

CROSSED PRODUCT ALGEBRAS ATTACHED TO WEIGHT ONE FORMS

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ABSTRACT. We completely determine the Brauer class of the crossed product algebra attached to a non-dihedral cusp form of weight one.

1. Introduction

Let f be a holomorphic cuspidal newform of weight at least one. In weights two and higher, there is a Grothendieck motive M_f (abelian variety A_f in weight two) attached to f . It is a fundamental fact, due to Momose and Ribet [8] in weight two and [1], [4] in higher weights, that the full algebra of endomorphisms of this motive has the structure of an explicit crossed product algebra X , when the form f is of non-dihedral type. Moreover, much, but not all, is known about the Brauer class of this algebra [9], [6], [4].

In this paper, we completely determine the Brauer class of X for non-dihedral cusp forms of weight one. The Brauer class in this case is closely related to the second Witt invariant of the trace form of a number field determined by the projective Artin representation associated to the form.

We show that, much as in higher weights, the Brauer class of X is completely determined at a prime of good reduction by the parity of a certain slope, when this slope is finite (Theorem 2). An interpretation of this slope in terms of the adjoint representation allows us to compute it, showing that, in contrast to what happens in higher weights, the Brauer class is essentially *unramified* at all primes of good reduction (Theorem 4, Corollary 7). At primes of bad reduction we show that the Brauer class is determined purely by the *nebentypus* of the form (Theorem 5, Corollary 7). Finally, as examples, we determine the Brauer class of X for all non-dihedral weight one forms of *prime* level (Corollary 9).

2. The crossed product algebra X

The crossed product algebra X we wish to study can be attached to non-dihedral forms in all weights. We recall the definition for forms of weight one.

Let $f = \sum a_n q^n \in S_1(N, \epsilon)$ be a normalized cuspidal newform of weight one, and nebentypus character ϵ . Let $E = \mathbb{Q}(a_n)$ be the number field generated by the Fourier

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coefficients of f . Let ρ_f be the finite image (Artin) Galois representation attached to f , as constructed in [2, Theorem 4.1], so

$$\rho_f : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(E),$$

and $\mathrm{trace}(\rho_f(\mathrm{Frob}_p)) = a_p$ and $\mathrm{det}(\rho_f(\mathrm{Frob}_p)) = \epsilon(p)$, for all primes $p \nmid N$. That ρ_f has a model over E follows from the oddness of ρ_f [2, p. 521, footnote]. We also note that the extension E/\mathbb{Q} is abelian, since E is generated by the traces of matrices of finite order. By a standard classification, the projective image of ρ_f is either

- a dihedral group, or,
- A_4 , S_4 or A_5 .

In the last three cases we say f is a non-dihedral form. More specifically, in these cases f is said to be of tetrahedral, octahedral, or icosahedral type, respectively.

Let f be of non-dihedral type. Let Γ be the group of inner twists contained in $\mathrm{Aut}(E/\mathbb{Q})$, i.e.,

$$\Gamma = \{\gamma \in \mathrm{Aut}(E) \mid f^\gamma = f \otimes \chi_\gamma, \text{ for some } E^*\text{-valued Dirichlet character } \chi_\gamma\}.$$

Define the E -valued 2-cocycle c on Γ by

$$c(\gamma, \delta) = \frac{G(\chi_\gamma^{-1})G(\chi_\delta^{-\gamma})}{G(\chi_{\gamma\delta}^{-1})} \in E^*,$$

for $\gamma, \delta \in \Gamma$. Let X be the crossed product algebra associated to the cocycle $c(\gamma, \delta)$ defined by

$$(1) \quad X = \bigoplus_{\gamma \in \Gamma} E \cdot x_\gamma,$$

where the x_γ are formal symbols satisfying the relations

$$\begin{aligned} x_\gamma \cdot x_\delta &= c(\gamma, \delta) \cdot x_{\gamma\delta} \\ x_\gamma \cdot e &= \gamma(e) \cdot x_\gamma, \end{aligned}$$

for $\gamma, \delta \in \Gamma$, $e \in E$. Then X is a central simple algebra over F , the fixed field of Γ in E . We note F contains $a_p^2 \epsilon^{-1}(p)$, for $p \nmid N$, and is the subfield of E generated by these quantities.

