Proc. Indian Acad. Sci. (Math. Sci.), Vol. 104, No. 3, August 1994, pp. 515–542.

Some remarks on the Jacobian question

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MS received 16 September 1993; revised 21 March 1994

Abstract. This revised version of Abhyankar's old lecture notes contains the original proof of the Galois case of the n-variable Jacobian problem. They also contain proofs for some cases of the 2-variable Jacobian, including the two characteristic pairs case. In addition, proofs of some of the well-known formulas enunciated by Abhyankar are actually written down. These include the Taylor Resultant Formula and the Semigroup Conductor formula for plane curves. The notes are also meant to provide inspiration for applying the expansion theoretic techniques to the Jacobian problem.

Keywords. Plane curves; Jacobian; automorphism; Newton-Puiseux expansion.

Introduction

What follows may best be described by altering the famous phrase of Zariski and Samuel – *this paper is the unborn parent of its child*. These notes, based on Abhyankar's lectures, were originally prepared by van der Put in 1972. Since he had to rush back to the Netherlands in the middle of their preparation, they were worked on by William Heinzer to some extent. But there were still unfinished parts and as a result, they stayed buried in Abhyankar's private papers for 21 years. Later, Abhyankar's work on the Jacobian problem was partly reported in the Tata Institute Notes by Balwant Singh [A1]. Avinash Sathaye rearranged the old notes and added some additional topics based on further discourses by Abhyankar. Paul Eakin and David Shannon have also contributed to the current form of these notes by their critical proof-reading and suggestions. Special thanks are also given to the referee whose comments have significantly improved the logical clarity of this exposition.

In the old tradition of Abhyankar's lecture-notes, these notes are also unread by Abhyankar in their final form. The responsibility for the exposition, therefore, rests with the note-takers.

Much has been published about the Jacobian problem in the meantime, but except for the Balwant Singh Notes, the novel technique of using a combination of Newton–Puiseux expansions at infinity and studying the resulting value-semigroups did not get much exposure. Our aim in reviving these old notes is to renew interest in these methods, which, to paraphrase Abhyankar's own words, "never really got stuck, but only got very tiring". Perhaps, this time one of the methods will be carried through!

The notes' main values are historical and motivational. We have tried to stay close to the original notes and hence there are no brand new theorems. In fact, they do not even give all the known results as found in [A1]. There are, however, proofs of some theorems enunciated elsewhere without proofs as described below.

These notes have two major parts.

In § 1 and § 2, we discuss the n-variable Jacobian problem. Thus, given n polynomials $(u)=(u_1,\ldots,u_n)$ in n variables $(x)=(x_1,\ldots,x_n)$, we assume that their jacobian $J_x(u)=\left(\dfrac{du_i}{dx_j}\right)$ is a nonzero constant in the ground field k. The problem is to deduce that $k[x_1,\ldots,x_n]=k[u_1,\ldots,u_n]$.

Of course, if the field has positive characteristic, it is well known that the Jacobian problem as stated has a negative answer. One standard example is

$$u_1=x_1+x_1^p,\quad u_2=x_2$$

where p is the characteristic. Clearly, similar examples exist in all dimensions. So we assume that k has characteristic 0.

In general, the problem is still unsolved.

We give a relatively simple proof under the additional assumption that the field extension $k(x)$ over $k(u)$ is essentially Galois. Explicitly this means that either the fields are equal or that there is a (nontrivial) Galois extension L of $k(u)$ contained in $k(x)$. A topological proof of this fact was first published by Campbell [Ca]. Most of the arguments presented below have already appeared in [A1]. However, in [A1], only the two dimensional theorem was deduced.

The second part deals with the two dimensional problem.

In § 3, we restrict to the two variables x,y and take two polynomials over a field k (of characteristic 0, of course) satisfying the Jacobi condition that $J_{x,y}(f,g)$ is a nonzero constant. Temporarily, by "degree", let us mean the total degree with respect to (x,y). It is evident that the jacobian of the highest degree parts of f, g will either be 0 or will give the highest degree terms of the jacobian $J_{x,y}(f,g)$. Moreover, in the latter case, the degree of the jacobian is exactly equal to the sum of the degrees of f,g minus the sum of the degrees of x,y. Much can be deduced by generalizing the idea of a degree by assigning weight a to x and weight b to y so that a monomial x^iy^j has weight $ai+bj$. We illustrate the use by disposing of the case when the "usual degrees" of f,g are either coprime or when their GCD is prime.

The last two sections were not part of the original notes.

In § 4, we discuss yet another viewpoint, also developed in [A1]. For this, it is convenient, though not quite necessary, to arrange that f,g are monic in, say y. By the basic "two points at infinity" Lemma (3.5), we get to assume that either the proof is finished or we may assume the usual degree forms to be powers of $y^s(y+cx)^t$ for some s,t. Then, we may think of the pair (f,g) as giving a parametrized plane curve over the ground field $k(x)$, where y is thought of as the parameter of the curve. This fits the mold of the Epimorphism Theorem calculations. One of the simplest observations deducible from this is that the Jacobian problem is equivalent to proving that this curve is nonsingular. We begin by giving the Taylor Resultant formula of Abhyankar, developed for this purpose in 1972, which calculates the "conductor" of this plane curve directly as a polynomial in x,f,g. This formula has since been stated without proof in [A2, page 153] and we take this opportunity to write down the proof, in view of the interest in the formula. In the remaining part of § 4, we write another formula for the conductor in terms of the special generators of value-semigroups developed by using the "Expansion Techniques" as in [A1]. This supplies most of the details for the formula in [A2, page 169].

In § 5, we expand on the theme of § 3 and give a solution of the Jacobian problem for the case of "two characteristic pairs". The results generalize the main results from § 3 and further illustrate the expansion techniques as applied to the problem.

Abhyankar has unpublished results disposing of cases when the plane curve (f, g) has "a small number" of singularities, but the calculations are too messy to be included in a paper of this nature.

1. The Jacobi condition

Notation. Let k be a field and let $x_1, \ldots, x_n$ be n indeterminates over k. Given polynomials $u_1, \ldots, u_n \in k[x_1, \ldots, x_n]$, we consider the Jacobian of $u_1, \ldots, u_n$ with respect to $x_1, \ldots, x_n$:

$$J_x(u) = \det\left(\frac{\partial u_i}{\partial x_j}\right).$$

We say that the polynomials (u) satisfy the JACOBI-CONDITION, or briefly JC, if we have that $J_x(u) = \mathrlap{-}\theta$. Here $\mathrlap{-}\theta$ stands for "Abhyankar's Nonzero", namely, any suitable nonzero element of the ground field k. Note that $\mathrlap{-}\theta$ may denote different numbers depending on the context, perhaps, even in the same equation.

Sometimes, we may replace the field k by a suitable domain and we may need the **generalized JC** which states that $J_x(u)$ is a unit in k.

Let Ω denote the universal module of differentials of the polynomial ring $k[x_1, \ldots, x_n]$ over k. Recall that Ω is a vector space generated by $dx_1, \ldots, dx_n$ over k. By $\wedge^n\Omega$, as usual, we will denote the nth exterior power of Ω, which is a one-dimensional vector space generated by $dx_1 \wedge \cdots \wedge dx_n$. Thus, another way of describing the Jacobian is by writing

$$du_1 \wedge \cdots \wedge du_n = J_x(u) dx_1 \wedge \cdots \wedge dx_n \in \wedge^n\Omega.$$

We can conveniently abbreviate this as $du = J_x(u)dx$.

We need some standard simple facts about the universal module of differentials to reformulate JC[1].

Properties of Differentials

(1) *Let A be a ring and B an A-algebra. Then there exists a B-module $\Omega_{B/A}$ and an A-derivation $d: B \to \Omega_{B/A}$ such that for every A-derivation D of B into a B-module M,*

[1] This material was part of the original notes and is left intact for historical reasons. For readers familiar with these concepts, it suffices to note that: for an A – algebra B let $\Omega_{B/A}$ denote, as usual, the universal module of differentials and let $d: B \to \Omega_{B/A}$ be the universal derivation. Such readers may safely proceed to Lemma 1.1.

there is an unique B-linear map $\alpha:\Omega_{B/A}\to M$, *such that* $D=\alpha\circ d$. *Indeed, this is the defining property of the universal module* $\Omega_{B/A}$.
(2) *If* $B=A[x_1,\ldots,x_n]$, *then* $\Omega_{B/A}=Bdx_1+\cdots+Bdx_n$, *where* $\{dx_1,\ldots,dx_n\}$ *is a free basis and d is given by* $df=\sum_i \frac{\partial f}{\partial x_i}dx_i$.
(3) *If* $S\supset T$ *are respectively multiplicative sets in B and A, then*

$$\Omega_{S^{-1}B/T^{-1}A}=\Omega_{S^{-1}B/A}=S^{-1}\Omega_{B/A}.$$

(4) *If I is an ideal in B, then*

$$\Omega_{(B/I)/A}=\frac{\Omega_{B/A}}{(I\Omega_{B/A}+BdI)}.$$

(5) *If B is essentially of finite type over A, i.e. B is a localization of a finitely generated ring over A, then* $\Omega_{B/A}$ *is a finite B-module.*
(6) *For a finitely generated field extension L of a field K, the condition that* $\Omega_{L/K}=0$ *is equivalent to L being a finite separable algebraic extension of K.*

Proof. We will briefly indicate the idea of the proof behind these. The first five properties are deduced by formal algebraic manipulations by proving the existence of $\Omega_{B/A}$ constructively.

For the sixth property above, let us indicate more details. Write, using standard field theory, $K\subset K_1\subset K_2\subset L$, where K_1 is a pure transcendental extension generated by a transcendence basis of L over K, K_2 is separable algebraic over K_1 and L is purely inseparable over K_2.

Suppose, that $\Omega_{L/K}=0$, or, in other words, every K-derivation of L into itself, is trivial.

If $L\neq K_2$, then there exists a nontrivial K_2-derivation of L into L and this contradicts the hypothesis. So, $L=K_2$, and thus L is separable algebraic over K_1.

If $K\neq K_1$, then, set one of the transcendence basis to be x. The K-derivation $\partial/\partial x$ of K_1 extends to the separable algebraic extension field L and again we get a nontrivial K-derivation of L, a contradiction. Thus $K=K_1$ and hence L is separable algebraic over K.

The converse is obvious!

Lemma 1.1. *Let* $B=k[x_1,\ldots,x_n]$ *be a polynomial ring over a field k and let* $u_1,\ldots,u_n$ *be polynomials. Set* $A=k[u_1,\ldots,u_n]$. *Let* $K=Qt(A)$ *and* $L=Qt(B)$ *be the respective quotient fields of A, B.*

(1) $A=B$ *implies* $J_x(u)=\cancel{0}$, *i.e.,* $0\neq J_x(u)=c$ *for some c in k.*
(2) L/K *is a finite separable extension if and only if* $J_x(u)\neq 0$.
(3) $J_x(u)\neq 0$ *implies that* $u_1,\ldots,u_n$ *are algebraically independent over k and converse holds if k has zero characteristic.*

Proof. (1) Since $A=B$, both the n-differentials $du=du_1\wedge\cdots\wedge du_n$ and $dx=dx_1\wedge\cdots\wedge dx_n$ are generators of the one-dimensional vector space $\wedge^n\Omega_{B/k}=\wedge^n\Omega_{A/k}$. Hence, they differ from each other by a nonzero constant in k, i.e. $du=cdx$ where $0\neq c\in k$. Clearly, $J_x(u)=c$ as stated.

(2) Since L is a finitely generated extension of K, we know that L/K is finite separable if and only if $\Omega_{L/K}=0$ if and only if $\Omega_{L/k}=\Omega_{K/k}$. The last condition is evidently equivalent to $u_1,\dots,u_n$ forming another free L-basis of $\Omega_{L/k}$ or equivalently, $du_1,\dots,du_n$, are independent over L.
(3) Follows from (2).

