})$ and $X = \cup X_i$ be a decomposition into connected components X_i and $d_i = \dim X_i$. Then $\mathrm{Corr}^r(X,Y) = \oplus_i CH^{d_i+r}(X_i \times Y)_{\mathbb{Q}}$ is the group of correspondences of degree r from X to Y.

We will use the standard framework of the category of Chow motives $\mathcal{M}_{rat}$ in this paper and refer to [Mu2] for details. We denote the category of motives $\mathcal{M}_{\sim}$, where $\sim$ is any equivalence, for instance $\sim$ is homological or numerical equivalence. When S is a smooth variety, we also consider the category of relative Chow motives $CH\mathcal{M}(S)$ which was introduced in [De-Mu] and [GHMu]. When $S =$ Spec $\mathbb{C}$ then the category $CH\mathcal{M}(S) = \mathcal{M}_{rat}$.

2.2. **Chow–Künneth decomposition for a variety.** Suppose X is a nonsingular projective variety over $\mathbb{C}$ of dimension n. Let $\Delta_X \subset X \times X$ be the diagonal. Consider the

Künneth decomposition of Δ in the Betti Cohomology:

$$\Delta_X = \oplus_{i=0}^{2n} \pi_i^{hom}$$

where $\pi_i^{hom} \in H^{2n-i}(X) \otimes H^i(X)$.

Definition 2.1. *The motive of X is said to have Künneth decomposition if each of the classes π_i^{hom} are algebraic and are projectors, i.e., π_i^{hom} is the image of an algebraic cycle π_i under the cycle class map from the rational Chow groups to the Betti Cohomology and satisfying $\pi_i \circ \pi_i = \pi_i$ and $\Delta_X = \oplus_{i=0}^{2n} \pi_i$ in the rational Chow ring of $X \times X$. The algebraic projectors π_i are called as the algebraic Künneth projectors.*

Definition 2.2. *The motive of X is furthermore said to have a Chow–Künneth decomposition if the algebraic Künneth projectors are orthogonal projectors, i.e., $\pi_i \circ \pi_j = \delta_{i,j}\pi_i$ and $\Delta_X = \oplus_{i=0}^{2n} \pi_i$ in the rational Chow ring of $X \times X$.*

3. Rational homogeneous bundles over a variety

In this section, we firstly recall the motive of a rational homogeneous variety and later construct relative Chow–Künneth projectors for a bundle of homogeneous varieties. The criterion of [GHMu2] can then be applied to obtain absolute Chow–Künneth projectors on the total space of the bundle. For this purpose, we need to show that the bundle is étale locally trivial and check patching conditions over the etale coverings. We begin by recalling the motive of a rational homogeneous variety.

3.1. **The motive of a rational homogeneous space.** Suppose F is a rational homogeneous variety. Then F is identified as a quotient G/P, for some reductive linear algebraic group G and P is a parabolic subgroup of G. Notice that F is a cellular variety, i.e., it has a cellular decomposition

$$\emptyset = F_{-1} \subset F_0 \subset ... \subset F_n = F$$

such that each $F_i \subset F$ is a closed subvariety and $F_i - F_{i-1}$ is an affine space.

Then we have

Lemma 3.1. [Ko, Theorem, p.363] *The Chow motive $h(F) = (F, \Delta_F)$ of F decomposes as a direct sum of twisted Tate motives*

$$h(F) = \bigoplus_{\omega} \mathbb{L}^{\otimes dim\ \omega}.$$

Here ω runs over the set of cells of F.

In particular, this says that the Chow–Künneth decomposition holds for F. Next we consider bundles of homogeneous spaces $Z \to S$ over a smooth variety S. We want to describe the Chow motive of Z in terms of the Chow motive of S, up to some Tate twists. For this , we need to show étale local triviality of $Z \to S$, and we discuss it in the next subsection.

3.2. **The étale local triviality of a rational homogeneous bundle.** Suppose $Z \to S$ is a smooth projective morphism and the base variety S is smooth and projective.

