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Research Paper

Wave Propagation in Structures with Periodic Defects

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In this paper, a Fourier transform based periodic approach is developed for wave-propagation analysis in a generic multi-coupled one-dimensional periodic structure. Dispersion relations in 1-D metallic waveguides with periodic defects are investigated. Defects considered here are horizontal cracks and staggered cracks, respectively. The formulation is based on the Bloch's theorem and uses the dynamic stiffness matrix of the sub-element obtained from spectral super element theory. Numerical investigations are carried out to study the dispersion characteristics and the evolution of band gaps in these structures.

Key Words: Wave Propagation; Periodic Structures; Bloch Theorem; Spectral Super Element; Periodic Defects

1. Introduction

Periodic structures are often encountered in engineering such as a bridge with repeated truss like structure, a periodically stiffened fuselage and wing of an aircraft, a periodically supported pipe conveying the fluid etc. Rivet holes in an aircraft stiffened connections are very common and they are periodic in nature. When the stress concentration in these holes exceeds the threshold values, they give rise to a number of cracks, which again is periodic in nature. Recently, with the advent of nano-structured materials such as carbon nano-tubes and their composites, the study of their periodic properties will help in exploring their functional properties. Thus periodic structures have been investigated intensively in the last decade.

Currently wave propagation analysis in periodic structures is performed using spectral finite element method [1], which is applicable for healthy structures only, and the conventional finite element method, which is applicable for any type of structures, without utilizing the periodic property of the structures. Since the wave propagation analysis deals with very high frequency waves, which imply very small wavelengths, finite element mesh size for such problems require that it should be at least of the order of minimum wavelength to capture the proper response. Thus if the structure to be analyzed is large enough with many periodic cells, then the system size for such problems will be too large and solving it will be computationally very expensive.

In this paper, the defects considered are the horizontal cracks, and staggered cracks. The procedure developed here can be easily extended to other defects such as elliptical or circular holes. Timoshenko beam theory is used here to obtain a healthy spectral frame element to model a sub-element of the periodic system. Equivalent Bernoulli-Euler beam theory can be recovered by setting the shear rigidity very high and making rotational inertia equal to zero. A single sub-element of the periodic system is modelled by two different methods, one by using spectral super element model [2] and the second using damaged spectral element model [3]. Using the Timoshenko beam theory, the eigenvalues and eigenvectors of the system is obtained, which are

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essential to determine the dispersion relation of the structure using periodic approach.

Research on periodic structures started in eighteenth century by Brillouin [4], wherein the study was made on their dynamic properties to understand the wave characteristics of lattice structures. An interesting characteristic of a periodic structure is the band-gap phenomenon, i.e., they exhibit certain frequency bands within which no wave propagates, resulting in filtering capabilities. Another important and useful feature of periodic system is the simplification of the calculation procedure due to periodic nature, as only one periodic subsystem needs to be analyzed and the wave propagation of a complete (infinite) profile strip can be deduced from the single subsystem results. Mead [5-7] proposed a general theory for wave propagation in mono-coupled and multi-coupled periodic systems. The mono-coupled periodic systems are connected by only one displacement variable in contrast to the multi-coupled systems. Signorelli *et al.* [8] investigated the wave propagation in a periodic truss system. This analysis is based on the transfer matrix of a single bay of the structure. The phase closure principle is invoked to predict natural frequencies of a fixed-free portion of the truss. At the same time, mono-coupled and undamped periodic systems were examined by wave, modal, receptance and finite element analyses by Keane *et al.* [9]. These analyses were aimed at providing an improved insight into the vibratory behavior of engineering structures, which exhibit periodicity and the effects of deviation from perfect periodicity were also demonstrated. The effect on the properties of periodic structure due to imperfections or near periodicity is discussed in detail by Benaroya *et al.* [10].

Spectral FEM is an established wave propagation analysis tool. It is essentially a finite element analysis tool in the frequency domain. Due to use of exact solution to frequency domain as interpolating function for element formulation and exact inertial distribution, the problem sizes in spectral element approach is orders of magnitude smaller compared to conventional FEM. In addition, spectral FEM provides understanding the nature of waves through wavenumber and group speeds computation, which is not possible directly through conventional FEM. There are two variants of spectral FEM, one based on Fourier transform and the other based on wavelet transform. Former uses Fast Fourier Transform (FFT) to go back and forth in time and frequency domain, while latter uses Daubechies compactly supported basis functions as wavelet functions. The details of Fourier based spectral FEM can be obtained from [1], while the details of wavelet based spectral element is found in the textbook [11]. The major disadvantage with spectral finite element method is its inherent difficulty in handling waveguides with arbitrary geometries and waveguides with defects. Although, a spectral element model for horizontal mid-plane or offset cracks are available (see [3]), such models for inclined or staggered cracks are not available. All the more, the wavenumbers and hence the group speeds are very difficult to obtain, especially for waveguides with defects. In this paper, we develop new methods based on Bloch theorem and FE model to obtain propagation constants in waveguides with periodic defects. It was mentioned earlier that a spectral finite element method is very useful to study wave propagation in an ideal structure without any defects but the concept of spectral super element for wave propagation in structures with local non-uniformities given by Gopalakrishnan and Doyle [2], which enhanced its utility and made it suitable for handling waveguides with defects.