Definition. Let $R\subset S$ be local rings with the maximal ideals $m(R)\subset m(S)$. We say that S is **unramified over** R (or S/R is **unramified**) if

(1) $S/m(S)$ is finite separable over $R/m(R)$ and
(2) $m(R)S+m(S)^2=m(S)$.

Note that the second condition implies that $m(R)S=m(S)$ in case $m(S)$ is finitely generated.

More generally, given an A-algebra B and a prime ideal p in B, we say that B is **locally unramified over** A at p, if $B_p/A_{p\cap A}$ is unramified. We say that B is **unramified** over A, if B is locally unramified over A at all primes p.

Lemma 1.2. *Let $R\subset S$ be local rings with $m(R)\subset m(S)$. Assume that S is essentially of finite type over R. Then S/R is unramified if and only if $\Omega_{S/R}=0$.*

Proof. Suppose that S/R is unramified. We know that $\Omega_{S/R}$ is a finite S-module. Consider the R-derivation

$$D:S\xrightarrow{d}\Omega_{S/R}\to\frac{\Omega_{S/R}}{m(S)\Omega_{S/R}}=M,\quad\text{say.}$$

Now any $x\in m(S)$ can be written as $x=\Sigma_i x_i m_i+\Sigma_j y_j z_j$, where $x_i\in S$, $y_i,z_j\in m(S)$ and $m_i\in m(R)$. It is easy to check that $D(x)=0$. Thus D induces a $R/m(R)$-derivation $\bar{D}:S/m(S)\to M$. Since $S/m(S)$ is finite and separable over $R/m(R)$ by hypothesis, we get that $\bar{D}=0$. It follows that $D=0$ and $M=0$. Thus, by Nakayama's lemma, we get $\Omega_{S/R}=0$.

Now assume that $\Omega_{S/R}=0$. Setting $S/m(S)=L$ and $R/m(R)=K$ we can deduce that $\Omega_{L/K}=0$. Thus L/K is a finite and separable extension. Now the complete local ring $T=S/(m(R)S+m(S)^2)$ has a coefficient field containing K. If $T\neq K$, then T has a residue ring of type $K[x]/(x^2)$. This last ring has a nontrivial K-derivation (and hence an A-derivation) $x(\partial/\partial x)$. This contradicts $\Omega_{S/R}=0$, so $T=K$ or equivalently $m(S)=m(R)S+m(S)^2$.

COROLLARY 1.3. *Let B be an A-algebra of essentially finite type. Then $\Omega_{B/A}=0$ if and only if B is locally unramified at every prime ideal p.*

More generally, $\Omega_{B/A}=0$ if and only if B is locally unramified at every maximal ideal p.

Proof. Since $\Omega_{B/A}$ is finitely generated, we have that $\Omega_{B/A}=0$ if and only if $(\Omega_{B/A})_p=0$ for all primes p in B if and only if $(\Omega_{B/A})_p=0$ for all maximal ideals p in B. Now we use $(\Omega_{B/A})_p=\Omega_{B_p/A_{p\cap A}}$ together with the above lemma.

PROPOSITION 1.4

Let $A=k[u_1,\dots,u_n]$ with polynomials $u_1,\dots,u_n$ in $B=k[x_1,\dots,x_n]$, a polynomial ring in n-variables over a field k. Then the following are equivalent:

(1) *The polynomials* $u_1,\ldots,u_n$ *satisfy the Jacobi-condition* $J_x(u) = \not\theta$.
(2) $\Omega_{B/A} = 0$.
(3) *B is locally unramified over A at every prime ideal p of B.*
(4) *B is locally unramified over A at every maximal ideal p of B.*

Proof. Obvious from the above discussion.

COROLLARY 1.5. *Let A, B be as in* (1.4) *and assume that the Jacobi-condition is satisfied for the polynomials* $u_1,\ldots,u_n$. *Let* $\bar{A}$ *denote the integral closure of A in B. Then we have the following*:

(1) *For every prime p in B,* $ht(p) = ht(q)$, *where* $q = p \cap A$ *and B is locally unramified over A at p. Moreover,* $\hat{B}_p$ *is a finite free* $\hat{A}_q$*-module.*[2]
(2) *If p is a height* 1 *prime of B, then* $\bar{A}_{p\cap\bar{A}} = B_p$.
(3) *If k is algebraically closed and m is a maximal ideal of B, then* $B_m = \bar{A}_{m\cap\bar{A}}$.
(4) *Let V be any dvr* (*rank* 1 *discrete valuation ring*) *of* $Qt(B)$ *which contains B. Then V is unramified over* $W = V \cap Qt(A)$.

Proof. (1) We already know that B_p is unramified over A_q. This implies that $\hat{B}_p$ is a finite $\hat{A}_q$-module. This, in turn, implies that q and p have the same heights. Moreover, both $\hat{B}_p$ and $\hat{A}_q$ are regular local rings, so by [Na, (25.16)] we get that $\hat{B}_p$ is a free $\hat{A}_q$-module.
(2) $\bar{A}_{p\cap\bar{A}}$ is a dvr contained in B_p, so by maximality of a dvr, it coincides with B_p.
(3) Using (1), we see that $A_{m\cap A} = B_m$. Let $C = \bar{A}_{m\cap\bar{A}}$. Then C is normal, hence analytically normal and hence contained in B_m. Since both C and B_m have the same quotient field, they are equal.
(4) Consider $R = W[x_1,\ldots,x_n] \subset V$ and set $p = R \cap m(V)$, where $m(V)$ is the maximal ideal of V. Since $\Omega_{B/A} = 0$, we get that $\Omega_{R/W} = 0$ and $\Omega_{R_p/W} = 0$. By Lemma 1.2, R_p is unramified over W. In particular R_p is a dvr and hence $V = R_p$.

2. The Galois case

In this section, we will use the following hypothesis, unless otherwise declared.

Hypothesis. Let k be a field, $B = k[x_1,\ldots,x_n]$ the polynomial ring in n variables over k and let $u_1,\ldots,u_n$ be polynomials in B. Set $A = k[u_1,\ldots,u_n]$. As before, we say that $u = (u_1,\ldots,u_n)$ satisfies JC (the Jacobi-condition), if $J_x(u) = \not\theta$.
Some of our results can be stated and proved under the following more general hypothesis.

Generalized Hypothesis. Let k be normal domain. Assume that k is prefactorial, i.e. assume that every height 1 prime of k is the radical of a principal ideal.
Let $B = k[x_1,\ldots,x_n]$ the polynomial ring in n variables over k and let $u_1,\ldots,u_n$ be polynomials in B. Set $A = k[u_1,\ldots,u_n]$.

[2] Here ht denotes the usual height of a prime ideal and $\hat{}$ denotes the usual completion.

PROPOSITION 2.1 (Birational Case).

Suppose that $Qt(A) = Qt(B)$ and u satisfies JC. *Then $A = B$*, i.e., *the Jacobian Theorem holds.*

Proof. Let q be any height 1 prime of A. Then $q = aA$ for a nonunit $a \in A$. Now, a is a nonunit in B. Write a factorization $a = p_1 \cdots p_r$, where $p_1 \cdots p_r$ are irreducible in B. Any one of them, say p_1 generates a height 1 prime ideal $p = p_1 B$ and $p \cap A \supset q$. By Corollary 1.5, we get that $p \cap A$ has height 1 and hence $p \cap A = q$. Hence, $A_q \subset B_p$ and since both are dvr with the same quotient field, they are equal. Consequently, $A_q \supset B$.

Now we have

$$A = \cap \{A_q | q \text{ is a height one prime}\} \supset B \supset A.$$

COROLLARY 2.2.

Let $R \subset S$ be noetherian domains such that:

(1) $\Omega_{S/R} = 0$.
(2) *$Qt(R) = Qt(S)$ and S/R is essentially of finite type.*
(3) *Any nonunit in R is a nonunit in S.*
(4) *R is normal and prefactorial.*

Then $R = S$.

Proof. The only place where the JC was used in the proof of the Proposition 2.1, was in the application of Corollary 1.5 to deduce that the contraction of a height 1 prime has height 1. This can be alternatively proved by the first two conditions of our Corollary; for details see the proof of part 1 of Corollary 1.5.

Remarks. (1) Proposition 2.1 remains valid under the generalized hypothesis, provided we assume that u satisfies the generalized JC also. This follows from Corollary 2.2 after noting that since k is prefactorial, $A = k[u_1, \ldots, u_n]$ is prefactorial. Such an example where k is the ring of integers was already discussed by O. Keller [Ke].

(2) The condition that R be prefactorial is essential. Indeed, take

$$R = k[x, xy, y(1 + xy)] \subset S = k[x, y].$$

Note that R is isomorphic to $k[u, v, w]/(uw - v(v + 1))$ and hence R is easily seen to be regular. To see that $\Omega_{S/R} = 0$, note that $dx, d(xy)$ and $d(y(1 + xy))$ generate $\Omega_{S/k}$. Clearly $\mathrm{Qt}(R) = \mathrm{Qt}(S)$ and obviously, $R \neq S$. Indeed, R is not prefactorial. To see this, consider the ideal $I = (u, 1 + v)R$. Suppose that it is the radical of a principal ideal in R. In S it extends to the ideal $(x, 1 + xy)S = (1)S$. Since only units in S are constants, radical of I and hence I itself would be the unit ideal in R. But the residue class ring R/I is isomorphic to $k[w]$, a contradiction!

PROPOSITION 2.3 (Galois Case)

Suppose that u satisfies JC and either $Qt(A) = Qt(B)$ or that there is a nontrivial tame Galois extension L of $Qt(A)$ contained in $Qt(B)$, then $A = B$.

Quick Proof. By Proposition 2.1, we may assume that $Qt(A) \neq Qt(B)$. Using the fact that the affine n-space is simply connected (i.e. there are no unramified proper tame extensions of $k(u_1, \ldots, u_n)$) and the purity of the branch locus, we deduce that there exists a height 1 prime q in A and a dvr V with quotient field L, such that $m(V) \cap A = q$ (where $m(V)$ is the maximal ideal of V) and V is ramified over A_q. Since L is a Galois extension of $Qt(A)$, all extensions of A_q to L are ramified over A_q. Hence, every extension of A_q to $Qt(B)$ is also ramified over A_q. Now, since A and B have the same units (nonzero elements of k), we must have at least one prime ideal p in B such that $p \supset q$. Choosing a minimal prime p with this property, we get that $ht(p) = 1$. Now by Corollary 1.5 $p \cap A = q$ and B_p is unramified over A_q. Since B_p is evidently an extension of A_q to $Qt(B)$, we get a contradiction!

COROLLARY 2.4

Proposition 2.3 *remains valid under the generalized hypothesis, provided we use the generalized JC also.*

Proof. Set $Qt(k) = K$ and let $B_1 = K[x_1, \ldots, x_n]$ and $A_1 = K[u_1, \ldots, u_n]$. Then by Proposition 2.3, we get that $A_1 = B_1$ and hence $Qt(B) = Qt(A)$. Now, we are done by Corollary 2.2.

Simpler Proof. We needed some celebrated theorems above to deduce that unless $Qt(B) = Qt(A)$, we must have a height 1 prime p in B, such that B_p is ramified over $A_{p \cap A}$. We give a simpler proof of this by reducing the proof to some rather well known theory of functions of one variable.