3. Brauer class of X

In this section we study the Brauer class of X as an element of the Brauer group of F .

3.1. Definition of α . In weights bigger than one, a crucial role is played by a certain map $\alpha : G_{\mathbb{Q}} \rightarrow E^*$. We now define an analogue of this map in the weight one setting.

The map α in higher weights is defined directly on the geometric object A_f or M_f , using the Skolem-Noether theorem [9, §2], [4, §4]. It seems difficult to extend this definition to weight one. However, a purely algebraic definition of α in higher weight was given by Papier (cf. [10, p. 192]), which extends nicely to the weight one setting, as follows.

For $\gamma \in \Gamma$, there is a unique Dirichlet character χ_γ such that $f^\gamma = f \otimes \chi_\gamma$, and hence

$$(2) \quad \rho_{f^\gamma} \sim \rho_f \otimes \chi_\gamma.$$

For $\gamma, \delta \in \Gamma$, the identity $\chi_{\gamma\delta} = \chi_\gamma\chi_\delta^\gamma$ shows that $\gamma \mapsto \chi_\gamma$ is a 1-cocycle. Specializing to $g \in G_\mathbb{Q}$, we see that $\gamma \mapsto \chi_\gamma(g)$ is a 1-cocycle as well. By Hilbert's theorem 90, $H^1(\Gamma, E^*)$ is trivial, i.e., there is an element $\alpha(g) \in E^*$ such that

$$(3) \quad \alpha(g)^{\gamma-1} = \chi_\gamma(g),$$

for all $\gamma \in \Gamma$. Clearly, $\alpha(g)$ is completely determined up to multiplication by elements of F^* . Varying $g \in G_\mathbb{Q}$, we obtain a well defined map

$$\tilde{\alpha} : G_\mathbb{Q} \rightarrow E^*/F^*.$$

Since each χ_γ is a character, $\tilde{\alpha}$ is a homomorphism. We can now easily verify all the relevant properties of α which hold in higher weight ([7, Theorem 1.1] and [11, Theorem 5.5]), though the proofs are a bit different.

Lemma 1. *The homomorphism $\tilde{\alpha}$ satisfies:*

- (1) $\tilde{\alpha}$ is unramified at all primes $p \nmid N$.
- (2) For all $g \in G_\mathbb{Q}$, we have $\alpha^2(g) \equiv \epsilon(g) \pmod{F^*}$.
- (3) If $a_p \neq 0$, then $\alpha(\text{Frob}_p) \equiv a_p \pmod{F^*}$.

Proof. (1) Since both ρ_{f^γ} and ρ_f are unramified at $p \nmid N$, we have $\chi_\gamma(g) = 1$ for all $g \in I_p$, by display (2). The relation (3) shows that $\alpha(g) \in F^*$, for all $g \in I_p$, so $\tilde{\alpha}$ is unramified at $p \nmid N$.
 (2) Taking determinants in display (2), we have $\chi_\gamma^2 = \epsilon^{\gamma-1}$. Again by equation (3) we have $\alpha^2 \equiv \epsilon \pmod{F^*}$.
 (3) If the trace of $\rho_f(g)$ for $g \in G_\mathbb{Q}$ is non-zero, then display (2) and equation (3) show that $\alpha(g) \equiv \text{trace}(\rho_f(g)) \pmod{F^*}$. In particular, if $p \nmid N$ and $a_p \neq 0$, then $\alpha(\text{Frob}_p) \equiv a_p \pmod{F^*}$. □