The propagation of elastic wave in periodic composites called phononic crystals (PCs) has received a great deal of attention in the recent past. Particularly, much interest is focused on the characteristics of the so-called phononic band gaps (PBG), in which elastic waves are all forbidden. The directional propagation characteristics of elastic wave during pass bands in two-dimensional thin plate phononic crystals are analyzed by using the lumped-mass method to yield the phase constant surface by Wen *et al.* [12]. Hong *et al.* [13] investigated the propagation characteristics of flexural waves in periodic grid structures designed with the idea of phononic crystals by combining the Bloch theorem with the finite element method. The same kind of analysis was done by Wen *et al.* [14] on periodic binary straight beam with different cross sections with the plane-wave expansion method.

Periodic structures and structures with defects have been investigated extensively but the periodicity of defect is not considered yet in detail to the best of author's knowledge. Thus in this study, a Fourier transform based periodic approach is developed for wave propagation in generic 1-D periodic waveguide and examples of 1-D structures with periodic defects are used to show the validity and prominence of periodic approach. In this periodic approach, the dynamic stiffness matrix of sub-element is needed in order to obtain the reduced dynamic stiffness matrix of the whole periodic structure. Based on the theory of wave propagation in multi coupled periodic systems, there are two approaches, namely determinantal equation approach and transfer matrix approach, and these can be used to obtain the modal parameters, which are required to formulate the reduced dynamic stiffness matrix of the system.

The outline of the paper is as follows. Section 2 gives a brief introduction to formulation of dynamic stiffness matrix using spectral super element approach. Section 3 briefly describes the computation procedure for computing the dispersion characteristics and numerical examples are presented in Section 4. This is followed by conclusions in Section 5.

2. Dynamic Stiffness Matrix of a Sub-Element

Dynamic stiffness matrix of sub-element of a periodic structure is needed in order to obtain the wavenumbers of the structure using periodic approach. It requires the dynamic stiffness matrix of spectral healthy frame element modelled using Timoshenko beam theory, which is discussed in [1]. Two different methods are used to obtain the dynamic stiffness matrix. First method is based on the damaged spectral finite element model, which was developed in [3]. This model represents the dynamics of the cracked beam exactly. However it has a major drawback that only horizontal cracks can be modelled using this approach. Second one is the spectral super element approach, in which region very near the defect is modelled using conventional 2-D finite elements, which is coupled to the regular 1-D healthy spectral element model. Again, this model is adopted from [2]. Due to versatility of FE model, practically, any type of defect having any orientation can be modelled with this approach. That is, this approach enables determination of dispersion characteristics of periodic defects of any type and hence in this paper, we propose to use the spectral super element approach to formulate the dynamic stiffness matrix of sub-element.

Spectral Super Element Approach

In this approach the region of defect, whether it is hole or crack, is meshed with conventional finite elements and its stiffness and mass matrices are dynamically condensed leaving only the connection degrees of freedom. These are further condensed to just waveguide connectivity thus making the element suitable for assembly with spectral frame element. The detailed formulation of spectral super element for 1-D waveguides is discussed in [2] and only relevant information is provided here for the sake of completeness.

As mentioned earlier, the defect region is modelled using finite elements. The equation of motion of this region is given by

$$[M]\{\ddot{u}\}+[C]\{\dot{u}\}+[K]\{u\}=\{P\} \tag{1}$$

where $[K]$ is the stiffness matrix, $[C]$ is the damping matrix, $[M]$ is the mass matrix, {u} is the vector of nodal degrees of freedom and {P} the vector of applied loads. The force and displacement vector can be transformed to frequency domain using Discrete Fourier Transform (DFT) [1] and the resulting equation will become

$$[\hat{K}]\{\hat{u}\}=\{\hat{P}\},\ [\hat{K}]=[K]+i\omega[C]-\omega^2[M] \tag{2}$$