Bertini Lemma 2.5. *Let F be an infinite field and let K be the quotient field of $F[z_1, \ldots, z_n]$, where $z_1, \ldots, z_n$ are indeterminates over F. Assume that $n \geq 2$. Let E be a finite separable algebraic extension of K, such that F is algebraically closed in E. Then for "almost all" linear combinations $y = \lambda_1 z_1 + \cdots + \lambda_n z_n$, we get that $F(y)$ is algebraically closed in E and E is a separable extension of $F(y)$.*

Proof. The separability of E over $F(y)$ is well known and we only demonstrate the algebraic closedness. For $\lambda \in F$ consider fields $K_\lambda = \overline{F(y_\lambda)}(z_2, \ldots, z_n)$, where the bar $\overline{}$ denotes algebraic closure in E and $y_\lambda = z_1 + \lambda z_2$.

Now we claim that if $K_\lambda = K_\mu$ for some $\lambda \neq \mu$, then $F(y_\lambda)$ is algebraically closed in E. Assuming the claim for a moment, we note that, since F is infinite and since there are only finitely many fields in between K and E, we have the result for almost all combinations $z_1 + \lambda z_2$. The result can then be deduced with a suitable technical interpretation of "almost all".

To prove the claim, set $w_1 = z_1 + \lambda z_2$, $w_2 = z_1 + \mu z_2$ and $w_i = z_i$ for $i \geq 3$. By assumption

$$\overline{F(w_1)}(w_2, w_3, \ldots, w_n) = \overline{F(w_2)}(w_1, w_3, \ldots, w_n).$$

Denote this field by L. Since F is algebraically closed in $\overline{F(w_2)}$, we get that $F(w_1, w_3, \ldots, w_n)$ is algebraically closed in $\overline{F(w_2)}(w_1, w_3, \ldots, w_n) = L$. In particular, $F(w_1)$ is algebraically closed in the field L. It follows that $F(w_1) = \overline{F(w_1)}$ as claimed.

COROLLARY 2.6

With the notation and assumptions of the Bertini Lemma (2.5), *there exist* $y_1, \ldots, y_n$, *linear combinations of* $z_1, \ldots, z_n$, *such that* $F[y_1, \ldots, y_n] = F[z_1, \ldots, z_n]$ *and* $F(y_2, \ldots, y_n)$ *is algebraically closed in E.*

Proof. Apply (2.5) a number of times.

Lemma 2.7. *Let K denote the quotient field of* $F[y]$. *Let E be a finite separable algebraic extension of K such that F is algebraically closed in E. Suppose that E/K is tamely ramified. If no height one prime of* $F[y]$ *is ramified in E then* $E = K$.

Proof. The algebraic closure $\bar{F}$ of F and E are linearly disjoint over F. Hence after replacing F by $\bar{F}$, K by $\bar{F}(y)$ and E by $E(\bar{F})$ we are reduced to the case where F is algebraically closed. The canonical divisor for both E and K is given by dy and according to [Ch, p. 106, Corollary 2] we have the formula

$$\deg_E(dy) = \deg_K(dy)[E:K] + \deg(\mathfrak{D}_{E/K})$$

where $\mathfrak{D}_{E/K}$ denotes the different of E/K. The only ramified discrete valuation of K/F is $F[y^{-1}]_{(y^{-1})}$. Since E/K is tamely ramified it follows that

$$\deg(\mathfrak{D}_{E/K}) \leqslant [E:K] - 1.$$

Let g be the genus of E/F. Then the above formula yields

$$-2 \leqslant 2g - 2 = -2[E:K] + \deg(\mathscr{D}_{E/K}) \leqslant -[E:K] - 1.$$

Hence $[E:K] = 1$ and $E = K$.

PROPOSITION 2.8

Let K denote the quotient field of the polynomial ring $k[z_1, \ldots, z_n]$ *and E a finite separable algebraic extension of K such that E/K is tamely ramified and k is algebraically closed in E. If* $E \neq K$, *then some height one prime of* $k[z_1, \ldots, z_n]$ *ramifies in E.*

Proof. Using (2.6) we consider $F[y]$ where $F = k(y_2, \ldots, y_n)$ and $y = y_1$. According to (2.7) some height one prime p of $F[y]$ ramifies in E. Then also $q = p \cap k[y_1, \ldots, y_n] = p \cap k[z_1, \ldots, z_n]$ has height one and ramifies in E.

Remark. (2.8) can replace the use of "purity of branch locus" and "simple connectedness" in the proof of (2.3).

3. Some results in two variables

Notation. In this section we suppose that k is an algebraically closed field of characteristic 0 and we study polynomials $f, g \in k[x, y]$ satisfying the Jacobi-condition JC:

$$df \wedge dg = \cancel{0}\, dx \wedge dy$$

where, as already explained, θ is the "Abhyankar's Nonzero", i.e. some nonzero element of k.

For $f = \Sigma f_{i,j} x^i y^j$ we consider $\mathrm{supp}(f) = \{(n,m) \in \mathbb{R}^2 | f_{n,m} \neq 0\}$.

The boundary polygon of the smallest convex set in $\mathbb{R}^2$ containing supp(f) will be called $N(f)$ = the Newton polygon of f. (This concept is somewhat similar to ordinary Newton polygon, but certainly not the same.) Let $E(f)$ denote the set of vertices of $N(f)$, or equivalently, $E(f)$ is the set of extreme points of $N(f)$.

In order to show that the Jacobi-condition for f and g implies a similarity of $N(f)$ and $N(g)$ we introduce degree functions and gradings of $k[x,y]$.

Given coprime integers $a, b \in \mathbb{Z}$ (i.e. integers satisfying $a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$), we form a grading $k[x,y] = \Sigma_{n=-\infty}^{\infty} H_{a,b}^n$ where $H_{a,b}^n$ is the k-vector space generated by all monomials $x^i y^j$ such that $ai + bj = n$. Every nonzero $f \in k[x,y]$ is written as Σf_i, where $f_i \in H_{a,b}^i$. The corresponding (a,b)-degree $\Delta_{a,b}$ is defined by $\Delta_{a,b}(f) = \max\{i | f_i \neq 0\}$. The degree form of f with respect to (a,b) is by definition equal to f_n with $n = \Delta_{a,b}(f)$. We use the notation $f_{a,b}^+$ to denote it. If $f = 0$ then usually its *degree* is taken to be undefined or sometimes equal to any desired number. The degree form is similarly, undefined or 0.

The reference to the weight a, b may be dropped, if the weights are clear from the context.

It is not hard to see that $(n,m) \in E(f)$ is equivalent to the existence of an (a,b)-degree such that $f_{a,b}^+ = \theta x^n y^m$. Moreover, every side of $N(f)$ corresponds to an (a,b)-degree for which $f_{a,b}^+$ is not a constant multiple of a monomial.

On $\wedge^2 \Omega_{k[x,y]/k}$ we introduce a similar grading associated with (a,b) by means of

$$\wedge^2 \Omega_{k[x,y]/k} = \sum H_{a,b}^n dx \wedge dy.$$

The corresponding degree-function is again denoted by $\Delta_{a,b}$. Further, for any $w \in \wedge^2 \Omega_{k[x,y]/k}$ we let w_i denote its homogeneous part (with respect to (a,b)) of order i.

The definitions concerning weights can easily be generalized to arbitrary real numbers (a,b). For analyzing Newton diagrams, however, we only need integer weights or sometimes, for clarity, we may use rational weights.

Lemma 3.1. *Let a, b be coprime integers. We write H^n for $H_{a,b}^n$ and Δ for $\Delta_{a,b}$. We will also drop reference to a, b from the various degree-form notations. Then we have:*

(1) $dH^n \wedge dH^m \subset H^{n+m-a-b} dx \wedge dy$.

(2) *For any $f, g \in k[x,y]$ one has*

$$(df \wedge dg)_{\Delta(f)+\Delta(g)-\Delta(xy)} = df^+ \wedge dg^+.$$

In particular, $\Delta(f) + \Delta(g) - \Delta(xy) \geqslant \Delta(df \wedge dg)$. Strict inequality holds if and only if $df^+ \wedge dg^+ = 0$.

(3) *If $f, g \in k[x,y]$ are nonzero polynomials, homogeneous with respect to (a,b), then $df \wedge dg = 0$ implies $f^{|\Delta(g)|} = \theta g^{|\Delta(f)|}$.*

(4) *If $f, g \in k[x,y]$ satisfy $d(xf) \wedge d(g) = \theta dx \wedge dy$, then $0 \neq f \in k$ and $g = \theta y + p(x)$ for some polynomial p. In particular, $k[f,g] = k[x,g] = k[x,y]$.*

(5) *If $f, g \in k[x,y]$ satisfy $df \wedge dg = \theta dx \wedge dy$ and f is homogeneous with respect to (a,b), then $k[f,g] = k[x,y]$ and either $deg(f) = 1$ or $f = \theta x + p(y)$ or $f = \theta y + p(x)$ for some polynomial p.*

Proof.

(1) For monomials $x^{i_1}y^{j_1}$ and $x^{i_2}y^{j_2}$ we have

$$d(x^{i_1}y^{j_1}) \wedge d(x^{i_2}y^{j_2}) = (i_1 j_2 - i_2 j_1)x^{i_1+i_2-1}y^{j_1+j_2-1}dx \wedge dy.$$

The statement in (1) follows easily from that.
(2) Let $n = \Delta(f)$ and $m = \Delta(g)$. Write

$$f = \sum_{i \leqslant n} f_i \text{ and } g = \sum_{j \leqslant m} g_j.$$

Clearly

$$df \wedge dg = \sum_d \sum_{i+j=d} df_i \wedge dg_j$$

and the term $\Sigma_{i+j=d} df_i \wedge dg_j$ is homogeneous (with respect to (a,b)) of order $d-a-b = d-\Delta(xy)$. This makes statement (2) obvious.
(3) Consider the 1-form $w = -(by)dx + (ax)dy$. It has the property that for homogeneous $h_1, h_2 \in k[x,y]$ (homogeneous with respect to (a,b)) and $h_2 \neq 0$ the following formula holds:

$$d\left(\frac{h_1}{h_2}\right) \wedge w = (\Delta(h_1) - \Delta(h_2))\left(\frac{h_1}{h_2}\right)dx \wedge dy.$$

Apply this to $v = f^{\Delta(g)}g^{-\Delta(f)}$. Then it gives $dv \wedge w = 0$.
We want to show that $v \in k$. If not, then $dv \neq 0$ and is linearly dependent with w (over the field $k(x,y)$). So $dv = hw$ with $0 \neq h \in k(x,y)$. Also $dv \wedge df = dv \wedge dg = 0$, hence $df \wedge w = dg \wedge w = 0$. This implies $\Delta(f) = \Delta(g) = 0$ and $v = 1$. Contradiction! Hence $v \in k$ and we have $f^{\Delta(g)} = \cancel{0}\, g^{\Delta(f)}$. Since f and g are actually polynomials, $\Delta(f)\Delta(g) \geqslant 0$ and (3) follows. We remark that the statement of (3) becomes uninteresting for $\Delta(f) = \Delta(g) = 0$.
(4) Assume first that f is nonconstant. Without loss of generality we may assume that $g(0,0) = 0$. Then neither f nor g is divisible by x since in either case $df \wedge dg$ would be divisible by x. Write $g = xp(x,y) + q(y)$ where $q \neq 0$, $q(0) = 0$. Take (a,b) such that $a < 0$ and $b = 1$. As usual, by $a \ll 0$, we mean a is assumed to be sufficiently negative. Thus, for $a \ll 0$, we have $\Delta(xp(x,y)) < 0$ and $\Delta(q(y)) = \deg_y q = s > 0$. Hence $g^+_{a,b} = \cancel{0}\, y^s$. A similar argument shows that $f^+_{a,b} = \cancel{0}\, y^t (t > 0)$ for $a \ll 0$ and $b = 1$. Hence $(xf)^+_{a,b} = \cancel{0}\, xy^t$. From (2) it follows that

$$d(xf)^+_{ab} \wedge d(g^+_{ab}) = v dx \wedge dy \text{ with } v \in k.$$

This is a contradiction. Hence $f \in k$, $f \neq 0$. The rest of (4) follows easily.
(5) Write $g = \Sigma g_i$. It follows from (1) that for some j, $df \wedge dg_j = \cancel{0}\, dx \wedge dy$. Put $h = g_j$. From (2) it follows that $\Delta(fh) = \Delta(xy)$, hence $fh = \Sigma\lambda_{\alpha,\beta}x^\alpha y^\beta$ and $\operatorname{supp}(fh) \subset \{(\alpha,\beta) \mid (\alpha-1)a + (\beta-1)b = 0\}$. We suppose (as we may) that $f(0,0) = 0$ and $h(0,0) = 0$. If x or y divides fh, then (4) yields $k[f,h] = k[x,y]$. It follows that $g = h + p(f)$ for some polynomial p and consequently $k[f,g] = k[x,y]$.
If neither x nor y divides fh, then there exists $n > 0$ and $m > 0$ with $(n,0)$, $(0,m) \in \operatorname{supp}(fh)$. Hence $(n-1)a = b$ and $a = (m-1)b$. So either $a = b = 1$ or $a = b = -1$. In both cases f and h are linear expressions in x and y. Clearly $k[f,h] = k[x,y]$ and $k[f,g] = k[x,y]$. The rest of the statement is easily checked.

