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Homogeneous operators and projective representations of the Möbius group: A survey

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Abstract. This paper surveys the existing literature on homogeneous operators and their relationships with projective representations of $PSL(2, \mathbb{R})$ and other Lie groups. It also includes a list of open problems in this area.

Keywords. Projective representations; homogeneous operators; reproducing kernels; Sz-Nagy–Foias characteristic functions.

1. Preliminaries

This paper is a survey of the known results on homogeneous operators. A small proportion of these results are as yet available only in preprint form. A miniscule proportion may even be new. The paper ends with a list of thirteen open problems suggesting possible directions for future work in this area. This list is not purported to be exhaustive, of course!

All Hilbert spaces in this paper are separable Hilbert spaces over the field of complex numbers. All operators are bounded linear operators between Hilbert spaces. If $\mathcal{H}$, $\mathcal{K}$ are two Hilbert spaces, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ will denote the Banach space of all operators from $\mathcal{H}$ to $\mathcal{K}$, equipped with the usual operator norm. If $\mathcal{H} = \mathcal{K}$, this will be abridged to $\mathcal{B}(\mathcal{H})$. The group of all unitary operators in $\mathcal{B}(\mathcal{H})$ will be denoted by $\mathcal{U}(\mathcal{H})$. When equipped with any of the usual operator topology $\mathcal{U}(\mathcal{H})$ becomes a topological group. All these topologies induce the same Borel structure on $\mathcal{U}(\mathcal{H})$. We shall view $\mathcal{U}(\mathcal{H})$ as a Borel group with this structure.

$\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the integers, the real numbers and the complex numbers, respectively. $\mathbb{D}$ and $\mathbb{T}$ will denote the open unit disc and the unit circle in $\mathbb{C}$, respectively, and $\bar{\mathbb{D}}$ will denote the closure of $\mathbb{D}$ in $\mathbb{C}$. Möb will denote the Möbius group of all biholomorphic automorphisms of $\mathbb{D}$. Recall that Möb $= \{\varphi_{\alpha,\beta} : \alpha \in \mathbb{T}, \beta \in \mathbb{D}\}$, where

$$\varphi_{\alpha,\beta}(z) = \alpha \frac{z - \beta}{1 - \bar{\beta} z}, \quad z \in \mathbb{D}. \tag{1.1}$$

For $\beta \in \mathbb{D}$, $\varphi_\beta := \varphi_{-1,\beta}$ is the unique involution (element of order 2) in Möb which interchanges 0 and β. Möb is topologized via the obvious identification with $\mathbb{T} \times \mathbb{D}$. With this topology, Möb becomes a topological group. Abstractly, it is isomorphic to $PSL(2, \mathbb{R})$ and to $PSU(1, 1)$.

The following definition from [6] has its origin in the papers [21] and [22] by the second named author.

DEFINITION 1.1

An operator T is called *homogeneous* if $\varphi(T)$ is unitarily equivalent to T for all φ in Möb which are analytic on the spectrum of T.

It was shown in Lemma 2.2 of [6] that

Theorem 1.1. *The spectrum of any homogeneous operator T is either $\mathbb{T}$ or $\bar{\mathbb{D}}$. Hence $\varphi(T)$ actually makes sense (and is unitarily equivalent to T) for all elements φ of* Möb.

Let $*$ denote the involution (i.e. automorphism of order two) of Möb defined by

$$\varphi^*(z) = \overline{\varphi(\bar{z})}, \quad z \in \mathbb{D}, \ \varphi \in \text{Möb}. \tag{1.2}$$

Thus $\varphi^*_{\alpha,\beta} = \varphi_{\bar{\alpha},\bar{\beta}}$ for $(\alpha, \beta) \in \mathbb{T} \times \mathbb{D}$. It is known that essentially (i.e. up to multiplication by arbitrary inner automorphisms), $*$ is the only outer automorphism of Möb. It also satisfies $\varphi^*(z) = \varphi(z^{-1})^{-1}$ for $z \in \mathbb{T}$. It follows that for any operator T whose spectrum is contained in $\bar{\mathbb{D}}$, we have

$$\varphi(T^*) = \varphi^*(T)^*, \quad \varphi(T^{-1}) = \varphi^*(T)^{-1}, \tag{1.3}$$

the latter in case T is invertible, of course. It follows immediately from (1.3) that the adjoint T^* – as well as the inverse T^{-1} in case T is invertible – of a homogeneous operator T is again homogeneous.

Clearly a direct sum (more generally, direct integral) of homogeneous operators is again homogeneous.

2. Characteristic functions

Recall that an operator T is called a *contraction* if $\|T\| \le 1$, and it is called *completely non-unitary* (cnu) if T has no non-trivial invariant subspace $\mathcal{M}$ such that the restriction of T to $\mathcal{M}$ is unitary . T is called a *pure contraction* if $\|Tx\| < \|x\|$ for all non-zero vectors x. To any cnu contraction T on a Hilbert space, Sz-Nagy and Foias associate in [25] a pure contraction valued analytic function θ_T on $\mathbb{D}$, called the *characteristic function* of T.

Reading through [25] one may get the impression that the characteristic function is only contraction valued and its value at 0 is a pure contraction. However, if θ is a contraction valued analytic function on $\mathbb{D}$ and the value of θ at some point is pure, its value at all points must be pure contractions. This is immediate on applying the strong maximum modulus principle to the function $z \to \theta(z)x$, where x is an arbitrary but fixed non-zero vector.

Two pure contraction valued analytic functions $\theta_i : \mathbb{D} \to \mathcal{B}(\mathcal{K}_i, \mathcal{L}_i)$, $i = 1, 2$ are said to *coincide* if there exist two unitary operators $\tau_1 : \mathcal{K}_1 \to \mathcal{K}_2$, $\tau_2 : \mathcal{L}_1 \to \mathcal{L}_2$ such that $\theta_2(z)\tau_1 = \tau_2\theta_1(z)$ for all $z \in \mathbb{D}$. The theory of Sz-Nagy and Foias shows that (i) two cnu contractions are unitarily equivalent if and only if their characteristic functions coincide, (ii) any pure contraction valued analytic function is the characteristic function of some cnu contraction. In general, the model for the operator associated with a given function θ is difficult to describe. However, if θ is an inner function (i.e., θ is isometry-valued on the boundary of $\mathbb{D}$), the description of the Sz-Nagy and Foias model simplifies as follows:

Theorem 2.1. *Let $\theta : \mathbb{D} \to \mathcal{B}(\mathcal{K}, \mathcal{L})$ be a pure contraction valued inner analytic function. Let $\mathcal{M}$ denote the invariant subspace of $H^2(\mathbb{D}) \otimes \mathcal{L}$ corresponding to θ in the sense of*

Beurling's theorem. That is, $\mathcal{M} = \{z \mapsto \theta(z) f(z) : f \in H^2(\mathbb{D}) \otimes \mathcal{K}\}$*. Then* θ *coincides with the characteristic function of the compression of multiplication by* z *to the subspace* $\mathcal{M}^\perp$.

From the general theory of Sz-Nagy and Foias outlined above, it follows that if T is a cnu contraction with characteristic function θ then, letting $T[\mu]$ denote the cnu contraction with characteristic function $\mu\theta$ for $0 < \mu \leq 1$, we find that $\{T[\mu] : 0 < \mu \leq 1\}$ is a continuum of mutually unitarily inequivalent cnu contractions (provided θ is not the identically zero function, of course). In general, it is difficult to describe these operators explicitly in terms of T alone. But, in [7], we succeeded in obtaining such a description in case θ is an inner function (equivalently, when T is in the class $C_{\cdot 0}$, i.e., $T^{*n}x \to 0$ as $n \to \infty$ for every vector x) – so that T has the description in terms of θ given in Theorem 2.1. Namely, for a suitable Hilbert space $\mathcal{L}$, T may be identified with the compression of M to $\mathcal{M}^\perp$, where $M : H^2_{\mathcal{L}} := H^2(\mathbb{D}) \otimes \mathcal{L} \to H^2_{\mathcal{L}}$ is multiplication by the co-ordinate function and $\mathcal{M}$ is the invariant subspace for M corresponding to the inner function θ. Let $M = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}$ be the block matrix representation of M corresponding to the decomposition $H^2_{\mathcal{L}} = \mathcal{M}^\perp \oplus \mathcal{M}$. (Thus, in particular, $T = M_{11}$ and M_{22} is the restriction of M to $\mathcal{M}$.) Finally, let $\mathcal{K}$ denote the co-kernel of M_{22}, $N : H^2_{\mathcal{K}} \to H^2_{\mathcal{K}}$ be multiplication by the co-ordinate function and let $E : H^2_{\mathcal{K}} \to \mathcal{M}$ be defined by $Ef = f(0) \in \mathcal{K}$. In terms of these notations, we have

Theorem 2.2. *Let* T *be a cnu contraction in the class* $C_{\cdot 0}$ *with characteristic function* θ*. Let* μ *be a scalar in the range* $0 < \mu < 1$ *and put* $\delta = \sqrt{1 - \mu^2}$*. Then, with respect to the decomposition* $\mathcal{M}^\perp \oplus \mathcal{M} \oplus H^2_{\mathcal{K}}$ *of its domain, the operator* $T[\mu] : H^2_{\mathcal{L}} \oplus H^2_{\mathcal{L}} \oplus H^2_{\mathcal{K}} \to H^2_{\mathcal{K}}$ *has the block matrix representation*

$$T[\mu] = \begin{pmatrix} M_{11} & 0 & 0 \\ \delta M_{21} & M_{22} & \mu E \\ 0 & 0 & N^* \end{pmatrix}.$$

In Theorem 2.9 of [6], it was noted that

Theorem 2.3. *A pure contraction valued analytic function* θ *on* $\mathbb{D}$ *is the characteristic function of a homogeneous cnu contraction if and only if* $\theta \circ \varphi$ *coincides with* θ *for every* φ *in* Möb.

From this theorem, it is immediate that whenever T is a homogeneous cnu contraction, so are the operators $T[\mu]$ given by Theorem 2.2. Some interesting examples of this phenomenon were worked out in [7]. See §6 for these examples.