3.2. The 2-cocycle undelying X . Let $G_F = \text{Gal}(\bar{\mathbb{Q}}/F)$. By [9, Proposition 1, p. 9], whose proof also holds in weight one, the class of X in $\text{Br}(F) = H^2(G_F, \mathbb{Q}^*)$ is given by the 2-cocycle

$$(g, h) \mapsto \chi_g(h),$$

for $g, h \in G_F$, where $\chi_g := \chi_\gamma$, for γ the image of g in Γ . By (3), this 2-cocycle is the same as the 2-cocycle given by

$$(g, h) \mapsto \frac{\alpha(h)^g}{\alpha(h)},$$

which in turn differs from the 2-cocycle

$$(4) \quad c(g, h) = \frac{\alpha(g)\alpha(h)}{\alpha(gh)}$$

by a coboundary. Hence, the class of X is given by the 2-cocycle $c(g, h)$ above.

Observe that the class of c is independent of the lift α of $\tilde{\alpha}$. Suppose α' is another lift of $\tilde{\alpha}$. Then $\alpha'(g) = \alpha(g) \cdot f(g)$, for some map $f : G_F \rightarrow F^*$. Let us denote the 2-cocycle obtained from α' by c' . Then c and c' differ by the map $(g, h) \mapsto \frac{f(g)f(h)}{f(gh)}$, which is clearly a 2-coboundary.

We also note that the class of c (hence X) is 2-torsion in the Brauer group of F . This follows immediately from part (2) of Lemma 1, noting that $c^2(g, h) = d(g)d(h)/d(gh)$ is a 2-coboundary, since $d(g) := \alpha^2(g)/\epsilon(g) \in F^*$.

3.3. Local Brauer class. To study the Brauer class of X , it suffices, as usual, to specify the Brauer class of $X_v := X \otimes_F F_v$ in $\text{Br}(F_v)$, for each place v of F , where F_v is the completion of F at v . The case of infinite v will be treated shortly. It is well-known that if v is finite, then

$$\text{inv}_v : \text{Br}(F_v) \simeq \mathbb{Q}/\mathbb{Z}$$

via the invariant map inv_v at v .

To aid in our computation of $\text{inv}_v(X_v)$, for v finite, it is useful to recall the explicit definition of the invariant map (cf. [13]), which we do now. Let I_v be the inertia subgroup of G_F at the prime v . Let $G_v^{\text{un}} = \text{Gal}(F_v^{\text{nr}}/F_v)$ be the Galois group of F_v^{nr} , the maximal unramified extension of F_v , over F_v . The inflation map

$$\text{Inf} : \text{H}^2(G_v^{\text{un}}, (F_v^{\text{nr}})^*) \rightarrow \text{Br}(F_v)$$

is well-known to be an isomorphism. Now, the surjective valuation $v : F^* \rightarrow \mathbb{Z}$ can be extended uniquely to $(F_v^{\text{nr}})^*$ which we continue to call v . This gives rise to a map

$$v : \text{H}^2(G_v^{\text{un}}, (F_v^{\text{nr}})^*) \rightarrow \text{H}^2(G_v^{\text{un}}, \mathbb{Z})$$

which we again denote by v . Also, the short exact sequence of abelian groups $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ gives rise to a long exact sequence of cohomology groups, with boundary map

$$\delta : \text{H}^1(G_v^{\text{un}}, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{H}^2(G_v^{\text{un}}, \mathbb{Z}),$$

which is an isomorphism, since $\text{H}^i(G_v^{\text{un}}, \mathbb{Q}) = 0$, for $i = 1, 2$. We recall the definition of δ . If $\chi : G_v^{\text{un}} \rightarrow \mathbb{Q}/\mathbb{Z}$ is a homomorphism, and $\tilde{\chi}$ is a lift of χ to \mathbb{Q} , then $\delta(\chi)$ is the \mathbb{Z} -valued 2-cocycle on G_v^{un} given by

$$(g, h) \mapsto \frac{\tilde{\chi}(g)\tilde{\chi}(h)}{\tilde{\chi}(gh)}.$$

Finally, there is a map, say Ev (for evaluation)

$$\text{Ev} : \text{H}^1(G_v^{\text{un}}, \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