Remark.

Part (2) above leads to the following natural.

Definition. We say that two polynomials f, g are (a,b)-**related** if

$$\Delta_{a,b}(f) + \Delta_{a,b}(g) - \Delta_{a,b}(xy) > \Delta_{a,b}(df \wedge dg)$$

and they are said to be (a,b)-**unrelated** otherwise.

We set the (a,b)-**deficiency** of (f,g) to be

$$\delta_{a,b}(f,g) = \Delta_{a,b}(f) + \Delta_{a,b}(g) - \Delta_{a,b}(xy) - \Delta_{a,b}(df \wedge dg).$$

Thus part (3) says that (a,b)-relatedness is equivalent to the (a,b)-degree-forms being powers of each other, or equivalently, to the (a,b)-deficiency being positive.

Part (4) of the above lemma can also be stated as: If $f, g \in k[x,y]$ satisfy the Jacobi-condition and if there exists a factor p of fg and a q with $k[p,q] = k[x,y]$, then $k[f,g] = k[x,y]$.

PROPOSITION 3.2.

(Similarity of Newton polygons) Let f, $g \in k[x,y]$ satisfy $df \wedge dg = \theta\, dx \wedge dy$ and $f(0,0) \neq 0 \neq g(0,0)$. Let their total degrees with respect to (x,y) be respectively n, m. Assume $n \geqslant 2$ and $m \geqslant 2$. Then:

(1) *If (a,b) is a degree such that $a \leqslant 0$ and $b \leqslant 0$, then $supp(f^{+}_{a,b})$ is either $\{(0,0)\}$ or lies entirely on the x-axis or the y-axis. The same holds for g.*

(2) *There exists a non-zero constant $c \in k$ such that for every degree (a,b) with $a > 0$ or $b > 0$ one has*

$$(f^{+}_{a,b})^m = c(g^{+}_{a,b})^n.$$

In particular $\Delta_{a,b}(f^m - cg^n) < max(m\Delta_{a,b}(f), n\Delta_{a,b}(g))$ for all degrees (a,b) with $a > 0$ or $b > 0$.

(3) $mN(f) = nN(g)$ and $mE(f) = nE(g)$. *The Newton polygons of f and g look like as in figure* 1.

(4) *Let (a,b) be a degree such that $a > 0$ or $b > 0$. Then there are (a,b)*-homogeneous elements $h_1, h_2 \in k[x,y]$ *such that* $df^{+}_{a,b} \wedge dh_1 = h_2 dx \wedge dy$, $\Delta(f^{+}_{a,b}) + \Delta(h_1) = \Delta(xy) + \Delta(h_2)$ *and for some* $p > 0$, $(f^{+}_{a,b})^p = \theta h_2^n$.

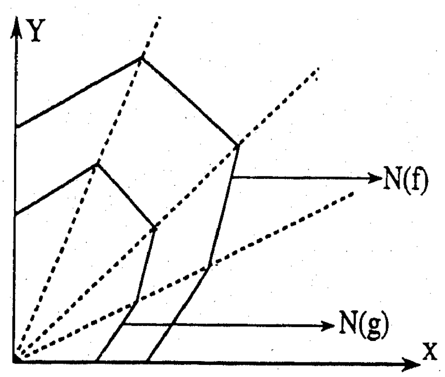

Figure 1.

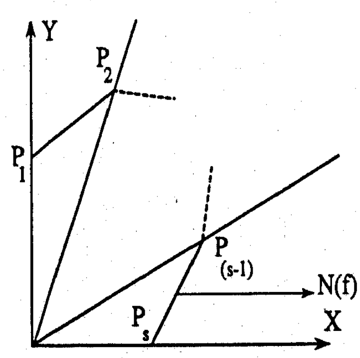

Figure 2.

Proof. Our assumptions $f(0,0)\neq 0$, $\deg f\geqslant 2$ and $df\wedge dg=\theta\,dx\wedge dy$ together with (3.1) part (4) imply that $f-f(0,0)$ is neither divisible by x nor y. So the Newton polygon of f looks like as in figure 2.

From the diagram, the statements in (1) follow.

Let (a,b) be a degree with $a>0$ or $b>0$. We claim that

$$\Delta_{a,b}(f)+\Delta_{a,b}(g)>\Delta_{a,b}(xy).$$

Consider separately the cases: (i) $a>0$, $b\leqslant 0$; (ii) $a\leqslant 0$, $b>0$; (iii) $a\geqslant b>0$; (iv) $b\geqslant a>0$.

Case (i): $\Delta_{a,b}(f)\geqslant a$, $\Delta_{a,b}(g)\geqslant a$, so $\Delta_{a,b}(fg)\geqslant 2a>\Delta_{a,b}(xy)$.

Case (iii): If $a>b$ the above argument still holds. If $a=b$, then $a=b=1$ and $\Delta_{a,b}$ equals the usual total degree on $k[x,y]$. It is given that $\deg(f)+\deg(g)\geqslant 4>\deg(xy)$.

Cases (ii) and (iv) are symmetric to (i) and (iii).

It follows from (3.1) that $df^+_{a,b}\wedge dg^+_{a,b}=0$ and $\Delta_{a,b}(f)>0$, $\Delta_{a,b}(g)>0$. Again using (3.1) we get

$$(f^+_{a,b})^{\Delta_{a,b}(g)}=\theta(g^+_{a,b})^{\Delta_{a,b}(f)}.$$

In particular, it follows that $f^+_{a,b}$ is a monomial if and only if $g^+_{a,b}$ is a monomial. Geometrically, this means that the line through $(0,0)$ and P_i (use the numbering of $E(f)$ as in the figure above) contains a point Q_i at $E(g)$ and that all points of $E(g)$ are obtained in this fashion. Also $f^+_{a,b}$ is not a monomial (i.e. represents a side of $N(f)$) if and only if $g^+_{a,b}$ is not a monomial. Hence the sides $\{P_i,P_{i+1}\}$ and $\{Q_i,Q_{i+1}\}$ are parallel. It follows that for some rational number $\lambda\neq 0$ one has $N(f)=\lambda N(g)$. Using $(a,b)=(1,1)$ one sees that $\lambda=n/m$.

Let c_i be the nonzero constant in k satisfying $(f^+_{a,b})^m=c_i(g^+_{a,b})^n$ where (a,b) is chosen such that $\operatorname{supp}(f^+_{a,b})=P_i$ and $\operatorname{supp}(g^+_{a,b})=Q_i$. For a degree (a,b) which corresponds to the side (P_i,P_{i+1}) of the Newton polygon of f we have $(f^+_{a,b})^m=\theta(g^+_{a,b})^n$. Comparing the monomials in this equation with maximal or minimal degree in x or y one finds $\theta=c_i=c_{i+1}$. Hence all c_i are equal to $c=c_1$. This proves (2) and (3).

For (4), put $T^1(g)=f^m-cg^n$. We claim that: $\delta_{a,b}(f,T^1(g))<\delta_{a,b}(f,g)$. This can be easily checked from $df\wedge dT^1(g)=\theta g^{n-1}dx\wedge dy$. If $\delta_{a,b}(f,T^1(g))=0$, then $h_1=T^1(g)^+_{a,b}$ and $h_2=\theta(g^+_{a,b})^{n-1}$ have the required properties. If $\delta_{a,b}(f,T^1(g))>0$, then (3.1) yields $df^+_{a,b}\wedge dT^1(g)^+_{a,b}=0$ and $(f^+_{a,b})^p=\theta(T^1(g)^+_{a,b})^n$ for some $p>0$. Put $T^2(g)=f^p-\theta T^1(g)^n$. Then, as before, we deduce that $\delta_{a,b}(f,T^2(g))<\delta_{a,b}(f,T^1(g))$. Since, the deficiency cannot decrease indefinitely, eventually some $T^r(g)$ has deficiency 0 and we get:

$$df\wedge dT^r(g)=\theta(gT^1(g)\cdots T^{r-1}(g))^{n-1}dx\wedge dy$$

and

$$\Delta(f)+\Delta(T^r(g))=\Delta(xy)+\Delta(gT^1(g)\cdots T^{r-1}(g))^{n-1}.$$

The elements $h_1=T^r(g)^+_{a,b}$ and $h_2=\theta((gT^1(g)\cdots T^{r-1}(g))^{n-1})^+_{a,b}$ have the required properties.

Remark. Part (4) of (3.2) can be phrased differently. The condition $(f^+_{a,b})^p=\theta h_2^n$ implies that $f^+_{a,b}$ and h_2 are powers of the same homogeneous element C. Putting $D=\theta h_1$, the equation reads $dC\wedge dD=C^{t+1}dx\wedge dy$, with $t\geqslant -1$. We study this equation separately in the sequel to this section.

Lemma 3.3. *Let C and D be (a, b)-homogeneous elements of $k[x, y]$. Write $C = x^{i_1} y^{j_1} \sigma(z)$ and $D = x^{i_2} y^{j_2} \tau(z)$ where $i_1, i_2, j_1, j_2 \in \mathbb{Z}$, $z = x^{-b} y^a$ and $\sigma, \tau \in k[z]$. Then*

$$dC \wedge dD = CD\left((i_1 j_2 - i_2 j_1) + z\left(\frac{\Delta(C)\tau'}{\tau} - \frac{\Delta(D)\sigma'}{\sigma}\right)\right)\frac{dx \wedge dy}{xy}.$$

Proof. Note that:

$$\left(\frac{dC}{C} \wedge \frac{dD}{D}\right) = \left(\frac{i_1 dx}{x} + \frac{j_1 dy}{y} + \frac{\sigma' dz}{\sigma}\right) \wedge \left(\frac{i_2 dx}{x} + \frac{j_2 dy}{y} + \frac{\tau' dz}{\tau}\right).$$

Using

$$\frac{dx}{x} \wedge \frac{dz}{z} = a\frac{dx \wedge dy}{xy} \text{ and } \frac{dy}{y} \wedge \frac{dz}{z} = b\frac{dx \wedge dy}{xy},$$

it is easy to establish the formula.

Lemma 3.4. *Let C and D be nonzero (a, b)-homogeneous elements of $k[x, y]$ such that $dC \wedge dD = \theta C^{t+1} dx \wedge dy$ with $t > 0$. If $\Delta(C) \neq 0$ and $\Delta(C)\Delta(xy) \geqslant 0$, then C^t divides D.*

Proof. Multiplying by C^{t-1} the equation becomes $d(C^t) \wedge dD = \theta C^{2t} dx \wedge dy$. After replacing C by C^t one sees that it is enough to deal with the case $t = 1$. Also, the nonzero constant can be absorbed in D.