As an interesting particular case of Theorem 2.3, one finds that any cnu contraction with a constant characteristic function is necessarily homogeneous. These operators are discussed in [11] and [6]. Generalizing a result in [6], Kerchy shows in [19] that

Theorem 2.4. *Let* θ *be the characteristic function of a homogeneous cnu contraction. If* $\theta(0)$ *is a compact operator then* θ *must be a constant function.*

(Actually Kerchy proves the same theorem with the weaker hypothesis that all the points in the spectrum of $\theta(0)$ are isolated from below.)

Sketch of Proof. Let $\theta : \mathbb{D} \to \mathcal{B}(\mathcal{K}, \mathcal{L})$ be the characteristic function of a homogeneous operator. Assume $C := \theta(0)$ is compact. Replacing θ by a coincident analytic function if neceesary, we may assume without loss of generality that $\mathcal{K} = \mathcal{L}$ and $C \geq 0$. By Theorem 2.3 there exists unitaries $U_z,\ V_z$ such that $\theta(z) = U_z C V_z$, $z \in \mathbb{D}$. Let $\lambda_1 > \lambda_2 > \cdots$ be the non-zero eigenvalues of the compact positive operator C. At this point Kerchy shows that (as a consequence of the maximum modulus principle for Hilbert space valued analytic functions) the eigenspace $\mathcal{K}_1$ corresponding to the eigenvalue λ_1 is a common reducing subspace for $U_z, V_z,\ z \in \mathbb{D}$ (as well as for C of course) and hence for $\theta(z),\ z \in \mathbb{D}$. So we can write $\theta(z) = \theta_1(z) \oplus \theta_2(z)$ where θ_1 is an analytic function into $\mathcal{B}(\mathcal{K}_1)$. Since θ_1 is a unitary valued analytic function, it must be a constant. Repeating the same argument with θ_2, one concludes by induction on n that the eigenspace $\mathcal{K}_n$ corresponding to the eigenvalue λ_n is reducing for $\theta(z),\ z \in \mathbb{D}$, and the projection of θ to each $\mathcal{K}_n$ is a constant function. Since the same is obviously true of the zero eigenvalue, we are done.

3. Representations and multipliers

Let G be a locally compact second countable topological group. Then a measurable function $\pi : G \to \mathcal{U}(\mathcal{H})$ is called a *projective representation* of G on the Hilbert space $\mathcal{H}$ if there is a function (necessarily Borel) $m : G \times G \to \mathbb{T}$ such that

$$\pi(1) = I,\ \pi(g_1 g_2) = m(g_1, g_2)\pi(g_1)\pi(g_2) \tag{3.1}$$

for all g_1, g_2 in G. (More precisely, such a function π is called a projective unitary representation of G; however, we shall often drop the adjective unitary since all representations considered in this paper are unitary.) The projective representation π is called an ordinary representation (and we drop the adjective 'projective') if m is the constant function 1. The function m associated with the projective representation π via (3.1) is called the multiplierof π. The ordinary representation π of G which sends every element of G to the identity operator on a one dimensional Hilbert space is called the identity (or trivial) representation of G. It is surprising that although projective representations have been with us for a long time (particularly in the Physics literature), no suitable notion of equivalence of projective representations seems to be available. In [7], we offered the following:

DEFINITION 3.1

Two projective representations $\pi_1,\ \pi_2$ of G on the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ (respectively) will be called equivalent if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ and a function (necessarily Borel) $f : G \to \mathbb{T}$ such that $\pi_2(\varphi)U = f(\varphi)U\pi_1(\varphi)$ for all $\varphi \in G$.

We shall identify two projective representations if they are equivalent. This has the some what unfortunate consequence that any two one dimensional projective representations are identified. But this is of no importance if the group G has no ordinary one dimensional representation other than identity representation (as is the case for all semi-simple Lie groups G.) In fact, the above notion of equivalence (and the resulting identifications) saves us from the following disastrous consequence of the above (commonly accepted) notion of projective representations: Any Borel function from G into $\mathbb{T}$ is a (one dimensional) projective representation of the group!!

3.1 *Multipliers and cohomology*

Notice that the requirement (3.1) on a projective representation implies that its associated multiplier m satisfies

$$m(\varphi, 1) = 1 = m(1, \varphi),\ m(\varphi_1, \varphi_2)m(\varphi_1\varphi_2, \varphi_3) = m(\varphi_1, \varphi_2\varphi_3)m(\varphi_2, \varphi_3) \quad (3.2)$$

for all elements $\varphi, \varphi_1, \varphi_2, \varphi_3$ of G. Any Borel function $m : G \times G \to \mathbb{T}$ satisfying (3.2) is called a *multiplier* of G. The set of all multipliers on G form an abelian group $M(G)$, called the multiplier group of G. If $m \in M(G)$, then taking $\mathcal{H} = L^2(G)$ (with respect to Haar measure on G), define $\pi : G \to \mathcal{U}(\mathcal{H})$ by

$$\Big(\pi(\varphi) f\Big)(\psi) = m(\psi, \varphi) f(\psi\varphi) \quad (3.3)$$

for φ, ψ in G, f in $L^2(G)$. Then one readily verifies that π is a projective representation of G with associated multiplier m. Thus each element of $M(G)$ actually occurs as the multiplier associated with a projective representation. A multiplier $m \in M(G)$ is called *exact* if there is a Borel function $f : G \to \mathbb{T}$ such that $m(\varphi_1, \varphi_2) = (f(\varphi_1) f(\varphi_2))/f(\varphi_1\varphi_2)$ for φ_1, φ_2 in G. Equivalently, m is exact if any projective representation with multiplier m is equivalent to an ordinary representation. The set $M_0(G)$ of all exact multipliers on G form a subgroup of $M(G)$. Two multipliers m_1, m_2 are said to be equivalent if they belong to the same coset of $M_0(G)$. In other words, m_1 and m_2 are equivalent if there exist equivalent projective representations $\pi_1,\ \pi_2$ whose multipliers are m_1 and m_2 respectively. The quotient $M(G)/M_0(G)$ is denoted by $H^2(G, \mathbb{T})$ and is called the second cohomology group of G with respect to the trivial action of G on $\mathbb{T}$ (see [24] for the relevant group cohomology theory). For $m \in M(G)$, $[m] \in H^2(G, \mathbb{T})$ will denote the cohomology class containing m, i.e., $[\ ] : M(G) \to H^2(G, \mathbb{T})$ is the canonical homomorphism.

The following theorem from [8] (also see [9]) provides an explicit description of $H^2(G, \mathbb{T})$ for any connected semi-simple Lie group G.

Theorem 3.1. *Let G be a connected semi-simple Lie group. Then $H^2(G, \mathbb{T})$ is naturally isomorphic to the Pontryagin dual $\widehat{\pi^1(G)}$ of the fundamental group $\pi^1(G)$ of G.*

Explicitly, if $\tilde{G}$ is the universal cover of G and $\pi : \tilde{G} \to G$ is the covering map (so that the fundamental group $\pi^1(G)$ is naturally identified with the kernel Z of π) then choose a Borel section $s : G \to \tilde{G}$ for the covering map (i.e., s is a Borel function such that $\pi \circ s$ is the identity on G, and $s(1) = 1$). For $\chi \in \widehat{Z}$, define $m_\chi : G \times G \to \mathbb{T}$ by

$$m_\chi(x, y) = \chi(s(y)^{-1}s(x)^{-1}s(xy)),\quad x, y \in G. \quad (3.4)$$

Then the main theorem in [8] shows that $\chi \mapsto [m_\chi]$ is an isomorphism from $\widehat{Z}$ onto $H^2(G, \mathbb{T})$ and this isomorphism is independent of the choice of the section s.

The following companion theorem from [8] shows that to find all the irreducible projective representations of a group G satisfying the hypotheses of Theorem 3.1, it suffices to find the ordinary irreducible representations of its universal cover $\tilde{G}$. Let Z be the kernel of the covering map from $\tilde{G}$ onto G. Let β be an ordinary unitary representation of $\tilde{G}$. Then we shall say that β is of *pure type* if there is a character χ of Z such that $\beta(z) = \chi(z)I$ for all z in Z. If we wish to emphasize the particular character which occurs here, we may also say that β is pure of type χ. Notice that, if β is irreducible then (as Z is central) by Schur's

Lemma β is necessarily of pure type. In terms of this definition, the second theorem in [8] says

Theorem 3.2. *Let G be a connected semi-simple Lie group and let $\tilde{G}$ be its universal cover. Then there is a natural bijection between (the equivalence classes of) projective unitary representations of G and (the equivalence classes of) ordinary unitary representations of pure type of $\tilde{G}$. Under this bijection, for each χ the projective representations of G with multiplier m_χ correspond to the representations of $\tilde{G}$ of pure type χ, and vice versa. Further, the irreducible projective representations of G correspond to the irreducible representations of $\tilde{G}$, and vice versa.*

Explicitly, if β is an ordinary representation of pure type χ of $\tilde{G}$ then define $f_\chi : \tilde{G} \to \mathbb{T}$ by $f_\chi(x) = \chi(x^{-1} \cdot s \circ \pi(x)),\ \ x \in \tilde{G}$. Define $\tilde{\alpha}$ on $\tilde{G}$ by $\tilde{\alpha}(x) = f_\chi(x)\beta(x)$. Then $\tilde{\alpha}$ is a projective representation of $\tilde{G}$ which is trivial on Z. Therefore there is a well-defined (and uniquely determined) projective representation α of G such that $\tilde{\alpha} = \alpha \circ \pi$. The multiplier associated with α is m_χ. The map $\beta \mapsto \alpha$ is the bijection mentioned in Theorem 3.2.

Finally, as was pointed out in [9], any projective representation (say with multiplier m) of a connected semi-simple Lie group can be written as a direct integral of irreducible projective representations (all with the same multiplier m) of the group. It follows, of course, that any multiplier of such a group arises from irreducible projective representations. It also shows that, in order to have a description of all the projective representations, it is sufficient to have a list of the irreducible ones and to know when two of them have identical multipliers. This is where Theorems 3.1 and 3.2 come in handy.