The dynamic stiffness matrix can be portioned as

$$\begin{bmatrix}[\hat{K}_{cc}] & [\hat{K}_{ci}]\\ [\hat{K}_{ic}] & [\hat{K}_{ii}]\end{bmatrix}\begin{Bmatrix}\{\hat{u}_c\}\\ \{\hat{u}_i\}\end{Bmatrix}=\begin{Bmatrix}\{\hat{P}_c\}\\ \{0\}\end{Bmatrix} \tag{3}$$

where $\{\hat{u}_c\}$ and $\{\hat{u}_i\}$ are the vector of connection and internal degrees of freedom respectively. After eliminating $\{\hat{u}_i\}$, the stiffness relation of super element can be written as

$$\{\hat{P}_c\} = [\hat{K}_{ss}]\{\hat{u}_c\}, [\hat{K}_{ss}] = [\hat{K}_{cc}] - [\hat{K}_{ci}][\hat{K}_{ii}]^{-1}[\hat{K}_{ic}] \quad (4)$$

This equation relates connection forces $\{\hat{P}_c\}$ to only connection displacements $\{\hat{u}_c\}$; all the internal degrees of freedom have been condensed out. The computation of requires $[\hat{K}_{ss}]$, requires $[\hat{K}_{ii}]^{-1}$ which is nearly of the order of $[\hat{K}]$ and is frequency dependent, thus its efficient computation is absolutely essential and that can be done as given in [2]. This process is not repeated here. There are several other efficient methods to dynamically condense the stiffness matrix which are discussed in detail in [15].

2.1.1 Connection to Spectral Element

A node in spectral waveguide which is a frame element is a point at a center of cross section where average displacement and resultant force is assumed to act. The degrees of freedom at each waveguide node are two displacements $(\hat{u}, \hat{v})$ and one rotation (ϕ), giving a total of three degrees of freedom. These displacements and corresponding forces can be represented as

$$\{\hat{u}\} = \begin{Bmatrix} \hat{u} \\ \hat{v} \\ \hat{\phi} \end{Bmatrix}, \qquad \{\hat{F}\} = \begin{Bmatrix} \hat{F} \\ \hat{V} \\ \hat{M} \end{Bmatrix} \quad (5)$$

In order to assemble super element with waveguide, it is necessary to further reduce the connection nodes as shown in Fig. 1 to give average displacement and resultant forces and this is accomplished as

$$\hat{u} = \frac{1}{A}\int_A \hat{u}_c dA,\ \hat{v} = \frac{1}{A}\int_A \hat{v}_c dA,\ \hat{\phi} = \frac{1}{I}\int_A \hat{u}_c y dA, \quad (6)$$

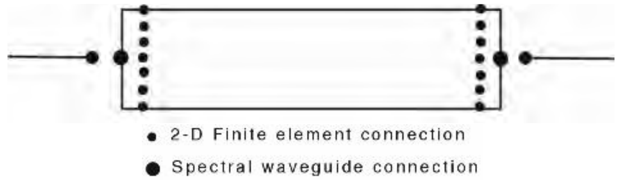

Fig. 1: Connection between super element and waveguide spectral element

where integration is over the connection area A and I is the moment of inertia of cross section. Using this approach $\{\hat{u}_c\}$ vector can be written as

$$\{\hat{u}_c\} = [T]^T \{\hat{u}\}^s \quad (7)$$

where **[*T*]** is the transformation matrix and stiffness relation of super element in terms of an equivalent waveguide element can be written as

$$\{P\}^S = [\hat{K}_{ss}]^s \{\hat{u}\}^s,\ [\hat{K}_{ss}]^s = [T][[\hat{K}_{ss}][T]^T \quad (8)$$

where $[\hat{K}_{ss}]^s$ is the reduced dynamic stiffness matrix of size [6x6] for sub-element of 1-D structure with periodic defects. The elements of transformation matrix [*T*] and the efficient computation of $[\hat{K}_{ss}]^s$ is as given in [2] and the steps and equations are not repeated here.