By étale local triviality, we mean that there exist étale morphisms $p_\alpha : U_\alpha \to S$ such that the pullback bundle

$$Z_{U_\alpha} := Z \times_S U_\alpha \to U_\alpha$$

is a Zariski trivial fibration and the images of p_α cover S, i.e., $\cup_\alpha p_\alpha(U_\alpha) = S$. Here α runs over some indexing set I. Consider a rational homogeneous bundle $f : Z \longrightarrow S$, i.e., π is a smooth projective morphism and any fibre $\pi^{-1}y$ is a rational homogeneous variety G/P. Here G is a reductive linear algebraic group and $P \subset G$ is a parabolic subgroup. Assume that S is a smooth complex projective variety.

In the following discussion, we note that an étale cover $\{U_\alpha\}$ as above exists for a rational homogeneous bundle $Z \to S$.

Lemma 3.2. *There are étale open sets $p_\alpha : U_\alpha \to S$ (satisfying $\cup_\alpha p_\alpha(U_\alpha) = S$), such that the pullback bundle $Z_{U_\alpha} \to U_\alpha$ is a Zariski trivial fibration.*

Proof. We need to note that the formal deformations of a rational homogeneous variety are trivial. This is just a consequence of the well-known Bott's vanishing theorem: $H^1(G/P, T) = 0$. The assertion on étale local triviality follows from [Se, Proposition 2.6.10].

□

Our aim is to obtain relative Chow–Künneth projectors for the bundle Z/S. For this purpose, we first construct relative projectors over the étale coverings of $Z \to S$ and check the patching conditions. This requires us to use the language of stacks which enables us to descend the projectors down to $Z \to S$. Hence in the following subsection, we recall some facts on regular stacks and the relationship of the rational Chow groups/cohomology of stacks with that of its coarse moduli space. These facts will be essentially applied to the simplest situation—the rational homogeneous bundle $Z \to S$. Also, the patching will be used for étale open sets of Z which are of the type $Z_{U_\alpha} := Z \times_S U_\alpha$, for étale morphisms $U_\alpha \to S$. In this context, it is possible to avoid stacks, since the regular stack associated to the étale coverings is again Z. But we use the stacks, essentially to say that the algebraic cells which live in the fibres of $Z \to S$ patch together over the étale coverings. This will be needed in the proof of Lemma 3.5.

We remark that more general patching statements might also hold for other varieties, using stacks. However we do not know concrete examples as yet, where it can be checked.

3.3. **Chow groups of an étale site.** Mumford, Gillet ([Mm],[Gi]) have defined Chow groups for Deligne–Mumford stacks and more generally for any algebraic stack $\mathcal{X}$. Furthermore, intersection products are defined whenever $\mathcal{X}$ is a regular stack . Let $\mathcal{X}$ be a regular stack. The coarse moduli space of $\mathcal{X}$ is denoted by X and $p : \mathcal{X} \to X$ be the projection. So from [Gi, Theorem 6.8], the pullback p^* and pushforward maps p_* establish a ring isomorphism of rational Chow groups

$$CH^*(\mathcal{X})_{\mathbb{Q}} \cong CH^*(X)_{\mathbb{Q}}. \tag{1}$$

This can be applied to the product $p \times p : \mathcal{X} \times \mathcal{X} \to X \times X$, to get a ring isomorphism

$$CH^*(\mathcal{X} \times \mathcal{X})_{\mathbb{Q}} \cong CH^*(X \times X)_{\mathbb{Q}}. \tag{2}$$

Assume that X is a smooth projective variety. Then these isomorphisms also hold in the rational singular cohomology of $\mathcal{X}$ and $\mathcal{X} \times \mathcal{X}$ (for example, see [Be]):

$$H^*(\mathcal{X}, \mathbb{Q}) \cong H^*(X, \mathbb{Q}). \tag{3}$$

and

$$H^*(\mathcal{X} \times \mathcal{X}, \mathbb{Q}) \cong H^*(X \times X, \mathbb{Q}). \tag{4}$$

Via these isomorphisms, we can pullback the Künneth decomposition of the diagonal class in $H^{2n}(X \times X, \mathbb{Q})$ to a decomposition of the diagonal class of $\mathcal{X}$ in $H^{2n}(\mathcal{X} \times \mathcal{X}, \mathbb{Q})$, and whose components we refer to as the Künneth components of $\mathcal{X}$.