3.2 *The multipliers on Möb*

Notice that for any element φ of the Möbius group, φ' is a non-vanishing analytic function on $\bar{\mathbb{D}}$ and hence has a continuous logarithm on this closed disc. Let us fix, once for all, a Borel determination of these logarithms. More precisely, we fix a Borel function $(z, \varphi) \mapsto \log \varphi'(z)$ from $\bar{\mathbb{D}} \times \text{Möb}$ into $\mathbb{C}$ such that $\log \varphi'(z) \equiv 0$ for $\varphi = \text{id}$. Now define $\arg \varphi'(z)$ to be the imaginary part of $\log \varphi'(z)$.

Define the Borel function $\mathbf{n} : \text{Möb} \times \text{Möb} \to \mathbb{Z}$ by

$$\mathbf{n}(\varphi_1^{-1}, \varphi_2^{-1}) = \frac{1}{2\pi}(\arg(\varphi_2\varphi_1)'(0) - \arg \varphi_1'(0) - \arg \varphi_2'(\varphi_1(0))).$$

For any $\omega \in \mathbb{T}$, define $m_\omega : \text{Möb} \times \text{Möb} \to \mathbb{T}$ by

$$m_\omega(\varphi_1, \varphi_2) = \omega^{\mathbf{n}(\varphi_1, \varphi_2)}.$$

The following proposition is a special case of Theorem 3.1. Detailed proofs may be found in [9].

PROPOSITION 3.1

For $\omega \in \mathbb{T},\ m_\omega$ is a multiplier of Möb. *It is trivial if and only if $\omega = 1$. Every multiplier on* Möb *is equivalent to m_ω for a uniquely determined ω in $\mathbb{T}$. In other words, $\omega \mapsto [m_\omega]$ is a group isomorphism between the circle group $\mathbb{T}$ and the second cohomology group* $H^2(\text{Möb}, \mathbb{T})$.

3.3 *The projective representations of the Möbius group*

Every projective representation of a connected semi-simple Lie group is a direct integral of irreducible projective representations (cf. [9], Theorem 3.1). Hence, for our purposes, it suffices to have a complete list of these irreducible representations of Möb. A complete list of the (ordinary) irreducible unitary representations of the universal cover of Möb was obtained by Bargmann (see [29] for instance). Since Möb is a semi-simple and connected Lie group, one may manufacture all the irreducible projective representations of Möb (with Bargmann's list as the starting point) via Theorem 3.2. Following [8] and [9], we proceed to describe the result. (*Warning*: Our parametrization of these representations differs somewhat from the one used by Bargmann and Sally. We have changed the parametrization in order to produce a unified description.)

For $n \in \mathbb{Z}$, let $f_n : \mathbb{T} \to \mathbb{T}$ be defined by $f_n(z) = z^n$. In all of the following examples, the Hilbert space $\mathcal{F}$ is spanned by an orthogonal set $\{f_n : n \in I\}$, where I is some subset of $\mathbb{Z}$. Thus the Hilbert space of functions is specified by the set I and $\{\|f_n\|, n \in I\}$. (In each case, $\|f_n\|$ behaves at worst like a polynomial in $|n|$ as $n \to \infty$, so that this really defines a space of function on $\mathbb{T}$.) For $\varphi \in$ Möb and complex parameters λ and μ, define the operator $R_{\lambda,\mu}(\varphi^{-1})$ on $\mathcal{F}$ by

$$(R_{\lambda,\mu}(\varphi^{-1})f)(z) = \varphi'(z)^{\lambda/2}|\varphi'(z)|^{\mu}(f(\varphi(z)), \ \ z \in \mathbb{T}, \ f \in \mathcal{F}, \ \varphi \in \text{Möb}.$$

Here one defines $\varphi'(z)^{\lambda/2}$ as $\exp \lambda/2 \log \varphi'(z)$ using the previously fixed Borel determination of these logarithms.

Of course, there is no *a priori* guarantee that $R_{\lambda,\mu}(\varphi^{-1})$ is a unitary (or even bounded) operator. But, when it is unitary for every φ in Möb, it is easy to see that $R_{\lambda,\mu}$ is then a projective representation of Möb with associated multiplier m_ω, where $\omega = e^{i\pi\lambda}$. Thus the description of the representation is complete if we specify I, $\{\|f_n\|^2, n \in I\}$ and the two parameters λ, μ. It turns out that almost all the irreducible projective representations of Möb have this form.

In terms of these notations, here is the complete list of the irreducible projective unitary representations of Möb. (However, see the concluding remark of this section.)

- *Principal series representations* $P_{\lambda,s}$, $-1 < \lambda \le 1$, s purely imaginary. Here $\lambda = \lambda$, $\mu = \frac{1-\lambda}{2} + s$, $I = Z$, $\|f_n\|^2 = 1$ for all n (so the space is $L^2(\mathbb{T})$).
- *Holomorphic discrete series representations* D_λ^+: Here $\lambda > 0$, $\mu = 0$, $I = \{n \in Z : n \ge 0\}$ and $\|f_n\|^2 = \frac{\Gamma(n+1)\Gamma(\lambda)}{\Gamma(n+\lambda)}$ for $n \ge 0$. For each f in the representation space there is an $\tilde{f}$, analytic in $\mathbb{D}$, such that f is the non-tangential boundary value of $\tilde{f}$. By the identification $f \leftrightarrow \tilde{f}$, the representation space may be identified with the functional Hilbert space $\mathcal{H}^{(\lambda)}$ of analytic functions on $\mathbb{D}$ with reproducing kernel $(1 - z\bar{w})^{-\lambda}$, $z, w \in \mathbb{D}$.
- *Anti-holomorphic discrete series representations* D_λ^-, $\lambda > 0$: D_λ^- may be defined as the composition of D_λ^+ with the automorphism $*$ of eq. (1.2): $D_\lambda^-(\varphi) = D_\lambda^+(\varphi^*)$, φ in Möb. This may be realized on a functional Hilbert space of anti-holomorphic functions on $\mathbb{D}$, in a natural way.
- *Complementary series representation* $C_{\lambda,\sigma}$, $-1 < \lambda < 1$, $0 < \sigma < \frac{1}{2}(1 - |\lambda|)$: Here $\lambda = \lambda$, $\mu = \frac{1}{2}(1 - \lambda) + \sigma$, $I = Z$, and

$$\|f_n\|^2 = \prod_{k=0}^{|n|-1} \frac{k \pm \frac{\lambda}{2} + \frac{1}{2} - \sigma}{k \pm \frac{\lambda}{2} + \frac{1}{2} + \sigma}, \ n \in Z,$$

where one takes the upper or lower sign according as n is positive or negative.

Remark 3.1. (a) All these projective representation of Möb are irreducible with the sole exception of $P_{1,0}$ for which we have the decomposition $P_{1,0} = D_1^+ \oplus D_1^-$. (b) The multiplier associated with each of these representations is m_ω where $\omega = \mathrm{e}^{-i\pi\lambda}$ if the representation is in the anti-holomorphic discrete series, and $\omega = \mathrm{e}^{i\pi\lambda}$ otherwise. It follows that the multipliers associated with two representations π_1 and π_2 from this list are either identical or inequivalent. Further, if neither or both of π_1 and π_2 are from the anti-holomorphic discrete series, then their multipliers are identical iff their λ parameters differ by an even integer. In the contrary case (i.e., if exactly one of π_1 and π_2 is from the anti-holomorphic discrete series), then they have identical multipliers iff their λ parameters add to an even integer. This is Corollary 3.2 from [9]. Using this information, one can now describe all the projective representations of Möb (at least in principle).

4. Projective representations and homogeneous operators

If T is an operator on a Hilbert space $\mathcal{H}$ then a projective representation π of Möb on $\mathcal{H}$ is said to be associated with T if the spectrum of T is contained in $\bar{\mathbb{D}}$ and

$$\varphi(T) = \pi(\varphi)^* T \pi(\varphi) \tag{4.1}$$

for all elements φ of Möb. Clearly, if T has an associated representation then T is homogeneous. In the converse direction, we have

Theorem 4.1. *If T is an irreducible homogeneous operator then T has a projective representation of Möb associated with it. This projective representation is unique up to equivalence.*

We sketch a proof of Theorem 4.1 below. The details of the proof may be found in [9]. The existence part of this theorem was first proved in [23] using a powerful selection theorem. This result is the prime reason for our interest in projective unitary representations of Möb. It is also the basic tool in the classification program for the irreducible homogeneous operators which is now in progress.

Sketch of Proof. Notice that the scalar unitaries in $\mathcal{U}(\mathcal{H})$ form a copy of the circle group $\mathbb{T}$ in $\mathcal{U}(\mathcal{H})$. There exist Borel transversals E to this subgroup, i.e., Borel subsets E of $\mathcal{U}(\mathcal{H})$ which meet every coset of $\mathbb{T}$ in a singleton. Fix one such (in the Proof of Theorem 2.2 in [9], we present an explicit construction of such a transversal). For each element φ of Möb, let E_φ denote the set of all unitaries U in $\mathcal{U}(\mathcal{H})$ such that $U^*TU = \varphi(T)$. Since T is an irreducible homogeneous operator, Schur's Lemma implies that each E_φ is a coset of $\mathbb{T}$ in $\mathcal{U}(\mathcal{H})$. Define $\pi :$ Möb $\to \mathcal{U}(\mathcal{H})$ by

$$\{\pi(\varphi)\} = E \cap E_\varphi.$$

It is easy to see that π, thus defined, is indeed a projective representation associated with T. Another appeal to Schur's Lemma shows that any representation associated with T must be equivalent to π. This completes the proof.

For any projective representation π of Möb, let $\pi^{\#}$ denote the projective representation of Möb obtained by composing π with the automorphism $*$ of Möb (cf. (1.2)). That is,

$$\pi^{\#}(\varphi) := \pi(\varphi^{*}), \quad \varphi \in \text{Möb}. \tag{4.2}$$

Clearly, if m is the multiplier of π, then $\bar{m}$ is the multiplier of $\pi^{\#}$. Also, from (1.3) it is more or less immediate that if π is associated with a homogeneous operator T then $\pi^{\#}$ is associated with the adjoint T^{*} of T. If, further, T is invertible, then $\pi^{\#}$ is associated with T^{-1} also.