3. Dispersion Characteristics

The sub-element of the structure is connected at two nodes, with three degrees of freedom at each node, with the rest of the structure resulting in a multi-coupled periodic system with six coupling coordinates. The reduced dynamic stiffness matrix of sub-element can be partitioned into left and right degree of freedoms and equation of motion of sub-element can be written as

$$\begin{Bmatrix} \{F\}_l \\ \{F\}_r \end{Bmatrix} = [\hat{K}] \begin{Bmatrix} \{u\}_l \\ \{u\}_r \end{Bmatrix} = \begin{bmatrix} [\hat{K}_{ll}] & [\hat{K}_{lr}] \\ [\hat{K}_{rl}] & [\hat{K}_{rr}] \end{bmatrix} \begin{Bmatrix} \{u\}_l \\ \{u\}_r \end{Bmatrix}. \quad (9)$$

Here, index l represents the quantities on the left hand side of the element and index r represents the quantities on the right. According to the Bloch's theorem, the quantities on the left are always related to the quantities on the right by an exponential factor, and one may write the following relation using the complex propagation constant $\mu = -\delta + i\varepsilon$ where ε is the phase constant and δ is the attenuation constant. That is,

$$\{\hat{u}_r\} = e^{-\mu}\{\hat{u}_l\} = e^{-ikL}\{\hat{u}_l\},\ \{\hat{F}_r\} = -e^{-\mu}\{\hat{F}_l\} = -e^{-ikL}\{\hat{F}_l\} \quad (10)$$

Here, k is the complex wavenumber with real part defining the phase difference per unit length and imaginary part defining the attenuation per unit length and L is the periodic length. Substitution of these Bloch relations into the equation of motion (Eq. (9)), results in a homogeneous matrix equation which has non-trivial solutions when the determinant of the matrix vanishes, which is given by

$$\left[[\hat{K}_{rl}] + [\hat{K}_{ll} + \hat{K}_{rr}]e^{-\mu} + [\hat{K}_{lr}]e^{-2\mu}\right] = 0 \quad (11)$$

At any frequency six values of μ can exist. The phase constant related to propagating waves is multi-valued. If ε_0 is the solution between 0 and π, then

$$\varepsilon_n = \varepsilon_0 + 2n\pi \ (n = 0, \pm 1, \pm 2, \pm 3 \ ...)$$

is also a solution of determinantal equation. Thus the real part of wavenumber which is the phase difference per unit length can be written as

$$k_n = \pm (\varepsilon_0 + 2n\pi)/L \quad (12)$$

As the real part of wavenumber is multi-valued, an infinite series of (harmonic) waves with the given wavenumbers exists in a periodic system. The positive and negative wavenumbers are related to left and right wards traveling waves, respectively. In analogy to the wavenumber definition for traveling waves, an imaginary wavenumber component for the decaying waves can be defined by

$$k_{decay} = \pm i\delta/L \quad (13)$$

In contrast to the multi-valued solution for the traveling waves, the decaying waves are single valued. Thus six wavenumbers, which corresponds to six waves traveling in waveguide, are calculated and separated out and the wavenumbers, which corresponds to axial and flexural wave, are plotted and studied for different configurations.

4. Numerical Results

In order to get better understanding of dispersion characteristics and band gaps in one dimensional structure with defects, examples with different types of defects are studied and based on the results conclusions will be drawn about the nature of wavenumber variation and band gaps phenomenon. Aluminum (E = 70GPa, G = 27GPa and ρ = 2700kg/m^3) is used as a material of beam and the various dimensions of the periodic cell are taken as length (L) = 2.1 m; width (B) = 0.05 m, thickness (t) = 0.01 m.

4.1 Beam with Periodic Single Horizontal Cracks

Here a periodic horizontal crack is situated in a beam at some offset with respect to beam axis as shown in Fig. 2(a). Here crack length of the horizontal crack is fixed as 2 cm and its offset(c) is varied from 0 to 1.5 cm. Four specific values of offset are 0, 0.5 cm, 1 cm, 1.5 cm; which are used in the analysis, for which dispersion characteristics are plotted and compared. The equivalent FE model for spectral super element approach, for zero offset case, is shown in Fig. 2(b).

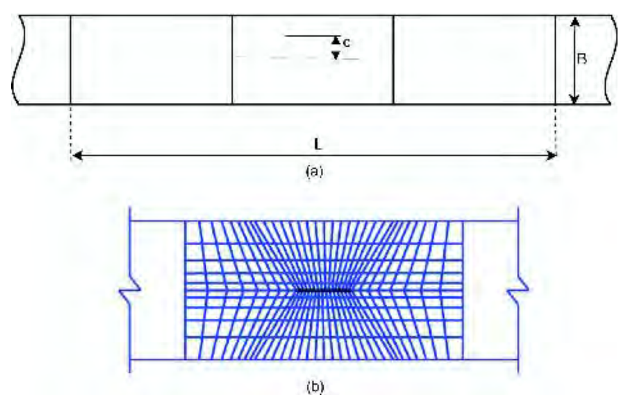

Fig. 2: (a) A periodic cell with axial crack with offset (b) Equivalent FE model of a beam with horizontal crack with zero offset