Given a smooth variety X, consider an atlas $\sqcup_{\alpha \in I} U_\alpha$ of X such that $p_\alpha : U_\alpha \to X$ is an étale morphism, for each $\alpha \in I$, and the images of p_α cover X. Then one can associate a Q-variety [Mm] to this atlas. Furthermore, by [Gi, Proposition 9.2], there is a regular stack $\mathcal{X}$ associated to this data such that X is its coarse moduli space, i.e., there is a projection

$$p : \mathcal{X} \to X.$$

In this case, we note that the regular stack $\mathcal{X}$ is the same as the variety X. Hence the isomorphisms in (1), (2), (3) and (4) trivially hold for the projection p. More precisely, we have

$$CH^*(\mathcal{X})_{\mathbb{Q}} = CH^*(X)$$

and

$$H^*(\mathcal{X}, \mathbb{Q}) = CH^*(X, \mathbb{Q}).$$

3.4. **The motive of a rational homogeneous bundle.** Suppose $Z \to S$ is a rational homogeneous bundle over a smooth projective variety S. Let S^{et} be the étale site on S, together with the natural morphism of the sites $f : S^{et} \to S$. Here S is considered with the Zariski site. Consider the pullback bundle

$$Z^{et} := Z \times_S S^{et} \to S^{et}$$

over S^{et}.

Since we are dealing with a rational homogeneous bundle, we can describe these covers explicitly as follows; by Lemma 3.2, the pullback bundles $Z_{U_\alpha} \to U_\alpha$, for $\alpha \in I$, are

Zariski trivial. In other words, $Z_{U_\alpha} = F \times U_\alpha$, where F is a typical fiber of $Z \to S$. Hence $Z_{U_\alpha} \to U_\alpha$ is a relative cellular variety for each $\alpha \in I$.

The description of the rational Chow groups of relative cellular spaces $\pi : X \to T$ is given by B. Koeck [Ko] (see also [Ne-Za, Theorem 5.9]), which is stated for the higher Chow groups:

Suppose $X \to T$ is a relative cellular space.

Then there is a sequence of closed embeddings

$$\emptyset = Z_{-1} \subset Z_0 \subset ... \subset Z_n = X \tag{5}$$

such that $\pi_k : Z_k \longrightarrow T$ is a flat projective T-scheme. Furthermore, for any $k = 0, 1, ..., n$, the open complement $Z_k - Z_{k-1}$ is T-isomorphic to an affine space $\mathbb{A}_T^{m_k}$ of relative dimension m_k. Denote $i_k : Z_k \hookrightarrow X$.

Theorem 3.3. *For any $a, b \in \mathbb{Z}$, the map*

$$\begin{array}{ccc} \bigoplus_{k=0}^{n} H_{a-2m_k}(T, b - m_k) & \longrightarrow & H_a(X, b) \\ (\alpha_0, ..., \alpha_n) & \mapsto & \sum_{k=0}^{n} (i_k)_* \pi_k^* \alpha_k \end{array}$$

is an isomorphism. Here $H_a(T, b) = CH_b(T, a - 2b)$ are the higher Chow groups of T.

Proof. See [Ko, Theorem, p.371]. □

The above theorem can equivalently be restated to express the rational Chow groups of X as

$$CH^r(X)_{\mathbb{Q}} = \bigoplus_{k=0}^{r} (\oplus_\gamma \mathbb{Q}[\omega_k^\gamma]).f^* CH^k(T)_{\mathbb{Q}}, \tag{6}$$

Here ω_k^γ are the $r-k$ codimensional relative cells and γ runs over the indexing set of $r-k$ codimensional relative cells in the T-scheme X.

We now apply this theorem to our situation: we have a homogeneous bundle $Z \to S$ and an étale atlas $S^{et} := \sqcup_\alpha U_\alpha \to S$, such that $Z_{U_\alpha} \to U_\alpha$ is trivial.