Multiplying both sides of the equation by $\dfrac{xy}{CD}$ and using the reductions mentioned above, it is equivalent to

$$\frac{xy}{CD} dC \wedge dD = \frac{Cxy}{D} dx \wedge dy.$$

Using the notation of (3.3) we find that the expression $\dfrac{Cxy}{D}$ equals:

$$\frac{x^{i_1+1-i_2} y^{j_1+1-j_2} \sigma}{\tau} = (i_1 j_2 - i_2 j_1) + z\left(\frac{\Delta(C)\tau'}{\tau} - \frac{\Delta(D)\sigma'}{\sigma}\right).$$

Moreover, we can easily prearrange $\sigma(0) \neq 0$ and $\tau(0) \neq 0$, since any factors of z can be absorbed in the monomials.

The monomial $x^{i_1+1-i_2} y^{j_1+1-j_2}$, must then be a rational function of z and clearly a monomial z^r for some integer r. Now clearly, the rational function of z on the right hand side does not have a pole at $z = 0$, so neither does the left hand side. Since r is the z-adic order of the left hand side, we get that r is a nonnegative integer.

Let $\lambda(\neq 0)$ be any root of σ with multiplicity $e_1 > 0$ and a root of τ with multiplicity $e_2 \geqslant 0$. If we can show $e_2 \geqslant e_1$, then we have proved that σ divides τ.

Write $\sigma = (z - \lambda)^{e_1} \sigma_1$ and $\tau = (z - \lambda)^{e_2} \tau_1$. One obtains:

$$\begin{aligned} &z^r (\sigma_1/\tau_1)(z - \lambda)^{e_1 - e_2} \\ &= (i_1 j_2 - i_2 j_1) + z(\Delta(C)\tau_1'/\tau_1 - \Delta(D)\sigma_1'/\sigma_1) + \frac{z(\Delta(C)e_2 - \Delta(D)e_1)}{z - \lambda}. \end{aligned}$$

If $e_1 > e_2$, then $\Delta(C)e_2 = \Delta(D)e_1$. Using $\Delta(D) = \Delta(C) + \Delta(xy)$ this yields $e_1\Delta(xy) = (e_2 - e_1)\Delta(C)$, which contradicts the hypothesis $\Delta(C)\Delta(xy) \geqslant 0$.

In order to show that C divides D we are left with showing $i_1 \leqslant i_2$ and $j_1 \leqslant j_2$. Or, in obvious notation, $(i_1, j_1) \leqslant (i_2, j_2)$.

If $r = 0$, then, $(i_1, j_1) + (1, 1) = (i_2, j_2)$ and we are done.

If $r > 0$, then $(i_1 j_2 - i_2 j_1) = 0$. Hence for integers λ_1, λ_2 (not both zero) one has $\lambda_1(i_1, j_1) = \lambda_2(i_2, j_2)$. Using again $\Delta(D) = \Delta(C) + \Delta(xy)$ this yields $(\lambda_1 - \lambda_2)\Delta(C) = \lambda_2\Delta(xy)$. If $\lambda_2\Delta(xy) = 0$ then $\lambda_1 = \lambda_2$ and we are finished. If $\lambda_2\Delta(xy) \neq 0$, then we may take $\lambda_2 > 0$. The assumption $\Delta(xy)\Delta(C) \geqslant 0$ gives $\lambda_1 > \lambda_2$ and that implies $(i_1, j_1) \leqslant (i_2, j_2)$.

Lemma 3.5. *Let $0 \neq C$ and D denote (a, b)-homogeneous elements such that $dC \wedge dD = C dx \wedge dy$. Suppose $a > 0$ and $b > 0$. Then C has "at most two points at infinity" which means: There are (a, b)-homogeneous elements x_1 and y_1 with $k[x, y] = k[x_1, y_1]$ such that $C = \theta x_1^i y_1^j$.*

Moreover, $D = \theta x_1 y_1$ and $i \neq j$. The only possibilities are:

(1) *$a = b = 1$ and x_1, y_1 are linear expressions in x and y.*
(2) *$a = 1$ and $b > 1$ and $x_1 = x$ and $y_1 = y + \theta x^b$.*
(3) *$a > 1$ and $b = 1$ and $x_1 = x + \theta y^a$ and $y_1 = y$.*
(4) *$a \geqslant 1$ and $b \geqslant 1$ and $x_1 = x$ and $y_1 = y$.*

Proof. The formula $\Delta(D) = \Delta(xy)$ implies that D must have the form $\lambda_1 xy + \lambda_2 x^s + \lambda_3 y^t$.

If $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$, then $a = b = 1$ and $s = t = 2$. D is obviously equal to $x_1 y_1$ where x_1 and y_1 are linear in x and y. Also, D cannot be a square, so x_1, y_1 are linearly independent.

If $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 = 0$, then $D = \theta x(y + \theta x^{s-1})$. Since $y + \theta x^{s-1}$ is (a, b)-homogeneous it follows that $a = 1$, $b = s - 1$. We take in this case $x_1 = x$ and $y_1 = y + \theta x^b$.

If $\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$, then similarly $D = \theta x_1 y_1$ with $x_1 = x + \theta y^a$, $y_1 = y$ and $b = 1$.

If $\lambda_1 \neq 0$ and $\lambda_2 = \lambda_3 = 0$, then take $x_1 = x$ and $y_1 = y$. In this case (a, b) can be arbitrary.

Finally, $\lambda_1 = \lambda_2 = 0$ or $\lambda_1 = \lambda_3 = 0$ is not possible.

In all cases we have $D = \theta x_1 y_1$. Write $C = \Sigma\lambda_{ij} x_1^i y_1^j$, with $\mathrm{supp}(C) \subset \{(i, j) | ai + bj = \Delta(C)\}$. The equation $dC \wedge dD = \theta C dx_1 \wedge dy_1$ becomes explicit: $x_1 C_{x_1} - y_1 C_{y_1} = \theta C$. This implies that for some integer l

$$\mathrm{supp}(C) \subset \{(i, j) | i - j = l\}.$$

Hence C is a monomial in x_1 and y_1.

Remarks. (1) The restriction in (3.5) given by $a > 0$ and $b > 0$ is necessary as is shown in the following counterexample: $(a, b) = (1, 0)$, $C = x^2 y(y + 1)^3$ and $D = xy(y + 1)$ have the property $dC \wedge dD = C dx \wedge dy$ and C is not a monomial in new variables x_1, y_1.
(2) Another example: $(a, b) = (3, -1)$ and $C = x(xy^3 + 1)^2$ and $D = xy(xy^3 + 1)$. Then $dC \wedge dD = C dx \wedge dy$.
(3) **However, the following generalization is valid.** Assume that C and D are (a, b)-homogeneous elements such that $dC \wedge dD = C dx \wedge dy$ and the highest y-degree terms

in C and D are unrelated. Assume further that $a > 0$. Then the conclusion of the Lemma holds. For a proof, note that the hypothesis implies that the highest y-degree term in D must be θxy since its jacobian with the highest y-degree term of C must reproduce itself up to a constant multiplier. It follows that we must have

$$D = \theta xy + \lambda_2 x^s.$$

The proof is easily finished. Let us further note that in case $a + b > 0$ and $b < 0$, we must have $\lambda_2 = 0$ and hence C, D are monomials in x, y already.

Theorem 3.6. *Let $f, g \in k[x, y]$ satisfy the Jacobi-condition and suppose that their degrees n and m satisfy $n \nmid m$ and $m \nmid n$. Then $\mathrm{GCD}(n, m)$ cannot be 1, a prime number or 4.*

Proof.

Suppose the contrary. Using (3.2) part (4) for the ordinary degree (1, 1) and the lemmas (3.4) and (3.5) one sees that after a linear change of variables, $f^+_{1,1} = \theta x^s y^t$ with $s < t$.

We will deduce that either our Theorem holds or by suitable automorphisms, if necessary, we can arrange that for some positive integers u, v

$$f^+_{1,1} = \theta (x^u y^v)^R \text{ and } g^+_{1,1} = \theta (x^u y^v)^S,$$

where $\mathrm{GCD}(R, S) = \mathrm{GCD}(u, v) = 1$. We will then deduce a final contradiction from this situation to finish the proof.

First we show that the case $s = 0$ can be reduced to the above situation.

If $s = 0$, then, the Newton polygon $N(f)$ has a side starting with $(0, t)$ corresponding to a degree (a, b) with $a > b > 0$. Reasoning as before, we find $b = 1$ and $f^+_{a,b} = \theta((x + \theta y^a)^u y^v)^R$ for some postive integer R. Similarly, $g^+_{a,b} = \theta((x + \theta y^a)^u y^v)^S$ for some positive integer S. Hence $(v + au)R = n$ and $(v + au)S = m$. So $(v + au)$ divides $\mathrm{GCD}(n, m)$, in other words, $(v + au)$ divides a prime number or 4.

There are only two possibilities:

(i) u, v are nonzero, $\mathrm{GCD}(u, v) = 1$ and $\mathrm{GCD}(R, S)$ equals 1 or 2

(ii) $a = u = 2$, $v = 0$ and $\mathrm{GCD}(R, S) = 1$ (this occurs only when $\mathrm{GCD}(n, m) = 4$).

We can clearly apply a suitable automorphism of the form: $x \to x + \theta y^a$, $y \to y$ so that the y-degree reduces, but the hypothesis of the Theorem continues to hold. We can also check that after the automorphism, the new (1, 1) degree forms for f, g become $\theta(x^u y^v)^R$, $\theta(x^u y^v)^S$ respectively. In the case (ii), the new $\mathrm{GCD}(n, m)$ becomes 2 and we can start the proof again with the assurance that we will not run into case (ii) again. Thus we have achieved the promised reduction.

Now we assume $s > 0$ and again reduce to the situation mentioned at the beginning of the proof.

Consider the side of $N(f)$ which starts with (s, t) and is directed towards the x-axis. For weights corresponding to this line, we must have $a > 0$, $a + b > 0$ and $b \leqslant 0$. As before we write $f^+_{a,b} = \theta C^R$ and $g^+_{a,b} = \theta C^S$ where C is (a, b)-homogeneous and *not* a monomial. Further $\theta(C^+_{1,1})^R = f^+_{1,1}$ and $\theta(C^+_{1,1})^S = g^+_{1,1}$. Write $C^+_{1,1} = x^u y^v$, $0 < u < v$.

Since $(u + v)R = n$ and $(u + v)S = m$ and $\mathrm{GCD}(n, m) = 1$, prime or 4, we find again that $\mathrm{GCD}(u, v) = 1$.

Thus, again, we have achieved the promised reduction.

Now we deduce the final contradiction.

Consider the equation $dC \wedge dD = Cdx \wedge dy$ with respect to the $(1,1)$-degree. If $\Delta_{1,1}(D) \leqslant 2$, then according to (3.5) we find that C is a monomial in x and y.

If $\Delta_{1,1}(D) > 2$, then $dC_{1,1}^{+} \wedge dD_{1,1}^{+} = 0$. Hence a power of $D_{1,1}^{+}$ is equal to a power of $C_{1,1}^{+}$. But since $\mathrm{GCD}(u, v) = 1$, we have in fact, $D_{1,1}^{+} = \theta(C_{1,1}^{+})^p$ for some p. Replace now D by $D^* = D - \theta C^p$. Then $dC \wedge dD^* = Cdx \wedge dy$, D^* is (a,b)-homogeneous and $\Delta_{1,1}(D^*) < \Delta_{1,1}(D)$. So finally one finds a $\hat{D}$ with $\Delta_{1,1}(\hat{D}) \leqslant 2$ and $dC \wedge d\hat{D} = Cdx \wedge dy$. This implies again the contradiction "C is a monomial".