Horizontal crack can be modelled using both the above-mentioned approaches, namely damaged spectral finite element approach (given in reference [3] and not explained in this paper) and spectral super element approach. Dispersion characteristics for flexural case, obtained for these two different approaches using Euler-Bernoulli beam theory (obtained by setting shear rigidity to infinity and

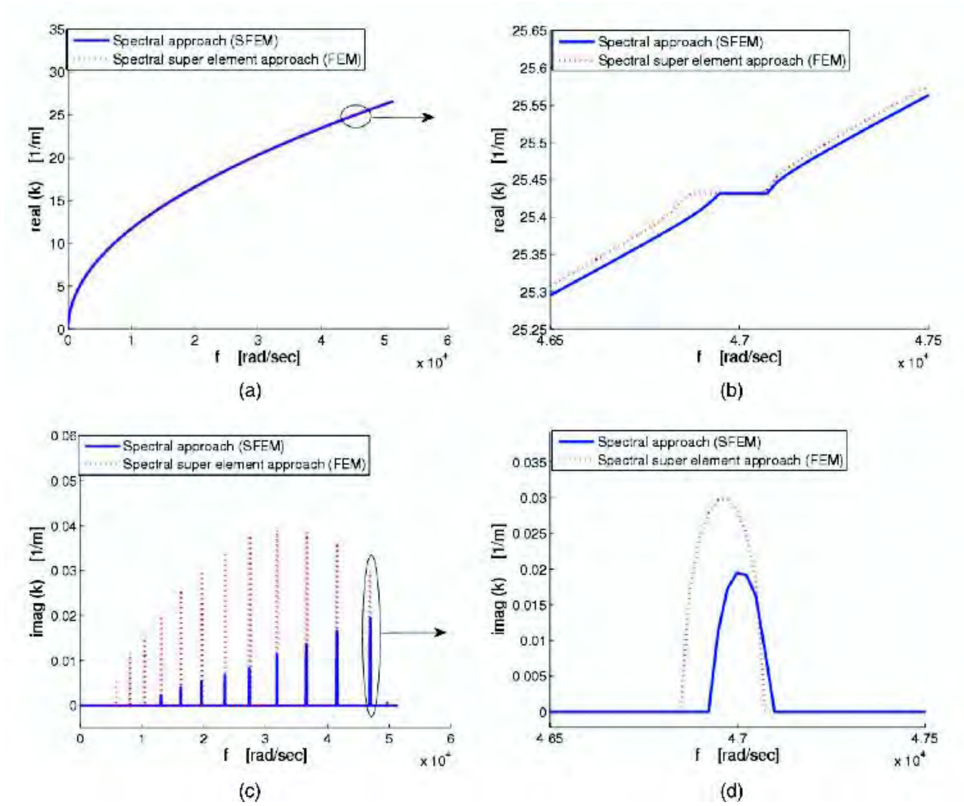

Fig. 3: **Comparison of flexural wavenumber and band gaps in beam with periodic horizontal crack obtained for two different approaches using Euler-Bernoulli beam theory. (a) Real part of flexural wavenumber. (b) Particular stop band in real part of flexural wavenumber. (c) Imaginary part of flexural wavenumber. (d) Corresponding stop band in imaginary part of flexural wavenumber**