4.1 *Classification of irreducible homogeneous operators*

Recall that an operator T on a Hilbert space $\mathcal{H}$ is said to be a *block shift* if there are non trivial subspaces V_n (indexed by all integers, all non-negative integers or all non-positive integers – accordingly T is called a bilateral, forward unilateral or backward unilateral block shift) such that $\mathcal{H}$ is the orthogonal direct sum of these subspaces and we have $T(V_n) \subseteq V_{n+1}$ for each index n (where, in the case of a backward block shift, we take $V_1 = \{0\}$). In [9] we present a proof (due to Ordower) of the somewhat surprising fact that in case T is an irreducible block shift, these subspaces V_n (which are called the blocks of T) are uniquely determined by T. This result lends substance to the following theorem.

For any connected semi-simple Lie group G takes a maximal compact subgroup $\mathbb{K}$ of G (it is unique up to conjugation). Let $\hat{\mathbb{K}}$ denote, as usual, the set of all irreducible (ordinary) unitary representation of $\mathbb{K}$ (modulo equivalence). Let us say that a projective representation π of G is *normalized* if $\pi|_{\mathbb{K}}$ is an ordinary representation of $\mathbb{K}$. (If $H^2(\mathbb{K}, \mathbb{T})$ is trivial, then it is easy to see that every projective representation of G is equivalent to a normalized representation). If π is normalized, then, for any $\chi \in \hat{\mathbb{K}}$, let V_χ denote the subspace of $\mathcal{H}_\pi$ (the space on which π acts) given by

$$V_\chi = \{v \in \mathcal{H}_\pi \ : \pi(k)v = \chi(k)v \ \forall k \in \mathbb{K}\}.$$

Clearly $\mathcal{H}_\pi$ is the orthogonal direct sum of the subspaces V_χ, $\chi \in \hat{\mathbb{K}}$. The subspace V_χ is called the $\mathbb{K}$-*isotypic subspace* of $\mathcal{H}_\pi$ of type χ.

In particular, for the group $G = \text{Möb}$, we may take $\mathbb{K}$ to be the copy $\{\varphi_{\alpha,0} : \alpha \in \mathbb{T}\}$ of the circle group $\mathbb{T}$. ($\mathbb{K}$ may be identified with $\mathbb{T}$ via $\alpha \mapsto \varphi_{\alpha,0}$.) For π as above and $n \in \mathbb{Z}$, let $V_n(\pi)$ denote the $\mathbb{K}$-isotypic subspace corresponding to the character $\chi_n : z \mapsto z^{-n}$, $z \in \mathbb{T}$. With these notations, we have the following theorem from [9].

Theorem 4.2. *Any irreducible homogeneous operator is a block shift. Indeed, if T is such an operator, and π is a normalized projective representation associated with T then the blocks of T are precisely the non-trivial $\mathbb{K}$-isotypic subspaces of π.*

(Note that if T is an irreducible homogeneous operator, then by Theorem 4.1 there is a representation π associated with T. Since such a representation is determined only up to equivalence, we may replace π by a normalized representation equivalent to it. Then the above theorem applies.)

A block shift is called a *weighted shift* if its blocks are one-dimensional. In [9] we define a *simple representation* of Möb to be a normalized representation π such that (i) the set $\mathcal{T}(\pi) := \{n \in \mathbb{Z} : V_n(\pi) \neq \{0\}\}$ is connected (in an obvious sense) and (ii) for each $n \in \mathcal{T}(\pi)$, $V_n(\pi)$ is one dimensional. If T is an irreducible homogeneous weighted shift, then, by the uniqueness of its blocks and by Theorem 4.2, it follows that any normalized representation π associated with T is necessarily simple. Using the list of irreducible projective representations of Möb given in the previous section (along with Remark 3.1(b) following this list) one can determine all the simple representations of Möb. This is done in Theorem 3.3 of [9]. Namely, we have

Theorem 4.3. *Up to equivalence, the only simple projective unitary representations of* Möb *are its irreducible representations along with the representations* $D^+_\lambda \oplus D^-_{2-\lambda}$, $0 < \lambda < 2$.

Since the representations associated with irreducible homogeneous shifts are simple, to complete a classification of these operators, it now suffices to take each of the representations π of Theorem 4.3 and determine all the homogeneous operators T associated with π. Given that Theorem 4.2 pinpoints the way in which such an operator T must act on the space of π, it is now a simple matter to complete the classification of these operators (at least it is simple in principle – finding the optimum path to this goal turns out to be a challenging task!). To complete a classification of all homogeneous weighted shifts (with non-zero weights – permitting zero weights would introduce uninteresting complications), one still needs to find the reducible homogeneous shifts. Notice that the technique outlined here fails in the reducible case since Theorem 4.1 does not apply. However, in Theorem 2.1 of [9], we were able to show that there is a unique reducible homogeneous shift with non-zero weights, namely the unweighted bilateral shift B. Indeed, if T is a reducible shift (with non-zero weights) such that the spectral radius of T is $= 1$, then it can be shown that $T^k = B^k$ for some positive integer k, and hence T^k is unitary. But Lemma 2.1 in [9] shows that if T is a homogeneous operator such that T^k is unitary, then T itself must be unitary. Clearly, B is the only unitary weighted shift. This shows that B is the only reducible homogeneous weighted shift with non-zero weights. When all this is put together, we have the main theorem of [9].

Theorem 4.4. *Up to unitary equivalence, the only homogeneous weighted shifts are the known ones* (*namely, the first five series of examples from the list in* §6).

Yet another link between homogeneous operators and projective representations of Möb occurs in [10]. Beginning with Theorem 2.3, in [10] we prove a product formula, involving a pair of projective representations, for the characteristic function of any irreducible homogeneous contraction. Namely we have

Theorem 4.5. *If T is an irreducible homogeneous contraction then its characteristic function $\theta : \mathbb{D} \to \mathcal{B}(\mathcal{K}, \mathcal{L})$ is given by*

$$\theta(z) = \pi(\varphi_z)^* C \sigma(\varphi_z), \quad z \in \mathbb{D}$$

where π and σ are two projective representations of Möb (*on the Hilbert spaces $\mathcal{L}$ and $\mathcal{K}$ respectively*) *with a common multiplier. Further, $C : \mathcal{K} \to \mathcal{L}$ is a pure contraction which intertwines $\sigma|_{\mathbb{K}}$ and $\pi|_{\mathbb{K}}$.*

Conversely, whenever $\pi,\ \sigma$ *are projective representations of* Möb *with a common multiplier and* C *is a purely contractive intertwiner between* $\sigma|_{\mathbb{K}}$ *and* $\pi|_{\mathbb{K}}$ *such that the function* θ *defined by* $\theta(z) = \pi(\varphi_z)^* C\sigma(\varphi_z)$ *is analytic on* $\mathbb{D}$, *then* θ *is the characteristic function of a homogeneous cnu contraction* (*not necessarily irreducible*).

(Here φ_z is the involution in Möb which interchanges 0 and z. Also, $\mathbb{K} = \{\varphi \in \text{Möb} : \varphi(0) = 0\}$ is the standard maximal compact subgroup of Möb.)

Sketch of Proof. Let θ be the characteristic function of an irreducible homogeneous cnu contraction T. For any φ in Möb look at the set

$$E_\varphi := \{(U, V) : U^*\theta(w)V = \theta(\varphi^{-1}(w))\ \forall w \in \mathbb{D}\} \subseteq \mathcal{U}(\mathcal{L}) \times \mathcal{U}(\mathcal{K}).$$

By Theorem 2.3, E_φ is non-empty for each φ. By Theorem 3.4 in [25], for $(U, V) \in E_\varphi$ there is a unitary operator $\tau(U, V)$ such that (i) $\tau(U, V)^* T\tau(U, V) = \varphi(T)$ and (ii) the restriction of $\tau(U, V)$ to $\mathcal{L}$ and $\mathcal{K}$ equal U and V respectively. Therefore, irreducibility of T implies that, for $(U, V),\ (U', V')$ in E_φ, $\tau(U', V')^*\tau(U, V)$ is a scalar unitary. Hence E_φ is a coset of the subgroup S (isomorphic to the torus $\mathbb{T}^2$) of $\mathcal{U}(\mathcal{L}) \times \mathcal{U}(\mathcal{K})$ consisting of pairs of scalar unitaries. As in the proof of Theorem 4.1, it follows that there are projective unitary representations π and σ with a common multiplier (on the spaces $\mathcal{L}$ and $\mathcal{K}$ respectively) such that $(\pi(\varphi), \sigma(\varphi)) \in E_\varphi$ for all φ in Möb. So we have

$$\pi(\varphi)^*\theta(w)\sigma(\varphi) = \theta(\varphi^{-1}(w)),\quad w \in \mathbb{D},\ \varphi \in \text{Möb}. \tag{4.3}$$

Now, choose $\varphi = \varphi_z$ and evaluate both sides of (4.3) at $w = 0$ to find the claimed formula for θ with $C = \theta(0)$. Also, taking $w = 0$ and $\varphi \in \mathbb{K}$ in (4.3), one sees that C intertwines $\sigma|_{\mathbb{K}}$ and $\pi|_{\mathbb{K}}$.