4. Some interesting calculations for plane curves

Preamble. We continue to use the previous notation. Explicitly, we fix polynomials f, g in $k[x, y]$, satisfying the Jacobi-condition. Without loss of generality, we may assume that f, g are monic in y. In fact, the best way of describing our setup is to start with the situation as deduced in the beginning of the proof of (3.6), where $f_{1,1}^{+} = (x^s y^t)$ with $s < t$ and make a linear change $x \to y + x$. Even this change is not quite necessary but avoids technical complications.

Set n to be the y-degree of f and m to be the y-degree of g.

Now the first corner of the Newton diagram (figure 3) will be along the line joining (s, t) to the origin.

We may, in this case, view the pair f, g as describing a parametric polynomial plane curve with parameter y over the field $k(x)$. We, therefore get a standard meromorphic Newton-Puiseux expansion of g in $\overline{k(x)}((\eta))$ where η is defined by $f = \eta^{-n}$ and $\overline{k(x)}$ denotes the algebraic closure of $k(x)$. Using the change of variables from x, y to x, η we compute $J_{x,\eta}(f, g)$ or the x-derivative of the η-expansion. Then it is easy to see that we have:

$$g = \eta^{-m} + \cdots + (\theta x + c)\eta^{n-1} + \cdots \in k[x]((\eta)).$$

In other words, the Newton-Puiseux expansion of g, indeed lives over the field $k(x)$. Moreover, all the terms of the expansion up to the displayed term of order $n-1$ are free of x.

Taylor Resultant Theorem 4.1. [A2, P. 153]. *Given any two nonconstant polynomials*

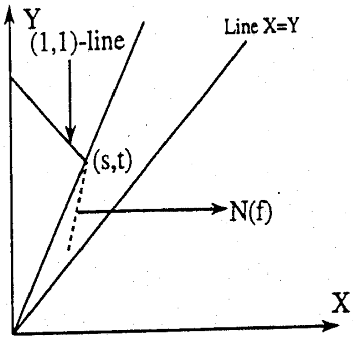

Figure 3.

$p(t)$, $q(t)$ over a field K of characteristic 0, *set*

$$\Theta(t) = \mathrm{Res}_\tau\left(\frac{p(t)-p(\tau)}{t-\tau}, \frac{q(t)-q(\tau)}{t-\tau}\right).$$

Set $A = K[p(t), q(t)]$ and let $\mathfrak{C}$ be the conductor of the ring A in the ring $A' = K[t]$. Then $\mathfrak{C} = \Theta(t)A'$. In particular, thinking of $p(t)$, $q(t)$ as parametrizing the plane curve

$$H(X, Y) = \mathrm{Res}_t(p(t) - X, q(t) - Y)$$

we get:

(1) *The parametrization is faithful if and only if $K(t) = Qt(A)$ or, equivalently, $\Theta(t) \neq 0$.*
(2) *If $\Theta \neq 0$, then the curve is nonsingular* (*at finite distance*) *if and only if $A = A'$ or, in other words, $\Theta = \cancel{\theta} \in K$.*

Proof. Let $p(t)$ have degree n and set $E = \mathrm{Qt}(K[p(t)])$. Then $K(t)$ is a separable algebraic extension of degree n over E. Let $t_1, \ldots, t_n = t$ be a full set of conjugates of t in some fixed algebraic closure of E. Note that we have the factorizations:

$$\frac{p(t)-p(\tau)}{t-\tau} = \prod_{1\ldots(n-1)} (\tau - t_i).$$

By the defining property of resultants, we get that

$$\Theta(t) = \cancel{\theta} \prod_{1\ldots(n-1)} \frac{q(t)-q(t_i)}{t-t_i}$$

where the nonzero constant is needed to handle signs and powers of the leading coefficient if any.

We also have from the definition of $H(X, Y)$ that

$$H(p(t), Y) = (-1)^n \prod_{1\ldots n} (Y - q(t_i))$$

and so

$$H_Y(p(t), q(t)) = (-1)^n \prod_{1\ldots(n-1)} (q(t) - q(t_i)).$$

Also, if we pick a new indeterminate W, then

$$p(W) - p(t) = \prod_{1\ldots n} (W - t_i), \quad \text{so } p'(t) = \prod_{1\ldots(n-1)} (t - t_i).$$

From the above calculations, it follows that

$$\Theta(t) = \cancel{\theta}\frac{H_Y(p(t), q(t))}{p'(t)}.$$

The Dedekind Formula for the conductor and the different exactly says that the fractionary ideal generated by the right hand side of the above equation is the conductor ideal $\mathfrak{C}$. This establishes the main formula.

The parametrization is faithful if and only if the field $K(t)$ coincides with $\mathrm{Qt}(A)$. Now the fields $K(t)$ and $\mathrm{Qt}(A)$ are distinct if and only if

$$[\mathrm{Qt}(A): K(p(t))] < n = [K(t): K(p(t))]$$

or in other words $q(t)$ has less than n conjugates over $K(p(t))$. But this last condition is equivalent to $q(t) = q(t_i)$ for some $i < n$, or, in other words, to $\Theta(t) = 0$.

The rest of the proof follows from the properties of the conductor.

Semigroups. We now present some connections between Newton-Puiseux expansions and certain value-semigroups associated with plane curves.

We begin with a field K whose characteristic does not divide a given integer M_0. We consider a (Laurent) power series

$$\phi(\eta) = \sum \alpha_i \eta^i \in K((\eta))$$

Without considering the genesis of this series, we build a set of associated sequences as follows.

First set $S = \mathrm{supp}(\phi) = \{i | \alpha_i \neq 0\}$. Set M_1 to be the order of ϕ, i.e. $\mathrm{ord}_\eta \phi(\eta) = \inf\{i | i \in S\}$. Also set $d_1 = |M_0|$, $d_2 = \mathrm{GCD}(M_1, d_1)$. Further, set $n_1 = d_1/d_2$.

Assume that we have already inductively constructed $M_1, \ldots, M_i$ as well as associated sequences $q_2, \ldots, q_i, d_1, \ldots, d_{i+1}$ and $n_1, \ldots, n_i$. We extend this construction as follows.

Set $M_{i+1} = \inf\{j > M_i | j \in S$ and j is not divisible by $d_{i+1}\}$. In case $M_{i+1} = \infty$ we set $h = i$ and declare the process finished. Otherwise, set $q_{i+1} = M_{i+1} - M_i$, $d_{i+2} = \mathrm{GCD}(M_{i+1}, d_{i+1})$ and $n_{i+1} = d_{i+1}/d_{i+2}$.

Of course, the inductive definition is then continued.

There are two more associated sequences which can be now defined in terms of the above.

$$s_0 = M_0 \text{ and } s_i = \Sigma_1^i q_j d_j \quad \text{for} \quad 1 \leqslant i \leqslant h.$$

Also, we set

$$r_0 = s_0 \text{ and } r_i = s_i/d_i \quad \text{for} \quad 1 \leqslant i \leqslant h.$$

It is easy to see that $\mathrm{GCD}(r_0, \ldots, r_i) = d_{i+1}$ and hence any integer b is an integral combination of $r_0, \ldots, r_i$ iff d_{i+1} divides b.

Strict generation. A combination $\Sigma a_i r_i$ is said to be a **strict combination** if

(1) $a_0 \geqslant 0$ and
(2) $0 \leqslant a_i < n_i$ for $1 \leqslant i \leqslant h$.

We say that $\{r_i\}$ form a **strict set of generators** if the semigroup generated by $r_0, \ldots, r_i$ consists entirely of their strict combinations.

There are two special cases of interest when we get a strict set of generators $\{r_i\}$ from the given ϕ and M_0. See [A1, Chapter 8].

Algebroid Case: Here $M_0 > 0$, $x = \eta^{M_0}$ and $y = \phi(\eta)$ describe the local Newton-Puiseux expansion for a plane curve at a point in the plane, provided the curve has only one branch at the point. In this case, $M_1 > 0$ and $\phi(\eta) \in K[[\eta]]$. If A denotes the local ring of the point on the curve, then the unique valuation of the curve centered at A coincides with the η-adic order and the semigroup of values of all the (nonzero) elements of A is the semigroup (strictly) generated by $\{r_i\}$.

Meromorphic Case: This is a special case of a meromorphic curve. Here, $M_0 < 0$, $x = \eta^{M_0}$ and $y = \phi(\eta)$ describe the expansion at infinity for a plane curve, provided the curve has only one branch (or place) at infinity. In this case, all r_i are negative. If A denotes the coordinate ring of the curve, then the η-adic order denotes the unique valuation at infinity (nonpositive on A) and the semigroup of orders of (nonzero) elements of A coincides with the semigroup (strictly) generated by $\{r_i\}$.

Thus, any properties deducible from strict generation apply to each of these situations.

Uniqueness of expression. Assume that $\{r_i\}$ is any sequence of nonzero integers with $0 \leqslant i \leqslant h$. Assume that $d_i = GCD(r_0, \ldots, r_{i-1})$ for $1 \leqslant i \leqslant h+1$ and that $n_i = d_i/d_{i+1}$ for $1 \leqslant i \leqslant h$. Let Γ denote the semigroup generated by $\{r_i\}$.

If b is any integer in the group generated by $\{r_i\}$ or equivalently, if d_{h+1} divides b, then we have a unique (partially strict) expression

$$b = \sum_0^h a_i r_i \text{ where } 0 \leqslant a_i < n_i \quad \text{for } 1 \leqslant i \leqslant h.$$

Moreover, if $\{r_i\}$ form a strict set of generators for Γ, then $b \in \Gamma$ iff $a_0 \geqslant 0$.

Proof. This is standard stuff as in [A1, Chapter 1]. The main idea is as follows. The remark about the condition for b to be in the group is obvious from the fact that the GCD of $r_0, \ldots, r_h$ is d_{h+1}. It is also obvious that $n_i r_i$ is divisible by d_i and hence is an integral combination of $r_0, \ldots, r_{i-1}$. Thus any given integral combination of b in terms of $r_0, \ldots, r_h$ can easily be transformed to the desired (partially strict) form. Uniqueness is deduced from the GCD properties by induction on the last r_i present in the expression.

The last assertion follows easily from the definition of strict generation.

A Symmetry Property. Assume that $\{r_i\}$ form a strict set of generators for a semigroup Γ. Set

$$\sigma = -r_0 + \sum_1^h (n_i - 1) r_i$$

Given any two integers u, v divisible by d_{h+1}, such that $u + v = \sigma$, one and only one of u, v belongs to Γ.

Proof. Clearly u, v belong to the group generated by $\{r_i\}$. Write the unique (partially strict) expression for u as

$$u = \sum_0^h u_i r_i \text{ where } 0 \leqslant u_i < n_i \quad \text{for } \quad 1 \leqslant i \leqslant h.$$

Then obviously the unique (partially strict) expression for v is given by

$$v = (-1 - u_0)r_0 + \sum_1^h (n_i - u_i - 1)r_i.$$

Thus, from the above criterion we have:

$$u \in \Gamma \text{ iff } u_0 \geqslant 0 \text{ and } v \in \Gamma \text{ iff } (-1 - u_0) \geqslant 0.$$

Clearly among the two integers u_0 and $-1 - u_0$ exactly one can be nonnegative, hence the result.

Conductor of a semigroup. If a semigroup consists of nonnegative integers only. **The conductor of the semigroup is defined to be** the least nonnegative integer such that it and all bigger integers are in the semigroup.

If a semigroup consists of nonpositive integers only, **the conductor of the semigroup is defined to be** the largest nonpositive integer such that it and all smaller integers are in the semigroup.