obtained by evaluating a homomorphism at the Frobenius at v . Then, by definition, the invariant map at v is given by

$$\text{inv}_v = \text{Ev} \circ \delta^{-1} \circ v \cdot \text{Inf}^{-1} : \text{Br}(F_v) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Since the class of X is 2-torsion in $\text{Br}(F)$, the class of X_v is also 2-torsion in $\text{Br}(F_v)$, so its image under inv_v will be in $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Identifying this group with $\mathbb{Z}/2$, we see that X_v is a matrix algebra over F_v if $\text{inv}_v(X_v) = 0 \pmod{2}$, and is a matrix algebra over a quaternion division algebra over F_v if $\text{inv}_v(X_v) = 1 \pmod{2}$.

3.4. Good primes. We can now calculate the local Brauer class X_v at a prime $v|p$ with $p \nmid N$ (good prime), using the explicit definition of the invariant map just given. For each such v , define

$$m_v := [F_v : \mathbb{Q}_p] \cdot v(a_p^2 \epsilon^{-1}(p)) \in \mathbb{Z} \cup \{\infty\}$$

where v is now normalized so that $v(p) = 1$.

The following theorem relating the ramification of X_v to the parity of m_v , when it is finite, is the weight one analogue of [4, Theorem 2.2] and [1, Theorem 1.0.4]. Although we will shortly show that in weight one, $m_v = 0$ almost always, when it is finite, we still spend some time on the proof of the theorem here, since it is simple, works in all weights, and includes the case $p = 2$.

Theorem 2. *Let $p \nmid N$ be a prime with $a_p \neq 0$, and let v be a place of F lying over p . Then $\text{inv}_v(X_v) = m_v \pmod 2$. Thus X_v is a matrix algebra over F_v if and only if the normalized slope $[F_v : \mathbb{Q}_p] \cdot v(a_p^2 \epsilon^{-1}(p))$ is even.*

Proof. X_v corresponds to the restriction of the 2-cocycle c in 4 to $G_v := \text{Gal}(\bar{F}_v/F_v)$. We compute $\text{inv}_v(c) = (\text{Ev} \circ \delta^{-1} \circ v \circ \text{Inf}^{-1})(c)$ step by step.

Since inflation is an isomorphism there exists a 2-cocycle \tilde{c} on G_v/I_v such that

$$\tilde{c}(gI_v, hI_v) = c(g, h)$$

up to a 2-coboundary. In fact, if we take the above equality to be the definition of \tilde{c} , then it is easy to check that \tilde{c} is a 2-cocycle on G_v/I_v , since c is a 2-cocycle, and that it is well defined. Indeed, since $\tilde{\alpha}$ is unramified at p (cf. part (1) of Lemma 1), we have $\tilde{\alpha}(gi) = \tilde{\alpha}(g)\tilde{\alpha}(i) = \tilde{\alpha}(g)$, for $g \in G_v$ and $i \in I_v$. Since we may replace α by any lift of $\tilde{\alpha}$, we may assume $\alpha(gi) = \alpha(g)$, for $g \in G_v$ and $i \in I_v$. Then, for $g, h \in G_v$ and $i, j \in I_v$, the quantities $c(g, h)$ and $c(gi, hj)$ coincide, and the cocycle $\tilde{c}(gI_v, hI_v)$ is independent of the choice of the representatives. Clearly the image of this particular \tilde{c} under the inflation map is c . Now, by definition, $v(\tilde{c})$ is the 2-cocycle induced by

$$(g, h) \mapsto v \left(\frac{\alpha(g)\alpha(h)}{\alpha(gh)} \right)$$

for $g, h \in G_F$. Recall $d(g) = \frac{\alpha^2(g)}{\epsilon(g)}$, for $g \in G_F$. Note $d(g) \in F^*$ by part (2) of Lemma 1. Then the 2-cocycle above is the same as the 2-cocycle induced by