Lemma 3.4. *Given a Zariski trivial homogeneous bundle $p_\alpha : Z_{U_\alpha} \to U_\alpha$, the rational Chow groups are described as follows:*

$$CH^r(Z_{U_\alpha})_{\mathbb{Q}} = \bigoplus_{k=0}^{r} (\oplus_\gamma \mathbb{Q}[\omega_k^\gamma]).p_\alpha^* CH^k(U_\alpha)_{\mathbb{Q}}.$$

Proof. Since the homogeneous bundle $p_\alpha : Z_{U_\alpha} \to U_\alpha$ is a Zariski trivial bundle, it is a relative cellular variety. Hence the above Theorem 3.3 can be applied and it gives a natural isomorphism

$$CH^r(Z_{U_\alpha})_{\mathbb{Q}} = \bigoplus_{k=0}^{r} (\oplus_\gamma \mathbb{Q}[\omega_k^\gamma]).f_\alpha^* CH^k(U_\alpha)_{\mathbb{Q}}.$$

Equivalently, since $Z_{U_\alpha} = F \times U_\alpha$, we have the equality (see [FMSS, Theorem 2]):

$$CH^r(Z_{U_\alpha})_{\mathbb{Q}} = CH^r(F \times U_\alpha)_{\mathbb{Q}} = \sum_{p,q,p+q=r} CH^p(F)_{\mathbb{Q}}.CH^q(U_\alpha)_{\mathbb{Q}}. \tag{7}$$

Here F is a typical fibre of $Z \to S$ which is cellular variety. This gives the assertion. $\square$

For our applications, it suffices to consider the piece $k = 0$, which consists of only the relative algebraic cells of codimension r, namely,

$$RCH^r(Z_{U_\alpha})_{\mathbb{Q}} := \oplus_\gamma \mathbb{Q}[\omega_0^\gamma].$$

In other words, we only look at the subgroup consisting of the direct summand

$$CH^r(F) \subset CH^r(F \times U_\alpha),$$

in (7).

A similar equality as in (7), holds in the rational singular cohomology of $Z_{U_\alpha} \to U_\alpha$. So we can also define the piece

$$RH^{2r}(Z_{U_\alpha})_{\mathbb{Q}} := \oplus_\gamma \mathbb{Q}[\omega_0^\gamma]$$

in the rational singular cohomology of Z_{U_α} and the piece

$$RH^{2r}(Z)_{\mathbb{Q}} := \oplus_\gamma \mathbb{Q}[\omega_0^\gamma]$$

as a subspace of the rational Betti cohomology $H^{2r}(Z, \mathbb{Q})$, generated by the relative analytic cells ω_0^γ. Here, we use the fact that $Z \to S$ is locally trivial in the analytic topology and there is a analytic cellular decomposition similar to (5).

Lemma 3.5. *The cycles ω_0^γ in $RCH^*(Z_{U_\alpha})_{\mathbb{Q}}$ patch together in the étale site to determine a subspace $RCH^*(Z)_{\mathbb{Q}}$ of $CH^*(Z)_{\mathbb{Q}}$, generated by the patched cycles and which maps isomorphically onto the subspace $RH^{2r}(Z)_{\mathbb{Q}} \subset H^{2r}(Z, \mathbb{Q})$, under the cycle class map*

$$CH^*(Z)_{\mathbb{Q}} \to H^{2*}(Z, \mathbb{Q}).$$

Proof. Note that the cycles $\omega_0^\gamma \in RCH^*(Z_{U_\alpha})_{\mathbb{Q}}$ patch together as analytic cycles in the étale site and determine a subspace $RH^{2r}(Z)_{\mathbb{Q}} \subset H^{2r}(Z, \mathbb{Q})$,

Since the fiber F is a cellular variety, there is a natural isomorphism

$$RCH^*(Z_{U_\alpha})_{\mathbb{Q}} \xrightarrow{\simeq} RH^{2*}(Z_{U_\alpha})_{\mathbb{Q}} \tag{8}$$

between the 0-th piece of the rational Chow group and the relative Betti cohomology, for each α.

Via the isomorphism in (8), the patching conditions required over the étale site, to define the piece $RCH^{2r}(Z)_{\mathbb{Q}}$ are the same as those for $RH^{2r}(Z)_{\mathbb{Q}}$. More precisely, the patching conditions are given in [Gi, §4]. The identification in (8) together with the fact that the patching conditions are fulfilled for the singular cohomology of the étale site, says that the cycles ω_0^γ patch together to give a class in $RH^{2r}(Z)_{\mathbb{Q}}$, and hence they also patch together to give a class in $RCH^{2r}(Z)_{\mathbb{Q}}$. These patched classes generate the $\mathbb{Q}$-subspace $RCH^{2r}(Z)_{\mathbb{Q}} \subset CH^*(Z)_{\mathbb{Q}}$ and which maps isomorphically onto the subspace $RH^{2r}(Z)_{\mathbb{Q}} \subset H^{2r}(Z,\mathbb{Q})$ under the cycle class map. $\square$

Corollary 3.6. *There is a canonical isomorphism*

$$RCH^r(Z)_{\mathbb{Q}} \simeq RH^{2r}(Z)_{\mathbb{Q}}.$$

between the rational Chow groups and the rational cohomology generated by the relative cells.