A Formula For The Conductor. Assume that $\{r_i\}$ form a strict set of generators for a semigroup Γ. To simplify notation, further assume that $d_{h+1} = 1$, i.e. that the group generated by $\{r_i\}$ consists of all integers.
Set as before:

$$\sigma = -r_0 + \sum_1^h (n_i - 1)r_i$$

If Γ consists entirely of nonnegative integers, then the number $c = \sigma + 1$ has the properties:

(1) *c is an even integer.*
(2) *Every integer bigger than or equal to c is in Γ and c is the smallest integer with this property. In other words, c is the conductor of Γ.*
(3) *There are exactly $c/2$ positive integers not in Γ.*

If Γ consists entirely of nonpositive integers, then the number $c = \sigma - 1$ has the properties:

(1) *c is an even integer.*
(2) *Every integer smaller than or equal to c is in Γ and c is the largest integer with this property. In other words c is the conductor of Γ.*
(3) *There are exactly $c/2$ negative integers not in Γ.*

Proof. Suppose first that Γ consists of nonnegative integers only. By assumption $-b \notin \Gamma$ if $b > 0$ and hence by the symmetry property, $\sigma - (-b) = \sigma + b \in \Gamma$.

Again by the symmetry property, $0 \in \Gamma$ and hence $\sigma - 0 = \sigma \notin \Gamma$. Thus, no number smaller than c has the desired property.

Also, σ must be odd, since if $\sigma = 2b$, then $b = \sigma - b$ will be both in Γ as well as outside Γ by the symmetry property. Thus all the c integers from 0 to $\sigma = c - 1$ can be paired off as u, $\sigma - u$ where exactly one of each pair is in Γ. This proves the remaining claim.

The case of nonpositive Γ is similar.

Length of the integral closure: Algebroid case. Assume that we have the Algebroid Case described above and A is the local ring of a point on the curve. Let A' be the integral closure of A in its quotient field (the function field of the curve) and let $\mathfrak{C}$ be the conductor of A' over A. Let Γ be the semigroup of values of (nonzero) elements of A in the unique valuation centered at the point and let c be the conductor of Γ as described above. Then the length of A'/A as an A-module is exactly $c/2$. By the well known Gorenstein property of such local rings, the length also coincides with the length of $A/\mathscr{C}$ as an A-module. In particular, the length is described by the formula

$$\frac{1-r_0+\Sigma_1^h(n_i-1)r_i}{2}.$$

This verifies the formula on page 169 of [A2]

Proof. To simplify matters, we assume that the field K is algebraically closed. It is clear that if we take the values of various nonzero elements of A' we get the semigroup of all nonnegative integers. List the $c/2$ nonnegative integers which do not belong to Γ as $u_1,\ldots,u_{c/2}$ and pick a sequence of $c/2$ elements of A' with values $u_1,\ldots,u_{c/2}$ respectively. It is easy to see that they form a basis of A'/A over K and in fact determine the length of A'/A.

The proof for the general K should be carried out by the already mentioned Dedekind formula or by the technique of extending the ground field. We omit these technical details.

Length of the integral closure: Meromorphic case. Assume that we have the Meromorphic case described above and $A=K[x,y]$ is the affine coordinate ring of a curve having one place at infinity. Let A' be the integral closure of A in its quotient field (the function field of the curve) and let $\mathscr{C}$ be the conductor of A' over A. Let Γ be the semigroup of values of (nonzero) elements of A in the unique valuation at infinity, i.e., the unique valuation of the function field not containing A. Let c be the conductor of Γ as described above. Then the length of A'/A as on A-module is exactly $c/2$. By the well-known Gorenstein property of such rings, the length also coincides with the length of $A/\mathscr{C}$ as an A-module. In particular, the length is described by the formula

$$\frac{1-r_0+\Sigma_1^h(n_i-1)r_i}{2}.$$

Proof. The proof is formally the same as in the Algebroid case.

5. The Newton Puiseux expansions for different weights

Preamble. We wish to generalize the Newton-Puiseux expansions discussed in the previous section by making the expansion which will respect a certain weight.

We assume that f, g are polynomials in x, y of degree at least 2 satisfying the Jacobi-condition.

Fix rational weights a, b and set $w=(a,b)$. Assume that $\Delta_w(f)>0$ and $\Delta_w(g)>0$. Moreover, we are generally interested in weights corresponding to Newton lines only, although most of the calculations go through for the general case. Then clearly, at least one of a, b is positive and since weights proportional by positive numbers yield the same degree forms, we may assume a, b to be coprime integers. By interchanging x, y if necessary, we are reduced to considering the cases $a = 0$, $b = 1$ or $a > 0$ with $\mathrm{GCD}(a, b) = 1$. The case when $a = 0$ is the case of considering the y-degree as the weight and the corresponding expansion is as discussed in the last section. We therefore assume that $a > 0$. We also make the following assumption, which is necessary for the validity of some of the technical results.

Special assumption: Assume that the weights a, b are such that $a + b > 0$ or that the weight of the monomial xy is positive.

Generally, this assumption is valid for weights along Newton lines starting above the line $X = Y$, since in the contrary case, the resulting Newton diagram will not cross the $X = Y$ line and the resulting polynomials f, g will be divisible by y and the Jacobi-condition fails. In our current set up, this will be true for the sequence of Newton lines starting from the end of the $(1, 1)$-line until the first line which crosses the $X = Y$ line.

We now set $y = zt^b$, $x = t^a$. Note that the change of variables from (x, y) to (z, t) causes the Jacobian to be multiplied by $\theta t^{(a+b-1)}$.

We now think of f, g as elements in the field $k(z)((\tau))$, the field of meromorphic power series in $\tau = t^{-1}$ over the field of rational functions in z. Note that the weight of any of the original polynomials can simply be read off as the highest power of t occurring, or, equivalently, the negative of the τ-order.

The change of variables to (z, τ) causes the original Jacobian to be multiplied by $\theta t^{(a+b-1)}\tau^{-2} = \theta\tau^{-(a+b+1)}$.

In particular, we can write $f_w^+ = \theta t^{\Delta_w(f)} P(z)$ where $P(z)$ is some polynomial. Consequently,

$$f = \theta P(z) t^{\Delta_w(f)} P^*(z, \tau) = \theta P(z) \tau^{-\Delta_w(f)} P^*(z, \tau)$$

where $P^*(z, \tau)$ is a polynomial in τ with coefficients in $k(z)$, thought of as an element of $k(z)[[\tau]]$ and in fact, it is a unit in the power series ring $k(z)[[\tau]]$. By taking its $-\Delta_w(f)$-th root, we can write

$$f = P(z)\eta^{-\Delta_w(f)}$$

where $\eta = \tau(P^*(z, \tau))^{-1/(\Delta_w(f))}$ is a new generator for the power series ring $k(z)((\tau))$ over $k(z)$.

Thus, the transformation from (x, y) to the new variables (z, η) multiplies the Jacobian by a unit times $\eta^{-(a+b+1)}$. Moreover, the w-degree can be computed as the negative of the η-order.

Given any power series

$$G = \sum a_i(z)\eta^i$$

we can compute its Jacobian with f to be

$$J_{z,\eta}(f, G) = \sum_i (iP'(z)a_i + \Delta_w(f)P(z)a_i')\eta^{i-1-\Delta_w(f)}.$$

In particular, if G is obtained from a polynomial in (x, y) by the change of variables explained above, we can check that f, G are w-related if and only if the term

$$-P'(z)Q(z)\Delta_w(G) + \Delta_w(f)P(z)Q'(z) \qquad (*)$$

equals 0, where the leading term of G is written as $Q(z)\eta^{-\Delta_w(G)}$.

Moreover, in the unrelated case, the expression $(*)$ coincides with a nonzero constant times the leading coefficient of the transformation of the usual jacobian of f, G.

If f, G are w-related, then solving the differential equation obtained by equating $(*)$ to 0, we deduce that

$$P^{\Delta_w(G)} = \theta Q^{\Delta_w(f)}.$$

Let us fix a polynomial $H = H(z)$ such that we can write for some positive integers v, δ'_0:

$$f = (H\eta^{-v})^{\delta'_0}$$

and such that this kind of expression does not hold for any polynomial of degree smaller than that of $H(z)$. Let us denote the expression $H\eta^{-v}$ by ζ.

Then the above relatedness condition gets replaced by "$Q(z)\eta^{\Delta_w(G)}$ is an integral power of ζ".

Thus, any power series G as above can be split in three parts:

$$G = \text{terms involving powers of } \zeta + a\eta^s + \text{higher terms}$$

where $a\eta^s$ is the first term unrelated with f. In particular, we have

$$-\Delta_w(f) + s = 1 + \operatorname{ord}_\eta(J_{z,\eta}(f, G))$$

Newton-Puiseux expansions for a given weight. 5.1 Applying the above substitutions to g, we see that it develops into a power series:

$$g = \sum a_i(z)\eta^i$$

We wish to set up the usual characteristic sequence associated with it, as commonly done in the expansion techniques.

Begin by setting M_1 to be the η-order of g and set $d_1 = \Delta_w(f)$ or the negative of the η-order of f. Set $d_2 = \mathrm{GCD}(M_1, d_1)$ and $n_1 = d_1/d_2$.

Assume that we have inductively defined $M_1, \ldots, M_i$ along with $d_1, \ldots, d_{i+1}$, $q_2, \ldots, q_i$ and $n_1, \ldots, n_i$. Then we set M_{i+1} to be the first exponent of η in the support of the expansion of g, which is not divisible by d_{i+1}. Set $d_{i+2} = \mathrm{GCD}(M_{i+1}, d_{i+1})$, $n_{i+1} = d_{i+1}/d_{i+2}$ and $q_{i+1} = M_{i+1} - M_i$. In case there is no such exponent, we declare the process finished and set $h = i$. We say that we have h characteristic pairs (M_i, d_i).

Strictly speaking, this whole construction depends on the choice of w, but we have chosen not to clutter up the notation by tacking on an extra subscript.

We can visually display the characteristic sequence by writing:

$$g = c_1\eta^{M_1} + \cdots + c_2\eta^{M_2} + \cdots + c_h\eta^{M_h} + \cdots$$

Associated with the above sequence is the sequence of "pseudoapproximate" roots, which are certain polynomials in f, g, which we now introduce.

Set $g_0 = f$ and $g_1 = g$. Define $\delta_0 = \Delta_w(f)$, $\delta_1 = -M_1 = \Delta_w(g)$. Also, set $\mu_i = \delta_0 + \delta_1 - \Sigma_2^i q_j$ for $i = 1, \ldots h$. For technical reasons, set $\mu_0 = \infty$. Note that μ_i is also equal to $\delta_0 - M_i$. Also set $\delta_i = n_{i-1}\delta_{i-1} - q_i$. It is a standard calculation to check the identity (for $1 \leqslant i \leqslant h$):

$$\mu_i = \delta_0 + \delta_i - \sum_1^{i-1} (n_j - 1)\delta_j.$$

Note that by the known transformations, the Jacobian of f, g with respect to z, η has η-order $-a-b-1$ and so the first unrelated term in the expansion of $g = g_1$ has η order $-a-b+\delta_0$. Thus, we can write

$$g = g_1 = \eta^{-\delta_1}(c_1 + \cdots + c\eta^{\delta_0 + \delta_1 - a - b} + \cdots)$$

where c is a nonzero polynomial in z and all earlier terms are related to f. In particular,

$$g_1 = \eta^{-\delta_1}(c_1 + \cdots + c\eta^{\mu_1 - a - b} + \cdots).$$

Now suppose that we have inductively built $g_0, \ldots, g_i$ such that for $1 \leqslant j \leqslant i$:

(1) The η-order of g_j is $-\delta_j$, so that its w-weight is precisely δ_j.
(2) The first term in the expansion of g_j which is unrelated to f has η-order $\mu_j - a - b - \delta_j$.
(3) g_j is related to f.

Then we try to build g_{i+1} as follows.