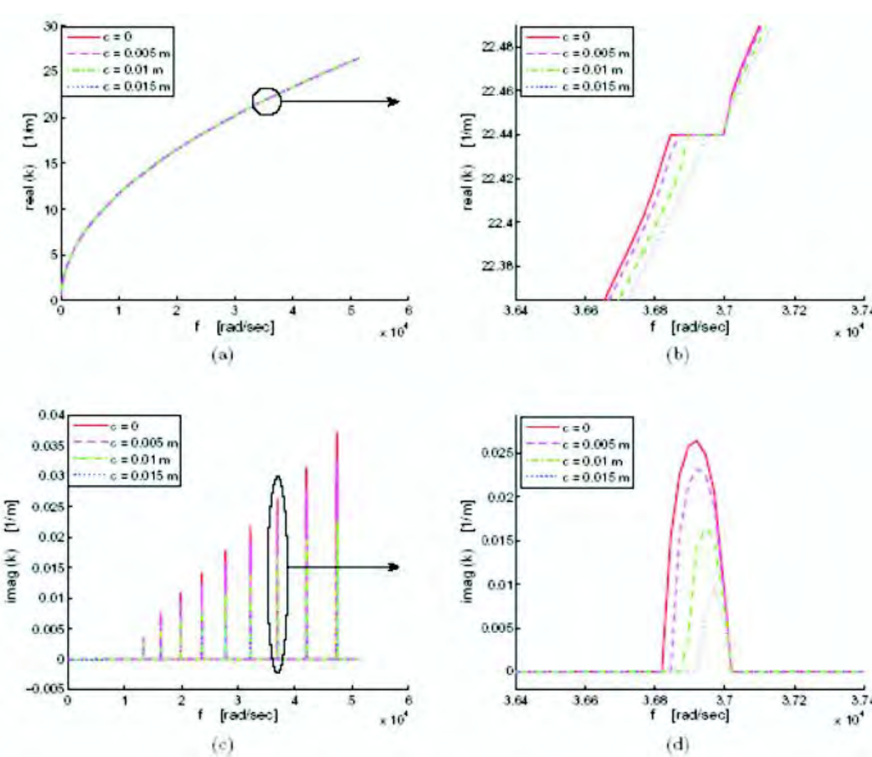

Fig. 4: **Flexural wavenumber and band gaps in beam with periodic horizontal crack, obtained using Euler-Bernoulli beam theory, for four different values of offset (c). (a) Real part of flexural wavenumber. (b) Particular stop band in real part of flexural wavenumber. (c) Imaginary part of flexural wavenumber. (d) Corresponding stop band in imaginary part of flexural wavenumber**

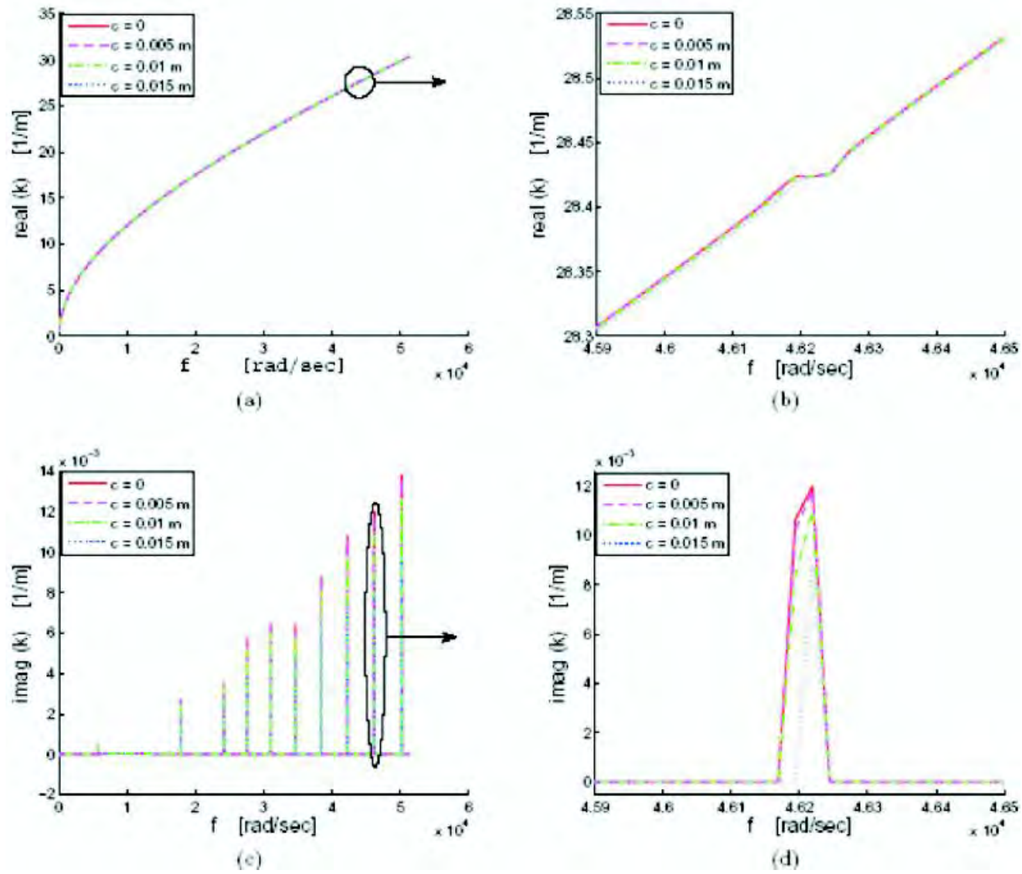

Fig. 5: Flexural wavenumber and band gaps in beam with periodic horizontal crack, obtained using Timoshenko beam theory, for four different values of offset(c). (a) Real part of flexural wavenumber. (b) Particular stop band in real part of flexural wavenumber. (c) Imaginary part of flexural wavenumber. (d) Corresponding stop band in imaginary part of flexural wavenumber

rotational inertia to zero), are compared for a particular case when crack is situated in the middle of beam (c = 0) and shown in Fig. 3. As shown in figure the real part of wavenumber is matching in both the cases, however the amplitude of imaginary part of wavenumber is less in case of spectral finite element approach. As far as the band gaps are concerned, results validates the spectral super element approach