For the converse, let $\theta(z) := \pi(\varphi_z)^* C\sigma(\varphi_z)$ be an analytic function. Since $C = \theta(0)$ is a pure contraction and $\theta(z)$ coincides with $\theta(0)$ for all z, θ is pure contraction valued. Hence θ is the characteristic function of a cnu contraction T. For $\varphi \in$ Möb and $w \in \mathbb{D}$, write $\varphi_w\varphi = k\varphi_z$ where $k \in \mathbb{K}$ and $z = (\varphi_w\varphi)^{-1}(0) = \varphi^{-1}(w)$. Then we have

$$\begin{aligned}
\pi(\phi)^*\theta(w)\sigma(\varphi) &= \pi(\varphi)^*\pi(\varphi_w)^* C\sigma(\varphi_w)\sigma(\varphi)\\
&= \pi(\varphi_w\varphi)^* C\sigma(\varphi_w\varphi)\\
&= \pi(k\varphi_z)^* C\sigma(k\varphi_z)\\
&= \pi(\varphi_z)^*\pi(k)^* C\sigma(k)\sigma(\varphi_z)\\
&= \pi(\varphi_z)^* C\sigma(\varphi_z)\\
&= \theta(\varphi^{-1}(w)).
\end{aligned}$$

(Here, for the second and fourth equality we have used the assumption that π and σ are projective representations with a common multiplier. For the penultimate equality, the assumption that C intertwines $\sigma|_{\mathbb{K}}$ and $\pi|_{\mathbb{K}}$ has been used.) Thus θ satisfies (4.3). Therefore $\theta \circ \varphi$ coincides with θ for all φ in Möb. Hence Theorem 2.3 implies that T is homogeneous.

5. Some constructions of homogeneous operators

Let us say that a projective representation π of Möb is a *multiplier representation* if it is concretely realized as follows. π acts on a Hilbert space $\mathcal{H}$ of E – valued functions on

Ω, where Ω is either $\mathbb{D}$ or $\mathbb{T}$ and E is a Hilbert space. The action of π on $\mathcal{H}$ is given by $(\pi(\varphi)f)(z) = c(\varphi, z)f(\varphi^{-1}z)$ for $z \in \Omega$, $f \in \mathcal{H}$, $\varphi \in$ Möb. Here c is a suitable Borel function from Möb $\times\, \Omega$ into the Borel group of invertible operators on E.

Theorem 5.1. *Let $\mathcal{H}$ be a Hilbert space of functions on Ω such that the operator T on $\mathcal{H}$ given by*

$$(Tf)(x) = xf(x),\ x \in \Omega,\ f \in \mathcal{H},$$

is bounded. Suppose there is a multiplier representation π of Möb *on $\mathcal{F}$. Then T is homogeneous and π is associated with T.*

This easy but basic construction is from Proposition 2.3 of [6]. To apply this theorem, we only need a good supply of what we have called multiplier representations of Möb. Notice that all the irreducible projective representations of Möb (as concretely presented in the previous section) are multiplier representations.

A second construction goes as follows. It is contained in Proposition 2.4 of [6].

Theorem 5.2. *Let T be a homogeneous operator on a Hilbert space $\mathcal{H}$ with associated representation π. Let $\mathcal{K}$ be a subspace of $\mathcal{H}$ which is invariant or co-invariant under both T and π. Then the compression of T to $\mathcal{K}$ is homogeneous. Further, the restriction of π to $\mathcal{K}$ is associated with this operator.*

A third construction (as yet unreported) goes as follows:

Theorem 5.3. *Let π be a projective representation of* Möb *associated with two homogeneous operators T_1 and T_2 on a Hilbert space $\mathcal{H}$. Let T denote the operator on $\mathcal{H} \oplus \mathcal{H}$ given by*

$$T = \begin{pmatrix} T_1 & T_1 - T_2 \\ 0 & T_2 \end{pmatrix}.$$

Then T is homogeneous with associated representation $\pi \oplus \pi$.

Sketch of proof. For φ in Möb, one verifies that

$$\varphi(T) = \begin{pmatrix} \varphi(T_1) & \varphi(T_1) - \varphi(T_2) \\ 0 & \varphi(T_2) \end{pmatrix}.$$

Hence it is clear that $\pi \oplus \pi$ is associated with T.

6. Examples of homogeneous operators

It would be tragic if we built up a huge theory of homogeneous operators only to find at the end that there are very few of them. Here are some examples to show that this is not going to happen.

- *The principal series example*. The unweighted bilateral shift B (i.e., the bilateral shift with weight sequence $w_n = 1$, $n = 0, \pm 1, \ldots$) is homogeneous. To see this, apply Theorem 5.1 to any of the principal series representations of Möb. By construction, all the principal series representations are associated with B.

- *The discrete series examples*. For any real number $\lambda > 0$, the unilateral shift $M^{(\lambda)}$ with weight sequence $\sqrt{\frac{n+1}{n+\lambda}}$, $n = 0, 1, 2, \ldots$ is homogeneous. To see this, apply Theorem 5.1 to the discrete series representation D_λ^+.

For $\lambda \geq 1$, $M^{(\lambda)}$ is a cnu contraction. For $\lambda = 1$, its characteristic function is the (constant) function 0 – not very interesting! But for $\lambda > 1$ we proved the following formula for the characteristic function of $M^{(\lambda)}$ (cf. [7]).

Theorem 6.1. *For $\lambda > 1$, the characteristic function of $M^{(\lambda)}$ coincides with the function θ_λ given by*

$$\theta_\lambda(z) = (\lambda(\lambda-1))^{-1/2} D_{\lambda-1}^+(\varphi_z)^* \partial^* D_{\lambda+1}^+(\varphi_z), \quad z \in \mathbb{D},$$

where ∂^* is the adjoint of the differentiation operator $\partial : \mathcal{H}^{(\lambda-1)} \to \mathcal{H}^{(\lambda+1)}$.

This theorem is, of course, an instance of the product formula in Theorem 4.5.

- *The anti-holomorphic discrete series examples*. These are the adjoints $M^{(\lambda)*}$ of the operators in the previous family. The associated representation is D_λ^-.

It was shown in [22] that

Theorem 6.2. *Up to unitary equivalence, the operators $M^{(\lambda)*}$, $\lambda > 0$ are the only homogeneous operators in the Cowen–Douglas class $B_1(\mathbb{D})$.*

This theorem was independently re-discovered by Wilkins in ([33], Theorem 4.1).

- *The complementary series examples*. For any two real numbers a and b in the open unit interval $(0, 1)$, the bilateral shift $K_{a,b}$ with weight sequence $\sqrt{\frac{n+a}{n+b}}$, $n = 0, \pm 1, \pm 2, \ldots$, is homogeneous. To see this in case $0 < a < b < 1$, apply Theorem 5.1 to the complementary series representation $C_{\lambda,\sigma}$ with $\lambda = a + b - 1$ and $\sigma = (b-a)/2$. If $a = b$ then $K_{a,b} = B$ is homogeneous. If $0 < b < a < 1$ then $K_{a,b}$ is the adjoint inverse of the homogeneous operator $K_{b,a}$, and hence is homogeneous.
- *The constant characteristic examples*. For any real number $\lambda > 0$, the bilateral shift B_λ with weight sequence $\ldots, 1, 1, 1, \lambda, 1, 1, 1, \ldots$, ($\lambda$ in the zeroth slot, 1 elsewhere) is homogeneous. Indeed, if $0 < \lambda < 1$ then B_λ is a cnu contraction with constant characteristic function $-\lambda$; hence it is homogeneous. Of course, $B_1 = B$ is also homogeneous. If $\lambda > 1$, B_λ is the inverse of the homogeneous operator B_μ with $\mu = \lambda^{-1}$, hence it is homogeneous. (In [6] we presented an unnecessarily convoluted argument to show that B_λ is homogeneous for $\lambda > 1$ as well.) It was shown in [6] that the representation $D_1^+ \oplus D_1^-$ is associated with each of the operators B_λ, $\lambda > 0$. (Recall that this is the only reducible representation in the principal series!)

In [6] we show that apart from the unweighted unilateral shift and its adjoint, the operators B_λ, $\lambda > 0$ are the only irreducible contractions with a constant characteristic function. In fact,

Theorem 6.3. *The only cnu contractions with a constant characteristic function are the direct integrals of the operators $M^{(1)}$, $M^{(1)*}$ and B_λ, $\lambda > 0$.*

Since all the constant characteristic examples are associated with a common representation, one might expect that the construction in Theorem 5.3 could be applied to any two of them to yield a plethora of new examples of homogeneous operators. Unfortunately, this is not the case. Indeed, it is not difficult to verify that for $\lambda \neq \mu$, the operator $\begin{pmatrix} B_\lambda & B_\lambda - B_\mu \\ 0 & B_\mu \end{pmatrix}$ is unitarily equivalent to $B_\sigma \oplus B_\delta$ where σ and δ are the eigenvalues of $(AA^*)^{1/2}$, $A = \begin{pmatrix} \lambda & \lambda - \mu \\ 0 & \mu \end{pmatrix}$.

Notice that the examples of homogeneous operators given so far are all weighted shifts. By Theorem 4.4, these are the only homogeneous weighted shifts with non-zero weights. Wilkins was the first to come up with examples of (irreducible) homogeneous operators which are not scalar shifts.

- *The generalized Wilkins examples*. Recall that for any real number $\lambda > 0$, $\mathcal{H}^{(\lambda)}$ denotes the Hilbert space of analytic functions on $\mathbb{D}$ with reproducing kernel $(z, w) \mapsto (1 - z\bar{w})^{-\lambda}$. (It is the Hilbert space on which the holomorphic discrete series representation D^+_λ lives.) For any two real numbers $\lambda_1 > 0, \lambda_2 > 0$, and any positive integer k, view the tensor product $\mathcal{H}^{(\lambda_1)} \otimes \mathcal{H}^{(\lambda_2)}$ as a space of analytic functions on the bidisc $\mathbb{D} \times \mathbb{D}$. Look at the Hilbert space $V_k^{(\lambda_1, \lambda_2)} \subseteq \mathcal{H}^{(\lambda_1)} \otimes \mathcal{H}^{(\lambda_2)}$ defined as the ortho-complement of the subspace consisting of the functions vanishing to order k on the diagonal $\Delta = \{(z, z) : z \in \mathbb{D}\} \subseteq \mathbb{D} \times \mathbb{D}$. Finally define the generalized Wilkins operator $W_k^{(\lambda_1, \lambda_2)}$ as the compression to $V_k^{(\lambda_1, \lambda_2)}$ of the operator $M^{(\lambda_1)} \otimes I$ on $\mathcal{H}^{(\lambda_1)} \otimes \mathcal{H}^{(\lambda_2)}$. The subspace $V_k^{(\lambda_1, \lambda_2)}$ is co-invariant under the homogeneous operator $M^{(\lambda_1)} \otimes I$ as well as under the associated representation $D^+_{\lambda_1} \otimes D^+_{\lambda_2}$. Therefore, by Theorem 5.2, $W_k^{(\lambda_1, \lambda_2)}$ is a homogeneous operator. For $k = 1$, $W_1^{(\lambda_1, \lambda_2)}$ is easily seen to be unitarily equivalent to $M^{(\lambda_1+\lambda_2)}$, see [7] and [14], for instance. But for $k \geq 2$, these are new examples.