$$(g, h) \mapsto \frac{1}{2} \cdot v \left(\frac{d(g)d(h)}{d(gh)} \right).$$

Consider now the map $\chi : \text{Gal}(F_v^{\text{nr}}/F_v) \rightarrow \mathbb{Q}/\mathbb{Z}$ defined by

$$\chi(g) = \frac{1}{2} \cdot v(d(g)) \pmod{\mathbb{Z}}.$$

Under the map δ the 1-cocycle χ maps to the 2-cocycle above, so $(\delta^{-1} \circ v \circ \text{Inf}^{-1})(c)$ is just χ . Hence

$$\text{inv}_v(c) = (\text{Ev} \circ \delta^{-1} \circ v \circ \text{Inf}^{-1})(c) = \chi(\text{Frob}_v) = \frac{1}{2} \cdot v(d(\text{Frob}_v)) \pmod{\mathbb{Z}}.$$

Let e_v and f_v be the ramification index and residue degree of F_v/\mathbb{Q}_p . Now Frob_v is the f_v -th power of Frob_p , so we have

$$\text{inv}_v(c) = \frac{1}{2} \cdot f_v \cdot v(d(\text{Frob}_p)) \pmod{\mathbb{Z}}.$$

And, by part (3) of Lemma 1, we have $\alpha(\text{Frob}_p) \equiv a_p \pmod{F^*}$, so that $d(\text{Frob}_p) = a_p^2/\epsilon(p) \pmod{(F^*)^2}$. It follows that

$$\text{inv}_v(c) = \frac{1}{2} \cdot f_v \cdot v(a_p^2/\epsilon(p)) \pmod{\mathbb{Z}}.$$

Finally the valuation considered in the statement of the theorem is normalized so that $v(p) = 1$, and differs from the valuation used in the proof by a factor of e_v . Noting $e_v f_v = [F_v : \mathbb{Q}_p]$, we obtain the theorem. \square

3.5. Adjoint representation. In this section we will compute the quantity m_v appearing in Theorem 2 using the adjoint representation. This will allow us to essentially completely determine the Brauer class of the algebra X at the good places $v|p$ with $a_p \neq 0$. The remaining good places v for which a_p vanishes are treated in the next section.

Recall that the adjoint representation

$$\text{Ad}(\rho_f) : G_{\mathbb{Q}} \rightarrow \text{GL}_4(E)$$

attached to the representation $\rho_f : G_{\mathbb{Q}} \rightarrow \text{GL}_2(E)$ is the 4-dimensional representation defined by the adjoint action:

$$\text{Ad}(\rho_f)(g)(X) = \rho_f(g)X\rho_f(g)^{-1},$$

for $X \in M_{2 \times 2}(E)$. It is important for us because of the following elementary but useful fact.

Lemma 3. *Say $p \nmid N$. Then*

$$\text{trace}(\text{Ad}(\rho_f)(\text{Frob}_p)) = \frac{a_p^2}{\epsilon(p)}.$$

Proof. If

$$\rho_f(\text{Frob}_p) \sim \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix},$$

then a small computation shows that

$$\text{Ad}(\rho_f)(\text{Frob}_p) \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha_p}{\beta_p} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\beta_p}{\alpha_p} \end{pmatrix}.$$

So

$$\text{trace}(\text{Ad}(\rho_f)(\text{Frob}_p)) = 2 + \frac{\alpha_p}{\beta_p} + \frac{\beta_p}{\alpha_p} = \frac{(\alpha_p + \beta_p)^2}{\alpha_p \beta_p} = \frac{a_p^2}{\epsilon(p)}.$$

\square

Theorem 4 (Good primes of finite slope). *Let f be a non-dihedral cuspidal newform in $S_1(N, \epsilon)$. Then*

X is unramified at v for all good primes $v|p$ with $p \nmid N$ and $a_p \neq 0$, except possibly in the case when $p = 2$ and the projective image of ρ_f is S_4 . Moreover, if the projective image of ρ_f is

- (1) A_4 , then $F = \mathbb{Q}$,
- (2) S_4 , then $F = \mathbb{Q}$,
- (3) A_5 , then $F = \mathbb{Q}(\sqrt{5})$.