Results for different offset values, using spectral finite element approach, with Euler-Bernoulli beam theory are shown in Fig. 4, while results obtained using Timoshenko beam theory are shown in Fig. 5. These results, obtained from both the beam theories, demonstrate that the band gap in flexural wavenumber reduces as the offset increases and it is maximum when crack is situated at the axis of beam and location of band gaps does not change with the offset provided the length of a crack remains same. However, crack offset does not introduce appreciable variation in the flexural wavenumber.

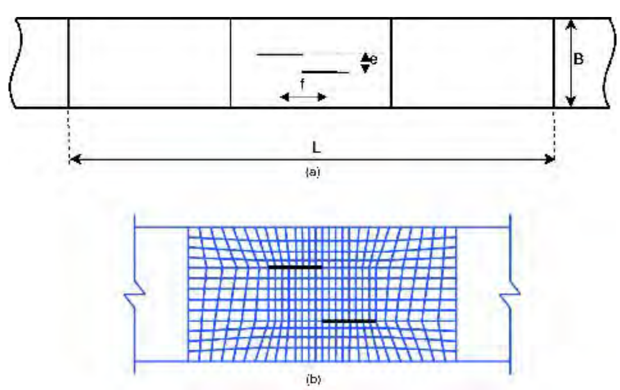

Fig. 6: (a) A periodic cell with two staggered cracks (b) Equivalent FE model of a beam with two staggered cracks