The operator $W_k^{(\lambda_1, \lambda_2)}$ may alternatively be described as multiplication by the co-ordinate function z on the space of $\mathbb{C}^k$-valued analytic functions on $\mathbb{D}$ with reproducing kernel

$$(z, w) \mapsto (1 - z\bar{w})^{-\lambda_1} \left(\left(\partial^i \bar{\partial}^j (1 - z\bar{w})^{-\lambda_2} \right) \right)_{0 \leq i, j \leq k-1}.$$

(Here ∂ and $\bar{\partial}$ denote differentiation with respect to z and $\bar{w}$, respectively.) Indeed (with the obvious identification of Δ and $\mathbb{D}$) the map $f \mapsto (f, f', \ldots, f^{(k-1)})|_\Delta$ is easily seen to be a unitary between $V_k^{(\lambda_1, \lambda_2)}$ and this reproducing kernel Hilbert space intertwining $W_k^{(\lambda_1, \lambda_2)}$ and the multiplication operator on the latter space. (This is a particular instance of the *jet construction* discussed in [15].) Using this description, it is not hard to verify that the adjoint of $W_k^{(\lambda_1, \lambda_2)}$ is an operator in the Cowen–Douglas class $B_k(\mathbb{D})$. The following is (essentially) one of the main results in [34].

Theorem 6.4. *Up to unitary equivalence, the only irreducible homogeneous operators in the Cowen–Douglas class $B_2(\mathbb{D})$ are the adjoints of the operators $W_2^{(\lambda_1, \lambda_2)}$, $\lambda_1 > 0, \lambda_2 > 0$.*

This is not the description of these operators given in [34]. But it can be shown that Wilkin's operator $T^*_{\lambda,\varrho}$ is unitarily equivalent to the operator $W_2^{(\lambda_1, \lambda_2)}$ with $\lambda = \lambda_1 + \lambda_2 + 1$, $\varrho = (\lambda_1 + \lambda_2 + 1)/(\lambda_2 + 1)$. Indeed, though his reproducing kernel $H_{\lambda,\varrho}$ looks a little different from the kernel (with $k = 2$) displayed above, a calculation shows that these two kernels have the same normalization at the origin (cf. [12]), so that the corresponding multiplication operators are unitarily equivalent. However, it is hard to see how Wilkins arrived at his examples $T^*_{\lambda,\varrho}$ while the construction of the operators $W_k^{(\lambda_1,\,\lambda_2)}$ given above has a clear geometric meaning, particularly in view of Theorem 5.2. But, as of now, we know that the case $k = 2$ of this construction provides a complete list of the irreducible homogeneous operators in $B_2(\mathbb{D})$ only by comparing them with Wilkins' list – we have no independent explanation of this phenomenon.

Theorem 6.1 has the following generalization to some of the operators in this series. (Theorem 6.1 is the special case $k = 1$ of this theorem.)

Theorem 6.5. *For $k = 1, 2, \dots$ and real numbers $\lambda > k$, the characteristic function of the operator $W_k^{(1,\,\lambda-k)}$ coincides with the inner analytic function $\theta_k^{(\lambda)} : \mathbb{D} \to \mathcal{B}(\mathcal{H}^{(\lambda+k)}, \mathcal{H}^{(\lambda-k)})$ given by*

$$\theta_k^{(\lambda)}(z) = c_{\lambda,k} D^+_{\lambda-k}(\varphi_z)^* \partial^{k*} D^+_{\lambda+k}(\varphi_z),\ z \in \mathbb{D}.$$

Here ∂^{k*} is the adjoint of the k-times differentiation operator $\partial^k : \mathcal{H}^{(\lambda-k)} \to \mathcal{H}^{(\lambda+k)}$ and $c_{\lambda,k} = \prod_{\ell=-(k-1)}^{k} (\lambda - \ell)^{-1/2}$.

Sketch of Proof. It is easy to check that $C := c_{\lambda,k}\partial^{k*}$ is a pure contraction intertwining the restrictions to $\mathbb{K}$ of $D^+_{\lambda+k}$ and $D^+_{\lambda-k}$. Since we already know (by Theorem 6.1) that θ_k^{λ} is an inner analytic function for $k = 1$, the recurrence formula

$$\theta_{k+1}^{(\lambda)} = \theta_1^{(\lambda-k)} \theta_{k-1}^{(\lambda)} \theta_1^{(\lambda+k)}$$

(for $k \geq 1$, $\lambda > k + 1$, with the interpretation that $\theta_0^{(\lambda)}$ denotes the constant function 1) shows that $\theta_k^{(\lambda)}$ is an inner analytic function on $\mathbb{D}$ for $\lambda > k$, $k = 1, 2, \dots$. Hence it is the characteristic function of a cnu contraction T in the class $C_{.0}$. By Theorem 2.1, T is the compression to $\mathcal{M}^\perp$ of the multiplication operator on $H^{(1)} \otimes \mathcal{H}^{(\lambda-k)}$, where $\mathcal{M}$ is the invariant subspace corresponding to this inner function. But one can verify that $\mathcal{M}$ is the subspace consisting of the functions vanishing to order k on the diagonal. Therefore $T = W_k^{(1,\,\lambda-k)}$.

- *Some perturbations of the discrete series examples*. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{f_k : k = 0, 1, \dots\} \cup \{h_{k,\ell} : k = 0, \pm 1, \pm 2, \dots\}$. For any three strictly positive real numbers λ, μ and δ, let $M^{(\lambda)}[\mu, \delta]$ be the operator on $\mathcal{H}$ given by

$$M^{(\lambda)}[\mu, \delta] f_k = \sqrt{\frac{k+1}{k+\lambda+1}} f_{k+1} + \sqrt{\frac{\delta}{k+\lambda+1}} h_{1,k+1},$$

$$M^{(\lambda)}[\mu, \delta] h_{0,\ell} = \mu h_{1,\ell},$$

and

$$M^{(\lambda)}[\mu, \delta] h_{k,\ell} = h_{k+1,\ell}, \quad \text{for } k \geq 1.$$

An application of Theorem 2.2 to the operators $M^{(\lambda)}$ in conjunction with an analytic continuation argument shows that these operators are homogeneous. This was observed in [7].

- *The normal atom.* Define the operator N on $L^2(\mathbb{D})$ by $(Nf)(z) = zf(z),\ z \in \mathbb{D},\ f \in L^2(\mathbb{D})$. The discrete series representation D_2^+ naturally lifts to a representation of Möb on $L^2(\mathbb{D})$. Applying Theorem 5.1 to this representation yields the homogeneity of N.

Using spectral theory, it is easy to see that the operators B and N are the only homogeneous normal operators of multiplicity one. In consequence, we have

Theorem 6.6. *Every normal homogeneous operator is a direct sum of* (*countably many*) *copies of B and N.*

Let us define an *atomic homogeneous operator* to be a homogeneous operator which can not be written as the direct sum of two homogeneous operators. Trivially, irreducible homogeneous operators are atomic. As an immediate consequence of Theorem 6.6, we have

COROLLARY 6.1

B and N are atomic (*but reducible*) *homogeneous operators.*

N is a cnu contraction. Its characteristic function was given in [7].

Theorem 6.7. *The characteristic function $\theta_N : \mathbb{D} \to \mathcal{B}(L^2(\mathbb{D}))$ of the operator N is given by the formula*

$$(\theta_N(z)f)(w) = -\varphi_w(z)f(w),\ \ z, w \in \mathbb{D},\ \ f \in L^2(\mathbb{D}).$$

(Here, as before, φ_w is the involution in Möb which interchanges 0 and w.)

The usual transition formula between cartesian and polar coordinates shows that $L^2(\mathbb{D}) = L^2(\mathbb{T}) \otimes L^2([0,1],\ r\mathrm{d}r)$. Since B may be represented as multiplication by the coordinate function on $L^2(\mathbb{T})$, it follows that the normal atom N is related to the other normal atom B by $N = B \otimes C$ where C is multiplication by the coordinate function on $L^2([0,1],\ r\mathrm{d}r)$. Clearly C is a positive contraction. Let $\{f_n :\ \ n \geq 0\}$ be the orthonormal basis of $L^2([0,1],\ r\mathrm{d}r)$ obtained by Gram–Schmidt orthogonalization of the sequence $\{r \mapsto r^n :\ \ n \geq 0\}$. (Except for scaling, f_n is given in terms of classical Jacobi polynomials by $x \mapsto P_n^{(0,1)}(2x-1)$, cf. [31].) Then the theory of orthogonal polynomials shows that (with respect to this orthonormal basis) C is a tri-diagonal operator. Thus we have

Theorem 6.8. *Up to unitary equivalence, we have $N = B \otimes C$ where the positive contraction C is given on a Hilbert space with orthonormal basis $\{f_n : n \geq 0\}$ by the formula*

$$Cf_n = a_n f_{n-1} + b_n f_n + a_{n+1} f_{n+1},\ \ n = 0, 1, 2, \ldots$$

where $(f_{-1} = 0)$ *and the constants $a_n,\ b_n$ are given by*

$$a_n = \frac{\sqrt{n(n+1)}}{4n+2},\ \ b_n = \frac{2(n+1)^2}{(2n+1)(2n+3)},\ \ n \geq 0.$$

7. Open questions

7.1 *Classification*

The primary question in this area is, of course, the classification of homogeneous operators up to unitary equivalence. Theorem 4.4 is a beginning in this direction. We expect that the same methodology will permit us to classify all the homogeneous operators in the Cowen–Douglas classes $B_k(\mathbb{D})$, $k = 1, 2, \ldots$. Work on this project has already begun. More generally, though there seem to be considerable difficulties involved, it is conceivable that extension of the same techniques will eventually classify all irreducible homogeneous operators. But, depending as it does on Theorem 4.1, this technique draws a blank when it comes to classifying reducible homogeneous operators. In particular, we do not know how to approach the following questions.