Proof. We have $\text{Ad}(\rho_f) = \text{Ad}^0(\rho_f) \oplus 1$, where $\text{Ad}^0(\rho_f)$ is the irreducible 3-dimensional representation, afforded by the trace zero matrices.

Say the projective image of ρ_f is A_4 . From the character table of A_4 , we see that A_4 has one 3-dimensional irreducible representation V , and three 1-dimensional representations (cf. [3, p. 20]). Thus

$$\text{Ad}(\rho_f) = V \oplus U$$

where U is the trivial representation. Computing the character of $V \oplus U$ on the four conjugacy classes of A_4 , and using Lemma 3 above, we obtain

$$\frac{a_p^2}{\epsilon(p)} = 4, 1, 1 \text{ or } 0.$$

Thus $F = \mathbb{Q}$ and by Theorem 2, X is unramified at all primes $v = p$, with $a_p \neq 0$ (note the 2-adic valuation of 4 is even!).

Suppose now that the projective image of ρ_f is S_4 . From the character table of S_4 (cf. [3, p. 19]), we see that S_4 has two 1-dimensional representations, U (trivial) and U' (sign), one 2-dimensional irreducible representation, and two 3-dimensional irreducible representations V and $V \otimes U'$. A small check shows that $\text{Ad}^0(\rho_f) = V \otimes U'$ (and not V), so that $\text{Ad}(\rho_f) = (V \otimes U') \oplus U$. Computing traces on the 5 conjugacy classes as above, we get this time that

$$\frac{a_p^2}{\epsilon(p)} = 4, 0, 1, 2 \text{ or } 0.$$

So again, $F = \mathbb{Q}$, and if $v = p$ is a good odd prime with $a_p \neq 0$, then by Theorem 2, X is unramified at v . Note that since the 2-adic valuation of 2 is odd, X may be ramified at the prime 2 (we shall later give examples where in fact X_2 is ramified).

Finally, let the projective image of ρ_f be A_5 . This time, there are two irreducible 3-dimensional representations Y and Z , and so $\text{Ad}(\rho_f) = Y \oplus U$ or $Z \oplus U$, where U is the trivial representation. Again, computing traces on the 5 conjugacy classes using the character table of A_5 (cf. [3, p. 29]) we get

$$\frac{a_p^2}{\epsilon(p)} = 4, 1, 0, \frac{3 \pm \sqrt{5}}{2} \text{ or } \frac{3 \mp \sqrt{5}}{2}.$$

Since $F = \mathbb{Q}(\frac{a_p^2}{\epsilon(p)})$, we conclude that $F = \mathbb{Q}(\sqrt{5})$. Now $\text{Norm}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(\frac{3 \pm \sqrt{5}}{2}) = 1$, so that $\frac{3 \pm \sqrt{5}}{2}$ are units in F , and $v(\frac{3 \pm \sqrt{5}}{2}) = 0$ for all primes v of F . Again, by Theorem 2, we conclude that X_v is unramified for all good places v of F lying above p with $a_p \neq 0$. \square

3.6. Symbols. In this section we give formulas for the Brauer class of X in terms of symbols, which will help us determine the Brauer class completely.

Let $\tilde{\rho}_f$ be the projectivization of ρ_f . Let D_K be the discriminant of the unique quadratic extension K/\mathbb{Q} in the S_4 -number field cut out by $\tilde{\rho}_f$, when $\tilde{\rho}_f(G_{\mathbb{Q}}) = S_4$. Define the 2-cocycle c_ϵ on $G_{\mathbb{Q}}$ by

$$c_\epsilon(g, h) = \frac{\sqrt{\epsilon(g)}\sqrt{\epsilon(h)}}{\sqrt{\epsilon(gh)}},$$

for $g, h \in G_{\mathbb{Q}}$. Then $[c_\epsilon]$ is 2-torsion in $\text{Br}(\mathbb{Q})$. Let $[c_\epsilon]$ and $[X]$ also denote the images in $\text{Br}(F)$ under the restriction map $\text{H}^2(G_{\mathbb{Q}}, \bar{F}^*) \rightarrow \text{H}^2(G_F, \bar{F}^*)$.