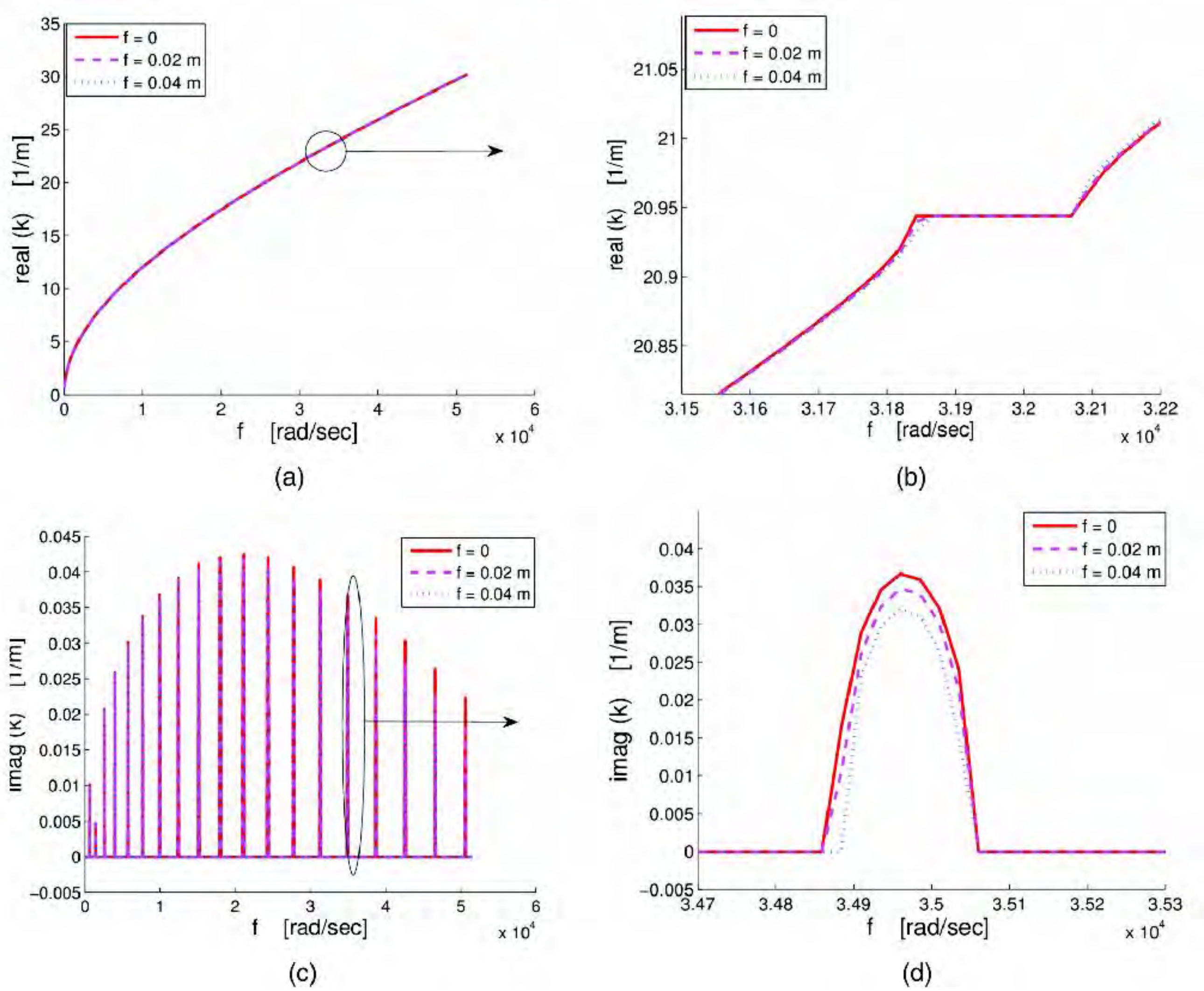

Fig. 7: Comparison of flexural wavenumber and band gaps in beam with periodic two staggered cracks using Timoshenko beam theory. (a) Real part of flexural wavenumber. (b) Particular stop band in real part of flexural wavenumber. (c) Imaginary part of flexural wavenumber. (d) Corresponding stop band in imaginary part of flexural wavenumber.

4.2 Beam with Periodic Horizontal Staggered Cracks

Beam with two periodic horizontal staggered cracks is shown in Fig. 6(a) and corresponding FE mesh for (*f* = *2cm*) in Fig. 6(b). Here crack lengths of both the horizontal cracks are same and taken as 2 cm and the vertical distance between the two cracks (*e*) is taken as 2 cm. The distance between the two cracks locations (*f*) is varied from 0 to 4 cm.

Fig. 7 show the flexural wavenumber variation obtained using Timoshenko beam theory respectively. The band gap in flexural wavenumber reduces as the distance between the two cracks location increases, however the effect of changing distance on the flexural wavenumber is very minimal.

5. Conclusions

A generic approach is developed, using spectral super element theory and Bloch's theorem, to study the dispersion characteristics and band gaps in one dimensional waveguides with arbitrary geometries and waveguides with periodic defects. The complete picture of dispersion characteristics and the band gaps is determined by waveguides with two different defect configurations. For these waveguides with defects, flexural wavenumber and their corresponding band gaps are studied. Keeping in mind that the band gap in any mode acts as a mechanical band pass filter for a wave in that mode, these defects can make the structure dynamically sound in a particular mode in frequency ranges, which match with the stop bands in the corresponding wavenumber. Thus if the structure is subjected to a harmonic load of particular frequency or a load with small frequency spectrum and that if this frequency spectrum lies in the stop bands, then no wave will propagate in the structure and structure will remain safe and sound. Hence, in some cases, if required, these defects can be intentionally introduced in a structure with some periodicity to make the structure dynamically sound.

References

1. Gopalakrishnan S, Chakraborty A and Roy D Mahapatra *Spectral Element Method* Springer-Verlag, United Kingdom (2007)
2. Gopalakrishnan S and Doyle JF *Comput Methods in Appl Mech and Engng* **121(1-4)** (1995) 77-90
3. Nag A, Roy Mahapatra D and Gopalakrishnan S *Struct Health Monitg* **1(1)** (2002) 105-126
4. Brillouin L *Wave Propagation in Periodic Structures* McGraw-Hill, New York (1946)
5. Mead DJ *J Sound and Vib* **40(1)** (1975) 1-18
6. Mead DJ *J Sound and Vib* **40(1)** (1975) 19-39
7. Mead DJ *J Sound and Vib* **190(3)** (1996) 495-524
8. Signorelli J and Von Flotow AH *J Sound and Vib* **126(1)** (1988) 127-144
9. Keane AJ and Price WG *J Sound and Vib* **128(3)** (1989) 423-450
10. Benaroya H *Comp Part B: Engng* **28(1-2)** (1997) 143-152
11. Gopalakrishnan S and Mira Mitra *Wavelet Methods for Dynamical Problems* Taylor and Francis Ltd (2010)
12. Wen J *et al. Phys Lett A* **364** (2007) 323-328
13. Hong WJ *et al. Chinese Phys B* **18(6)** (2009) 2404-2408
14. Wen J *et al. J Appl Phys* **97** (2005)
15. Sastry CVS, Mahapatra DR, Gopalakrishnan S and Ramamurthy TS *Comput Methods in Appl Mech and Engng* **192** (2003) 1821-1840.