$\square$

Let $n := \dim(Z/S)$.

Corollary 3.7. *The bundle $Z \to S$ has a relative Chow–Künneth decomposition, in the sense of* [GHMu].

Proof. This is an application of Lemma 3.5, applied to the relative product $Z \times_S Z \to S$. We notice that the relative orthogonal Künneth projectors in $H^{2n}(Z \times_S Z, \mathbb{Q})$ lift to relative orthogonal projectors in $H^{2n}(Z_{U_\alpha} \times_{U_\alpha} Z_{U_\alpha}, \mathbb{Q})$ and which add to the relative diagonal cycle. Now we note that the relative diagonal $\Delta_{Z/S}$ and its orthogonal Künneth components actually lie in the piece $RH^{2n}(Z \times_S Z)_{\mathbb{Q}}$ (generated by the relative algebraic cells) and under the isomorphisms in (3), (4), lift to an orthogonal decomposition

$$\Delta_{Z/S} = \sum_{i=0}^{2n} \Pi_i \,\in\, RH^{2n}(Z \times_S Z)_{\mathbb{Q}}$$

over the étale site, i.e, over $Z_{U_\alpha} \times_{U_\alpha} Z_{U_\alpha}$, for each $\alpha \in I$. Now apply Corollary 3.6 to the product space $Z_{U_\alpha} \times_{U_\alpha} Z_{U_\alpha} \to U_\alpha$, to lift the above orthogonal projectors to orthogonal algebraic projectors in $RCH^n(Z_{U_\alpha} \times_{U_\alpha} Z_{U_\alpha})_{\mathbb{Q}}$, and these patch to give relative Chow–Künneth projectors and a relative Chow–Künneth decomposition

$$\Delta_{Z/S} = \sum_{i=0}^{2n} \Pi_i \,\in\, CH^n(Z \times_S Z)_{\mathbb{Q}}.$$

$\square$

Proposition 3.8. *Suppose $Z \to S$ is a rational homogeneous bundle over a smooth variety S. Then the motive of the bundle $Z \to S$ is expressed as a sum of tensor products of summands of the motive of S with the twisted Tate motive. More precisely, the motive of Z can be written as*

$$h(Z) = \bigoplus_i h^i(Z)$$

where $h^i(Z) = \bigoplus_{j+k} r_{\omega_\alpha}.\mathbb{L}^j \otimes h^k(S)$. Here r_{ω_α} is the number of j-codimensional cells on a fibre $\mathbb{F}$

In particular, if S has a Chow–Künneth decomposition then Z also admits an absolute Chow–Künneth decomposition.

Proof. By Corollary 3.7, we know that the bundle Z/S has a relative Chow–Künneth decomposition. Since the map $Z \to S$ is a smooth morphism and the fibres of $Z \to S$ have only algebraic cohomology, we can directly apply the criterion in [GHMu2, Main theorem 1.3], to get absolute Chow–Künneth projectors for Z and the decomposition stated above (for example, see [Iy, Lemma 3.2, Corollary 3.3], together with [Ak-Jo] for étale quotients of abelian varieties). □

Remark 3.9. *Suppose X is a smooth projective variety with a nef tangent bundle. Then by* [Ca-Pe],[DPS]*, we know that there is an étale cover $X' \to X$ of X such that $X' \to A$ is a smooth morphism over an abelian variety A, whose fibres are smooth Fano varieties with a nef tangent bundle. It is an open question* [Ca-Pe, p.170]*, whether such a Fano variety is a rational homogeneous variety. A positive answer to this question, together with Proposition 3.8, will give absolute Chow–Künneth projectors for all varieties with a nef tangent bundle. See also* [Hw, section 4] *for a discussion on new cases (for instance, for all varieties upto dimension four) where this question is answered positively.*

4. Chow–Künneth decomposition for log homogeneous varieties

Log homogeneous varieties were introduced by M. Brion [Br]. Suppose X is a smooth projective variety and $D \subset X$ is a normal crossing divisor. Then X is said to be log homogeneous with respect to D if the logarithmic tangent bundle $\mathcal{T}_X(-D)$ is generated by its global sections. Then X is almost homogeneous under the connected automorphism group $G := Aut^0(X, D)$, with boundary D. The G-orbits in X are exactly the strata defined by D, in particular their number is finite.