Question 1. Is every homogeneous operator a direct integral of atomic homogeneous operators?

Question 2. Are B and N the only atomic homogeneous operators which are not irreducible?

We have seen that the homogeneous operator N can be written as $N = B \otimes C$. In this connection, we can ask:

Question 3. Find all homogeneous operators of the form $B \otimes X$. More generally, find all homogeneous operators which have a homogeneous operator as a 'tensor factor'.

Another possible approach towards the classification of irreducible homogeneous contractions could be via Theorem 4.5. (Notice that any irreducible operator is automatically cnu.) Namely, given any two projective representations π and σ of Möb having a common multiplier, we can seek to determine the class $\mathcal{C}(\pi, \sigma)$ of all operators $C : \mathcal{H}_\sigma \to \mathcal{H}_\pi$ such that (i) C intertwines $\sigma|_{\mathbb{K}}$ and $\pi|_{\mathbb{K}}$ and (ii) the function $z \mapsto \pi(\varphi_z)^* C \sigma(\varphi_z)$ is analytic on $\mathbb{D}$. Clearly $\mathcal{C}(\pi, \sigma)$ is a subspace of $\mathcal{B}(\mathcal{H}_\sigma, \mathcal{H}_\pi)$, and Theorem 4.5 says that any pure contraction in this subspace yields a homogeneous operator. Further, this method yields all irreducible homogeneous contractions as one runs over all π and σ. This approach is almost totally unexplored. We have only observed that, up to multiplication by scalars, the homogeneous characteristic functions listed in Theorem 6.5 are the only ones in which both π and σ are holomorphic discrete series representations. (But the trivial operation of multiplying the characteristic function by scalars correspond to a highly non-trivial operation at the level of the operator. This operation was explored in [7].) So a natural question is:

Question 4. Determine $\mathcal{C}(\pi, \sigma)$ at least for irreducible projective representations π and σ (with a common multiplier).

Note that Theorem 6.5 gives the product formula for the characteristic function of $W_k^{(\lambda_1, \lambda_2)}$ for $\lambda_1 = 1$. But for $W_k^{(\lambda_1, \lambda_2)}$ to be a contraction it is sufficient (though not necessary) to have $\lambda_1 \geq 1$. So on a more modest vein, we may ask:

Question 5. What is the (explicit) product formula for the characteristic functions of the operators $W_k^{(\lambda_1,\lambda_2)}$ for $\lambda_1 > 1$?

Recall that a cnu contraction T is said to be in the class C_{11} if for every nonzero vector x, $\lim_{n\to\infty} T^n x \neq 0$ and $\lim_{n\to\infty} T^{*n} x \neq 0$. In [19], Kerchy asks:

Question 6. Does every homogeneous contraction in the class C_{11} have a constant characteristic function?

7.2 *Möbius bounded and polynomially bounded operators*

Recall from [30] that a Hilbert space operator T is said to be *Möbius bounded* if the family $\{\varphi(T) : \varphi \in \text{Möb}\}$ is uniformly bounded in norm. Clearly homogeneous operators are Möbius bounded, but the converse is false. In [30], Shields proved:

Theorem 7.1. *If T is a Möbius bounded operator then $\|T^m\| = O(m)$ as $m \to \infty$.*

Sketch of proof. Say $\|\varphi(T)\| \le c$ for $\varphi \in$ Möb. For any $\varphi \in$ Möb, we have an expansion $\varphi(z) = \sum_{m=0}^{\infty} a_m z^m$, valid in the closed unit disc. Hence,

$$a_m T^m = \int_{\mathbb{T}} \varphi(\alpha T)\alpha^{-m}\, \mathrm{d}\alpha,$$

where the integral is with respect to the normalized Haar measure on $\mathbb{T}$. Therefore we get the estimate $|a_m|\|T^m\| \le c$ for all m. Choosing $\varphi = \varphi_{1,\beta}$, we see that for $m \ge 1$, $|a_m| = (1-r^2)r^{m-1}$ where $r = |\beta|$. The optimal choice $r = \sqrt{(m-1)/(m+1)}$ gives $|a_m| = O(1/m)$ and hence $\|T^m\| = O(m)$.

On the basis of this Theorem and some examples, we may pose:

Conjecture. For any Möbius bounded operator T, we have $\|T^m\| = O(m^{1/2})$ as $m \to \infty$.

In [30], Shields already asked if this is true. This question has remained unanswered for more than twenty years. One possible reason for its intractability may be the difficulty involved in finding non-trivial examples of Möbius bounded operators. (Contractions are Möbius bounded by von Neumann's inequality, but these trivially satisfy Shield's conjecture.) As already mentioned, non-contractive homogeneous operators provide non-trivial examples. For the homogeneous operator $T = M^{(\lambda)}$ with $\lambda < 1$, we have $\|T^m\| = \sqrt{\frac{\Gamma(\lambda)\Gamma(m+1)}{\Gamma(m+\lambda)}}$ and hence (by Sterling's formula) $\|T^m\| \sim c m^{(1-\lambda)/2}$ with $c = \Gamma(\lambda)^{1/2}$. Thus the above conjecture, if true, is close to best possible (in the sense that the exponent $1/2$ in this conjecture cannot be replaced by a smaller constant). An analogous calculation with the complementary series examples $C(a, b)$ (with $0 < a \neq b < 1$) leads to a similar conclusion. This leads us to ask:

Question 7. Is the conjecture made above true at least for homogeneous operators T?

(It is conceivable that the operators $T_{\lambda,s}$ introduced below contain counter examples to Shield's conjecture in its full generality.)

Recall that an operator T, whose spectrum is contained in $\bar{\mathbb{D}}$, is said to be *polynomially bounded* if there is a constant $c > 0$ such that $\|p(T)\| \leq c$ for all polynomial maps $p : \mathbb{D} \to \mathbb{D}$. (von Neumann's inequality says that this holds with $c = 1$ iff T is a contraction.) Clearly, if T is similar to a contraction then T is polynomially bounded. Halmos asked if the converse is true, i.e., whether every polynomially bounded operator is similar to a contraction. In [28], Pisier constructed a counter-example to this conjecture. (Also see [13] for a streamlined version of this counter-example.) However, one may still hope that the Halmos conjecture is still true of some 'nice' classes of operators. In particular, we ask

Question 8. Is every polynomially bounded homogeneous operator similar to a contraction? For that matter, is there any polynomially bounded (even power bounded) homogeneous operator which is not a contraction?

Notice that the discrete series examples show that homogeneous operators (though Möbius bounded) need not even be power bounded. So certainly they need not be polynomially bounded.

7.3 *Invariant subspaces*

If T is a homogeneous operator with associated representation π, then for each invariant subspace $\mathcal{M}$ of T and each $\varphi \in$ Möb, $\pi(\varphi)(\mathcal{M})$ is again T-invariant. Thus Möb acts on the lattice of T-invariant subspaces via π. We wonder if this fact can be exploited to explore the structure of this lattice. Further, if T is a cnu contraction, then the Sz-Nagy–Foias theory gives a natural correspondence between the invariant subspaces of T and the 'regular factorizations' of its characteristic function (cf. [25]). Since we have nice explicit formulae for the characteristic functions of the homogeneous contractions $M(\lambda)$, $\lambda > 1$, may be these formulae can be exploited to shed light on the structure of the corresponding lattices.

Recall that Beurling's theorem describes the lattice of invariant subspaces of $M^{(1)}$ in terms of inner functions. Recently, it was found ([18] and [1]) that certain partial analogues of this theorem are valid for the Bergman shift $M^{(2)}$ as well. We may ask:

Question 9. Do the theorems of Hedenmalm and Aleman *et al* generalize to the family $M^{(\lambda)}$, $\lambda \geq 1$ of homogeneous unilateral shifts?

7.4 *Generalizations of homogeneity*

In the definition of homogeneous operators, one may replace unitary equivalence by similarity. Formally, we define a *weakly homogeneous operator* to be an operator T such that (i) the spectrum of T is contained in $\bar{\mathbb{D}}$ and (ii) $\varphi(T)$ is similar to T for every φ in Möb. Of course, every operator which is similar to a homogeneous operator is weakly homogeneous. In [11] it was asked if the converse is true. It is not – as one can see from the following examples:

Example 1. Take $\mathcal{H} = L^2(\mathbb{T})$ and, for any real number in the range $-1 < \lambda \leq 1$ and any complex number s with $\mathrm{Im}(s) > 0$, define $P_{\lambda,s}$: Möb $\to \mathcal{B}(\mathcal{H})$ by

$$P_{\lambda,s}(\varphi^{-1})f = \varphi'^{\lambda/2}|\varphi'|^{(1-\lambda)/2+s} f\circ\varphi, \quad f\in\mathcal{H}.$$

For purely imaginary s, these are just the principal series unitary projective representations discussed earlier. For s outside the imaginary axis, $P_{\lambda,s}$ is not unitary valued. But, formally, it still satisfies the condition (3.1) with $m = m_\omega, \quad \omega = \mathrm{e}^{i\pi\lambda}$. In consequence, $P_{\lambda,s}$ is an invertible operator valued function on Möb.