By a **standard monomial** in $g_0, \ldots, g_i$ we mean a monomial of the form $\Pi_j g_j^{\alpha_j}$ where $0 \leqslant \alpha_j < n_j$ for $1 \leqslant j \leqslant i$, while $0 \leqslant \alpha_0$. We will conveniently shorten the notation to write g^α for $\Pi_j g_j^{\alpha_j}$.

For any $1 \leqslant j \leqslant i$ the η-order of g_j is $-\delta_j$, while the η-order of the first term in g_j unrelated to f is $\mu_j - \delta_j - a - b$. Thus, g_j is related to f if and only if this term is not the leading term of g_j, i.e. $-\delta_j < \mu_j - \delta_j - a - b$, or equivalently, $a + b < \mu_j$. We know this for $1 \leqslant j \leqslant i$ already.

We begin by a trial value $v = g_i^{n_i}$.

We keep on modifying v until it becomes g_{i+1}.

(1) If the w-leading form of v cannot be expressed as the w-leading form of ag^α for some $a \in k$ and some standard monomial g^α in $g_0, \ldots, g_i$, then we declare $g_{i+1} = v$ and stop this modification.
(2) If there is a standard monomial g^α in $g_0, \ldots, g_i$ such that v has the same w-leading form as cg^α for some $c \in k$, then we modify v to $v - cg^\alpha$. Note that the w-degree of v decreases in this process and so the modification has to eventually stop.

We need to verify that g_{i+1} has the correct order and indeed that it is the next "pseudoapproximate root" as in the expansion techniques.

For $f = g_0$, we might consider the difference between the leading term and the first term unrelated to f to be ∞, and hence equals $\mu_0 - a - b = \infty$.

If $1 \leqslant j \leqslant i$, then for g_j, the difference between the order of the leading term and the first unrelated term is exactly $\mu_j - a - b$. It is clear that the difference is the same for any power of g_j. Moreover, since the expression $\mu_j - a - b$ is a decreasing function of j, the difference for any standard monomial g^α is easily seen to be the infimum of $\mu_j - a - b$ such that $\alpha_j \neq 0$.

Now, in the buildup of g_{i+1} described above, the first step of cancelling a monomial causes this "gap of the first unrelated term" to decrease from $\mu_i - a - b$. Any subsequent modifications only increase the η-order without affecting the first unrelated term, since the modifying terms now all have bigger gaps.

Thus the final gap for g_{i+1} is $\mu_i - a - b - q$ for some q. We wish to show that $q = q_{i+1}$.

By the standard theory, $q \leqslant q_{i+1}$ since the modification cannot be pushed beyond that even with coefficients in $k(z)$. If $q < q_{i+1}$, then the η order of our g_{i+1} is in the semigroup generated by the η-orders of $g_0, \ldots, g_i$. The fact that our modification process stopped means that the multiplier coefficient needed is not in k. It is then evident that the highest z-degree term in g_{i+1} cannot be related to that of f. From the remarks in (3.5) and our special assumptions, it follows that the leading form of f must be a monomial. Since we have also assumed that we have weights whose degree form is a line, we are done.

Clearly, this process then continues, until we reach h or we reach an unrelated g_i. If $i = h$ and g_h is still related to f, then we can continue the modification until we reach an unrelated g_{h+1}. However, it cannot correspond to any characteristic term (since we have gone past all such terms) and we can deduce that the leading form of f must have been a monomial. We summarize this in:

Pseudoapproximate roots along a Newton Line 5.2: *Assume that we have weight $w = (a, b)$ with $a > 0$ and $a + b > 0$ such that the degree form of f is not a monomial. Then, in the above notation, there is a sequence of pseudoapproximate roots $g_0, \ldots, g_i$ for some $i \leqslant h$ such that each g_j is a polynomial in f, g over k. Moreover, g_i is w-unrelated to f and we have $\mu_i = a + b$.*

Definition. Let w be any rational weight such that either $w = (0, 1)$ or $w = (a, b)$ with $a > 0$. We will say that f has i pseudoapproximate roots along w, if we can construct the sequence $g_0, \ldots, g_i$ as described above. Note that the number i can be smaller than the usual number h given by the expansion techniques and can even be $h + 1$ when the degree form relative to w is a monomial.

Now we consider the variation of the number of pseudoapproximate roots corresponding to two consecutive Newton Lines. Let $w_1 = (a_1, b_1)$ and $w_2 = (a_2, b_2)$ be the consecutive weights, so that $b_2/a_2 < b_1/a_1$. Also assume that the common corner for the Newton diagram of f is a point (s_1, s_2) above the line $X = Y$ (i.e. $s_1 < s_2$). The point corresponds to the lowest y-degree term for $f^+_{w_1}$ and the highest y-degree term for $f^+_{w_2}$. Let the sequence $g_1, \ldots, g_i$ be constructed for the weight w_1, as shown above.

Then, $\Delta_{w_1}(f) = a_1 s_1 + b_1 s_2$ and $\Delta_{w_2}(f) = a_2 s_1 + b_2 s_2$. Set

$$\lambda = \frac{a_2 s_1 + b_2 s_2}{a_1 s_1 + b_1 s_2}$$

and

$$\lambda^* = \frac{a_2 + b_2}{a_1 + b_1}.$$

Recall that g_j is w_1-related to f if and only if $\mu_j/(a_1 + b_1) > 1$ and, in fact, we have:

$$\frac{\mu_1}{a_1 + b_1} > \frac{\mu_2}{a_1 + b_1} > \cdots \frac{\mu_i}{a_1 + b_1} = 1.$$

From the alternate formula for μ_j, it is clear that μ_j corresponding to w_2 is $\lambda\mu_j$. To see if g_j is also w_2-related to f, we need to check if $\lambda\mu_j/(a_2+b_2)>1$ or

$$\frac{\mu_j}{a_1+b_1}>\lambda^*/\lambda.$$

With our hypothesis, it is easy to check that $\lambda^*/\lambda>1$ and so the condition for relatedness gets tighter as we move from w_1 to w_2. Indeed, if i^* is the last pseudoapproximate root for the weight w_2, then we must have,

$$\frac{\lambda\mu_1}{a_2+b_2}>\frac{\lambda\mu_2}{a_2+b_2}>\cdots>\frac{\lambda\mu_{i^*}}{a_2+b_2}=1$$

or, in other words:

$$\frac{\lambda}{\lambda^*}\frac{\mu_1}{a_1+b_1}>\frac{\lambda}{\lambda^*}\frac{\mu_2}{a_1+b_1}>\cdots>\frac{\lambda}{\lambda^*}\frac{\mu_{i^*}}{a_1+b_1}=1.$$

Thus, we have $\mu_{i^*}=(a_1+b_1)\lambda^*/\lambda$. Naturally, $i^*<i$. Consider the possibility that $i^*<i-1$. We choose a weight $w_3=(a_3,b_3)$ such that

$$\frac{a_3s_1+b_3s_2}{a_1s_1+b_1s_2}\frac{\mu_{i-1}}{a_3+b_3}=1$$

or, in other words,

$$\frac{\mu_{i-1}}{a_1+b_1}=\frac{a_1s_1+b_1s_2}{a_3s_1+b_3s_2}\frac{a_3+b_3}{a_1+b_1}.$$

Using (the consequence of the special assumption) $a_1+b_1>0$, it is easy to verify that the expression $\psi(a,b)=\dfrac{a_1s_1+b_1s_2}{as_1+bs_2}\dfrac{a+b}{a_1+b_1}$ is a monotonic decreasing function of the ratio b/a and consequently, the weight line corresponding to w_3 lies between the lines for w_1, w_2. Since, the w_1, w_2 lines were assumed consecutive, this implies that the degree form of f for the weight w_3 must be the same monomial $\cancel{\theta}x^{s_1}y^{s_2}$. Also, clearly, for w_3 we have exactly one less pseudoapproximate root than for w_1.

What we have obtained is the result:

Variation along Newton lines 5.3. *Assume that two consecutive Newton Lines of f share a common vertex and let $x^{s_1}y^{s_2}$ be the monomial pointing towards the vertex, chosen as described above. Further assume that:*

(1) *The two weights $w_1=(a_1,b_1)$ and $w_2=(a_2,b_2)$ are such that a_1,a_2 are positive.*
(2) $b_2/a_2<b_1/a_1$.
(3) $s_1<s_2$.

Then f has at least one less pseudoapproximate root along w_2 than along w_1. Moreover, there exists an intermediate weight $w_3=(a_3,b_3)$ with

$$\frac{b_2}{a_2}\leqslant\frac{b_3}{a_3}\leqslant\frac{b_1}{a_1}$$

such that we have exactly one less pseudoapproximate root along w_3 than along w_1.

Note. Note that we are discussing the concept of pseudoapproximate roots corresponding to a given weight and we quit developing the pseudoapproximate roots as soon as we reach an unrelated root corresponding to the weight. Thus, as we start with the starting weight $(0, 1)$ and consider various values (a, b) with the ratio b/a steadily decreasing, we march along the Newton diagram. What we have shown here is that we keep on getting fewer and fewer pseudoapproximate roots related to f until we cross the line $X = Y$. Afterwards, generally the function $\psi(a, b)$ turns increasing and the number of roots tends to increase. In fact, the sign of the derivative of the function $\psi(a, b)$ is determined by the sing of $(s_1 - s_2)/(a + b)$ and after crossing the line $X = Y$ the sign turns positive, unless the special assumption also fails and $a + b$ turns negative.

It is possible to make an independent argument to show that the highest y-degree term of the unrelated pseudoapproximate root must become unrelated to that of f below the $X = Y$ line. Thus, in view of the remarks in (3.5), the special assumption must fail after crossing the $X = Y$ line. The analysis of Newton Lines in this region after the $X = Y$ line is not relevant for the remaining part and hence no further discussion is provided here.

Lemma 5.4. **Lower bound on the number of pseudoapproximate roots.** *Let a weight $w = (a, b)$ corresponding to a Newton Line of f satisfy $0 < a$ and $b < a$. Then f, g must be w-related. In other words, the number of pseudoapproximate roots for the weight w is at least* 2.

Proof. This is only a special case of the proof in the beginning of (3.2) where the result is proved when either a or b are positive.

Case of two characteristic terms 5.5. *Assume that f and g have at most two characteristic terms for the* $(1, 1)$ *weight and satisfy the rest of the conditions described in the preamble. Then the Jacobian theorem holds for f, g.*

Proof. In view of the various results from the earlier sections, it is clear that we are reduced to considering the case where the first corners after the $(1, 1)$ weight line for f, g are of the form (pt_1, pt_2), (qt_1, qt_2) respectively, where $0 < t_1 < t_2$ and p, q are coprime. Let $w = (a, b)$ be the weight for the next line. Clearly, we have $0 < a$ and $b < a$.

By Lemma 5.3, we can have at most one pseudoapproximate root along w. This means f, g must be w-unrelated.

On the other hand by Lemma 5.4, we get that f, g must be w-related. This is a contradiction.

References

[A1] Abhyankar S S, Expansion Techniques in Algebraic Geometry, Tata Institute of Fundamental Research, Bombay, 1977

[A2] Abhyankar S S, Algebraic geometry for Scientists and Engineers, *Mathematical Surveys* (35) AMS., 1990

[Ca] Campbell L A, A Condition For a Polynomial To Be Invertible, *Math. Ann.* **205** (1973), 243–248

[Ch] Chevalley C, Algebraic Functions of One variable, *Mathematical Surveys* (6) AMS, 1951

[Ke] Keller O, Ganze Cremona Transformationen, *Mh. Math. Phys.* **47** (1993), 299–306

[Na] Nagata M, *Local Rings*, John Wiley, New York, 1962

[Ra] Raynaud M, *Anneaux Locaux Henseliens*, Springer Verlag Lecture Notes in Mathematics (169), 1939