Theorem 5. *The class of X in $\text{Br}(F)$ is as follows. If the image of $\tilde{\rho}_f$ is*

- (1) A_4 , then $[X] = [c_\epsilon]$,
- (2) S_4 , then $[X] = [c_\epsilon] \cdot (2, D_K)$, where (a, b) is the symbol for $F = \mathbb{Q}$,
- (3) A_5 , then $[X] = [c_\epsilon]$.

Proof. We use a general formula from [6] which works for weight one forms as well:

$$[X] = [c_\epsilon] \cdot [c_d],$$

where $[c_\epsilon] \in \text{Br}(F)$ is as above, and $[c_d] \in \text{Br}(F)$ is the product of the symbols

$$[c_d] = (t_1, d_1) \cdots (t_n, d_n),$$

where t_i and d_i are determined as follows. Note that $d = \alpha^2/\epsilon$ induces a continuous map

$$d : G_{\mathbb{Q}} \rightarrow F^*/F^{*2},$$

where F^*/F^{*2} has the discrete topology. Thus $G_{\mathbb{Q}}/\ker(d) \simeq \text{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/2)^m$, for some elementary 2-extension K of \mathbb{Q} . Now, for each i between 1 and m , let $\sigma_i \in \text{Gal}(K/\mathbb{Q})$ denote the element corresponding to $(0, 0, \dots, 1, \dots, 0) \in (\mathbb{Z}/2)^m$, with a 1 in the i -th spot. Then $t_j \in \mathbb{Q}^*$ is defined by $\sigma_i(\sqrt{t_j}) = (-1)^{\delta_{ij}}\sqrt{t_j}$. We lift each σ_i to an element in $G_{\mathbb{Q}}$ which we continue to call σ_i , and set $d_i := d(\sigma_i) \in F^*/F^{*2}$.

We now claim that $\ker(\tilde{\rho}_f) \subset \ker d$. This is immediate, since if $\tilde{\rho}_f(g) = 1$, for some $g \in G_{\mathbb{Q}}$, then $\rho_f(g)$ is a scalar, so $\text{trace}(\rho_f(g))^2/\det(\rho_f(g)) = 4$. Since the trace is non-zero, this last expression is also equal to $\alpha^2(g)/\epsilon(g)$ up to an element of F^{*2} , and hence $d(g) = 1 \pmod{F^{*2}}$, proving the claim. Thus, there is a surjection

$$G_{\mathbb{Q}}/\ker(\tilde{\rho}_f) \twoheadrightarrow G_{\mathbb{Q}}/\ker(d).$$

The group on the left is A_4 , S_4 or A_5 , and the group on the right is an elementary 2-group. We conclude that in the A_4 and A_5 cases, the 2-group is trivial, and hence d is trivial. This finishes the proof in these cases. In the S_4 case, the group on the right must be $\mathbb{Z}/2$, or trivial. By our earlier computation of the character of the adjoint representation and the Chebotarev density theorem, there is a prime $p \nmid N$ such that $a_p/\sqrt{\epsilon(p)} = \pm\sqrt{2}$. By part (3) of Lemma 1, $d(\text{Frob}_p) = 2 \pmod{F^*}$ is non-trivial for such a prime p . Hence, the group on the right is $\mathbb{Z}/2$. The resulting quadratic field cut out by d must be K . Thus $m = 1$, $t_1 = D_K$, and $d_1 = 2$ up to a square in F^* . This shows that the class of c_d is given by the symbol $(2, D_K)$, completing the proof in the S_4 case as well. \square