For λ and s as above, let $T_{\lambda,s}$ denote the bilateral shift on $L^2(\mathbb{T})$ with weight sequence

$$\frac{n+(1+\lambda)/2+s}{n+(1+\lambda)/2-s},\ n\in\mathbb{Z}.$$

When s is purely imaginary, these weights are unimodular and hence $T_{\lambda,s}$ is unitarily equivalent to the unweighted bilateral shift B. In [9] it is shown that, in this case the principal series representation $P_{\lambda,s}$ is associated with $T_{\lambda,s}$ as well as to B. That is, we have

$$\varphi(T_{\lambda,s}) = P_{\lambda,s}(\varphi)^{-1}T_{\lambda,s}P_{\lambda,s}(\varphi) \tag{7.1}$$

for purely imaginary s. By analytic continuation, it follows that eq. (7.1) holds for all complex numbers s. Thus $T_{\lambda,s}$ is weakly homogeneous for $\mathrm{Im}(s) > 0$. It is easy to see that $\|T^m_{\lambda,s}\| \geq \|T^m_{\lambda,s}f_0\| \geq \frac{|\Gamma(m+a)\Gamma(b)|}{|\Gamma(m+b)\Gamma(a)|}$ where $a = (1+\lambda)/2+s$, $b = (1+\lambda)/2-s$ and f_0 is the constant function 1. Hence by Sterling's formula, we get

$$\|T^m_{\lambda,s}\| \geq cm^{2\mathrm{Re}(s)}$$

for all large m (and some constant $c > 0$). If $T_{\lambda,s}$ were similar to a homogeneous operator, it would be Möbius bounded and hence by Theorem 7.1 we would get $\|T^m_{\lambda,s}\| = O(m)$ which contradicts the above estimate when $\mathrm{Re}(s) > 1/2$. Therefore we have

Theorem 7.2. *The operators $T_{\lambda,s}$ is weakly homogeneous for all $\lambda,\ s$ as above. However, for $\mathrm{Re}(s) > 1/2$, this operator is not Möbius bounded and hence is not similar to any homogeneous operator.*

Example 2 (due to Ordower). For any homogeneous operator T, say on the Hilbert space $\mathcal{H}$, let $\tilde{T}$ denote the operator $\begin{pmatrix} T & I \\ 0 & T\end{pmatrix}$. For any φ in a sufficiently small neighbourhood of the identity, $\varphi(\tilde{T})$ makes sense and one verifies that $\varphi(\tilde{T}) = \begin{pmatrix} \varphi(T) & \varphi'(T) \\ 0 & \varphi(T)\end{pmatrix}$. If U is a unitary on $\mathcal{H}$ such that $\varphi(T) = U^*TU$ then an easy computation shows that the operator $L = U\varphi'(T)^{1/2}\oplus U\varphi'(T)^{-1/2}$ satisfies $L\tilde{T}L^{-1} = \varphi(T)$. Thus $\varphi(\tilde{T})$ is similar to $\tilde{T}$ for all φ in a small neighbourhood. Therefore an obvious extension of Theorem 1.1 shows that $\tilde{T}$ is weakly homogeneous. Since $\|\varphi(\tilde{T})\| \geq \|\varphi'(T)\|$ and since the family $\varphi',\ \varphi\in$ Möb is not uniformly bounded on the spectrum of T, it follows that $\tilde{T}$ is not Möbius bounded. Therefore we have

Theorem 7.3. *For any homogeneous operator T, the operator $\tilde{T}$ is weakly homogeneous but not Möbius bounded. Therefore this operator is not similar to any homogeneous operator.*

These two classes of examples indicate that the right question to ask is

Question 10. Is it true that every Möbius bounded weakly homogeneous operator is similar to a homogeneous operator?

For purely imaginary s, the homogeneous operators $T_{\lambda,s}$ and B share the common associated representation $P_{\lambda,s}$; hence one may apply the construction in Theorem 5.3 to this pair. We now ask

Question 11. Is the resulting homogeneous operator atomic? Is it irreducible? More generally, are there instances where Theorem 5.3 lead to atomic homogeneous operators?

Another direction of generalization is to replace the group Möb by some subgroup G. For any such G, one might say that an operator T is G-homogeneous if $\varphi(T)$ is unitarily equivalent to T for all 'sufficiently small' φ in G. (If G is connected, the analogue of Theorem 1.1 holds.) The case $G = \mathbb{K}$ has been studied under the name of 'circularly symmetric operators'. See, for instance, [17] and [3]. Notice that if S is a circularly symmetric operator then so is $S \otimes T$ for any operator T – showing that this is a rather weak notion and no satisfactory classification can be expected when the group G is so small. A more interesting possibility is to take G to be a Fuchsian group. (Recall that a closed subgroup of Möb is said to be Fuchsian if it acts discontinuously on $\mathbb{D}$.) Fuchsian homogeneity was briefly studied by Wilkins in [33]. He examines the nature of the representations (if any) associated with such an operator.

Another interesting generalization is to introduce a notion of homogeneity for commuting tuples of operators. Recall that a bounded domain Ω in $\mathbb{C}^d$ is said to be a *bounded symmetric domain* if, for each $\omega \in \Omega$, there is a bi-holomorphic involution of Ω which has ω as an isolated fixed point. Such a domain is called irreducible if it cannot be written as the cartesian product of two bounded symmetric domains. The irreducible bounded symmetric domains are completely classified modulo biholomorphic equivalence (see [2] or [16] for instance) – they include the unit ball $I_{m,n}$ in the Banach space of all $m \times n$ matrices (with operator norm). Let G_Ω denote the connected component of the identity in the group of all bi-holomorphic automorphisms of an irreducible bounded symmetric domain Ω. If $\mathbf{T} = (T_1, \ldots, T_d)$ is a commuting d-tuple of operators then one may say that $\mathbf{T}$ is homogeneous if, for all 'sufficiently small' $\varphi \in G_\Omega$, $\varphi(\mathbf{T})$ is (jointly) unitarily equivalent to $\mathbf{T}$. (Of course, this notion depends on the choice of Ω – for most values of d there are several choices – so, to be precise, one ought to speak of Ω-homogeneity). Theorem 1.1 generalizes to show that, in this setting, the Taylor spectrum of $\mathbf{T}$ is contained in $\bar{\Omega}$ (and is a G_Ω-invariant closed subset thereof). Also, if $\mathbf{T}$ is an irreducible homogeneous tuple (in the sense that its components have no common non-trivial reducing subspace), then Theorem 4.1 generalizes to yield a projective representation of G_Ω associated with it. Therefore, many of the techniques employed in the single variable case have their several variable counterparts. But these are yet to be systematically investigated. One difficulty is that for $d \geq 2$, the (projective) representation theory of G_Ω (which is a semi-simple Lie group) is not as well understood as in the case $\Omega = \mathbb{D}$. But this also has the potential advantage that when (and if) this theory of homogeneous operator tuples is investigated in depth, the operator theory is likely to have significant impact on the representation theory.

With each domain Ω as above is associated a kernel B_Ω (called the Bergman kernel) which is the reproducing kernel of the Hilbert space of all square integrable (with respect to Lebesgue measure) analytic functions on Ω. The Wallach set $W = W_\Omega$ of Ω is the set of all $\lambda > 0$ such that $B_\Omega^{\lambda/g}$ is (a non-negative definite kernel and hence) the reproducing kernel of a Hilbert space $\mathcal{H}^{(\lambda)}(\Omega)$. (Here g is an invariant of the domain Ω called its genus, cf. [2].) It is well-known that the Wallach set W can be written as a disjoint union $W_d \cup W_c$ where the 'discrete' part W_d is a finite set (consisting of r points, where the 'rank' r of Ω is the number of orbits into which the topological boundary of Ω is broken by the action of G_Ω) and the 'continuous' part W_c is a semi-infinite interval.

The constant functions are always in $\mathcal{H}^{(\lambda)}(\Omega)$ but, for $\lambda \in W_d$, $\mathcal{H}^{(\lambda)}(\Omega)$ does not contain all the analytic polynomial functions on Ω. It follows that for $\lambda \in W_d$ multiplication by the co-ordinate functions do not define bounded operators on $\mathcal{H}^{(\lambda)}(\Omega)$. However, it was conjectured in [4] that for $\lambda \in W_c$, the d-tuple $\mathbf{M}^{(\lambda)}$ of multiplication by the d co-ordinates is bounded. (In [5], this conjecture was proved in the cases $\Omega = I_{m,n}$. In general, it is known that for sufficiently large λ the norm on $\mathcal{H}^{(\lambda)}(\Omega)$ is defined by a finite measure on $\bar{\Omega}$, so that this tuple is certainly bounded in these cases.) Assuming this conjecture, the operator tuples $\mathbf{M}^{(\lambda)}$, $\lambda \in W_c$, constitute examples of homogeneous tuples – this is in consequence of the obvious extension of Theorem 5.1 to tuples. In [4] it was shown that the Taylor spectrum of this tuple is $\bar{\Omega}$ and

Theorem 7.4. *Up to unitary equivalence, the adjoints of the tuples* $\mathbf{M}^{(\lambda)}$, $\lambda \in W_c$, *are the only homogeneous tuples in the Cowen–Douglas class* $B_1(\Omega)$.

For what values of $\lambda \in W_c$ is the tuple $\mathbf{M}^{(\lambda)}$ sub-normal? This is equivalent to asking for the values of λ for which the norm on $\mathcal{H}^{(\lambda)}(\Omega)$ is defined by a measure. In [4] we conjecture a precise answer. Again, the special case $\Omega = I_{m,n}$ of this conjecture was proved in [5].

Regarding homogeneous tuples, an obvious meta-question to be asked is

Question 12. Formulate appropriate generalizations to tuples of all the questions we asked before of single homogeneous operators – and answer them!

A d-tuple $\mathbf{T}$ on the Hilbert space $\mathcal{H}$ is said to be *completely contractive with respect to* Ω if for every polynomial map $P : \Omega \to I_{m,n}$, $P(\mathbf{T})$ is contractive when viewed as an operator from $\mathcal{H} \otimes \mathbb{C}^n$ to $\mathcal{H} \otimes \mathbb{C}^m$. $\mathbf{T}$ is called *contractive with respect to* Ω if this holds in the case $m = n = 1$. In general one may ask whether contractivity implies complete contractivity. In general the answer is 'no' for all $d \geq 5$ [27]. However one has a positive answer in the case $\Omega = \mathbb{D}$. But an affirmative answer (for special classes of tuples) would be interesting because complete contractivity is tantamount to existence of nice dilations which make the tuple in question tractable. For instance, we have an affirmative answer for subnormal tuples. We ask

Question 13. Is every contractive homogeneous tuple completely contractive?

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