A classification of log homogeneous varieties is given by Brion which says:

Theorem 4.1. *Any log homogeneous variety X can be written uniquely as $G \times^I Y$, where*

1) G is connected algebraic group,

2) $I \subset G$ is a closed subgroup containing G_{aff} as a subgroup of finite index,

3) choose any Levi subgroup $L \subset G_{aff}$, Y is a complete smooth I-variety containing an open L-stable subset Y_L such that the L-variety Y is spherical and the projection

$$X \to G/I =: A \tag{9}$$

is the Albanese morphism.

Proof. See [Br, Theorem 3.2.1]. □

Recall that a smooth spherical variety Y is a G-variety such that the Borel subgroup B of G has an open dense orbit in Y. It is known that Y contains a finite number of

B-orbits. Since we are looking at varieties defined over $\mathbb{C}$, it follows that a spherical variety is a linear variety (in the sense of [To, Addendum, p.5]). In particular, we have

Lemma 4.2. *Suppose Y is a smooth complete spherical variety. Then there is an isomorphism*

$$CH^i(Y) \stackrel{\simeq}{\to} H^{2i}(Y, \mathbb{Z})$$

for each i.

Proof. See [FMSS, Corollary to Theorem 2]. □

Lemma 4.3. *Suppose Y is a smooth complete spherical variety. Then Y has a Chow–Künneth decomposition.*

Proof. This follows from Lemma 4.2 and the construction of orthogonal projectors given in [Iy-Ml, Lemma 5.2]. □

We will show that X has a Chow–Künneth decomposition under the following assumption;

Theorem 4.4. *Suppose X is a log homogeneous variety. Then the variety X has a Chow–Künneth decomposition. Moreover, the motive of X is expressed as a sum of tensor products of the summands of the motive of its Albanese reduction, with the twisted Tate motive.*

Proof. With notations as in Theorem 4.1, suppose the spherical variety Y is a Fano variety. Then, by [Bi-Br, Proposition 4.2 i)], we have the vanishing $H^1(Y, T_Y) = 0$. In particular, this implies that the formal deformations of Y are trivial. Hence, by [Se, Proposition 2.6.10], the Albanese fibration in (9) is étale locally trivial. In general, consider the Albanese fibration

$$X = G \times^I Y \to G/I = A$$

which is easily seen to be etale locally trivial. The following explanation is due to B. Totaro: notice that all the fibers of this morphism are isomorphic to Y. In more detail, this morphism is etale locally trivial because the morphism $G \to G/I$ is etale locally trivial, which is a standard fact about the quotient of an algebraic group by a smooth closed subgroup. See the discussion of homogeneous spaces in [Bo, 6.14].

Hence we can apply the methods from the previous section. By Lemma 4.3, relative Chow–Künneth projectors can be constructed for Zariski trivialisations of (9) over étale covers $U_\alpha \to A$. Hence the proof of Proposition 3.8 applies to this situation. Indeed, Lemma 3.4 holds for a relative spherical variety over U_α. This can be applied to the Albanese fibration in (9) over étale morphisms where it is Zariski trivial. In this case, the following piece of the rational Chow ring $RCH^*(U_\alpha \times Y)_\mathbb{Q}$ is identified with the Chow ring $CH^*(Y)_\mathbb{Q}$. A formula similar to (6) holds for the Chow groups of $U_\alpha \times Y$, since Y is cellular, see [FMSS, Theorem 2]. Hence, by Lemma 4.2, $CH^*(U_\alpha \times Y)_\mathbb{Q} \simeq H^{2*}(U_\alpha \times Y)_\mathbb{Q}$.

Similarly Lemma 3.5 and Corollary 3.6 hold for (9) over étale morphisms. The rest of the arguments are the same as given for a rational homogeneous bundle. □

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