3.7. Relation with Witt invariant. The Brauer class of the algebra X is closely related to the 2nd Witt invariant $w_2(Q_{K_1}) \in \text{Br}(\mathbb{Q})$ of the quadratic form $Q_{K_1}(x) = \text{Tr}_{K_1/\mathbb{Q}}(x^2)$, studied in [14] and [5]. In the A_5 case, we let $w_2(Q_{K_1})$ also denote the image of $w_2(Q_{K_1})$ in $\text{Br}(F)$ under the restriction map: $H^2(G_{\mathbb{Q}}, \bar{F}^*) \rightarrow H^2(G_F, \bar{F}^*)$. Here K_1 is a number field of degree 4 (A_4 and A_5 cases) or 6 (A_5 case) with Galois closure equal to the fixed field of the kernel of the projective representation $\tilde{\rho}_f$ (and is determined up to isomorphism by this property).

Proposition 6. *The Brauer class of X is related to the image in $\text{Br}(F)$ of the Witt invariant $w_2(Q_{K_1})$ as follows. If the image of $\tilde{\rho}_f$ is*

- (1) A_4 , then $[X] = w_2(Q_{K_1})$,
- (2) S_4 , then $[X] = w_2(Q_{K_1}) \cdot (-1, D_K)$, where (a, b) is the symbol for $F = \mathbb{Q}$,
- (3) A_5 , then $[X] = w_2(Q_{K_1})$.

Proof. We use notation from [5]. By Tate’s theorem on p. 560 and the remark on p. 561 of *loc. cit.*, we have $\rho^* \beta_- = \rho^* c_2 = \epsilon^* c_1$ which is our $[c_\epsilon]$. The result now follows from [5, Theorem 3.6] and Theorem 5 above, noting that in the octahedral case, the discriminant of the quartic field K_1 differs from that of the quadratic field K by a square. □

3.8. Brauer class. We now state the following strengthening of Theorem 4, which includes the prime of good reduction for which a_p vanishes, and gives complete information at the remaining primes as well.

Corollary 7 (Brauer class). *X is unramified at all primes of good reduction, except possibly in the S_4 case when $p = 2$. At primes of bad reduction, the ramification of X is controlled by the parity of the local factors of ϵ . Finally, X is ramified at all the infinite places.*

Proof. It is known that $[c_\epsilon]$ is ramified at the infinite places since ϵ is odd, and at finite places $v|p$ of F , we have $[c_\epsilon]_v$ is trivial in $\text{Br}(F_v)$ if and only if $\epsilon_v(-1) = 1$ [6]. Moreover, in the S_4 case, K is also easily checked to be the quadratic field cut out by the homomorphism $\epsilon : S_4 \rightarrow \mathbb{Q}(\epsilon)^*/\mathbb{Q}(\epsilon)^{*2}$ induced by the determinant of ρ_f , and $p|D_K$ implies $p|N$. The result now follows from Theorem 5. □

3.9. Prime level. As an example we completely determine the Brauer class of X for forms of *prime* level. We have the following result [12, Theorem 7], which shows that only the octahedral and icosahedral cases occur in prime level.

Theorem 8. *Say $f \in S_1(p, \epsilon)$ is a non-dihedral cuspidal newform of prime level p . Then*

- (a) p is odd and $p \not\equiv 1 \pmod{8}$
- (b) If $p \equiv 5 \pmod{8}$, then ϵ is of order 4 and the projective image of ρ_f is S_4
- (c) If $p \equiv 3 \pmod{4}$, then ϵ is of order 2 and the projective image of ρ_f is S_4 or A_5 .

Following [12], we break case (c) into two cases (c1) and (c2), depending on whether the projective image of ρ_f is S_4 or A_5 . Theorem 5 gives:

Corollary 9. *For a weight one form of prime level p as above, the Brauer class of X in $\text{Br}(F)$ is:*

- (b) $[X] = (-1, -2)$
- (c1) $[X] = (-2, -p)$
- (c2) $[X] = (-1, -p),$

where (a, b) is the symbol for F .

In particular, there are octahedral forms of odd level for which X is ramified at the good prime 2.

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