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NOTE

MINIMUM WEIGHT WORDS OF BINARY CODES ASSOCIATED WITH FINITE PROJECTIVE GEOMETRIES

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Let $PG(n, s)$, $s = 2^\alpha$ and $n \geqslant 2$, denote the Desarguesian projective space of projective dimension n over the Galois field F_s. The set of its subsets with set theoretic symmetric difference as addition is a vector space over F_2. For $1 \leqslant t \leqslant n-1$, let $C_t(n, s)$ denote its subspace generated by the t-flats of $PG(n, s)$ and for $w \subseteq PG(n, s)$, let $|w|$ denote the cardinality (or weight) of w. Our object in this note is to present a purely geometric proof of the following theorem proved independently by Smith [5] and Delsarte et al. [2].

Theorem. *For $s = 2^\alpha$, $n > 1$ and $0 < t < n$, the words of $C_t(n, s)$ of least non-zero weight are precisely the t-flats of $PG(n, s)$.*

Some crucial parts of the proof are contained in the following lemmas.

Lemma 1.

(a) *If f_1 and f_2 are t-flats in $PG(n, s)$ and $x_0 \in f_1 \cap f_2$, then $f_1 + f_2$ can be expressed in $C_t(n, s)$ as a sum of an even number of t-flats, each excluding x_0.*

(b) *If $w \in C_t(n, s)$ and $x \in PG(n, s) \setminus w$, then w is a sum of some t-flats, each excluding x.*

Proof. (a) We prove this by induction on t. First consider the case when $t = 1$. Restricting our attention to the plane containing f_1 and f_2, we may assume that $n = 2$. Since the Desarguesian projective plane of order s admits ovals of size $s+1$ [3, p. 147] and its automorphism group is doubly transitive on its lines, it possesses an oval θ of size $s+1$, containing x_0 such that f_1 is a tangent to θ at x_0. Let $f_2 \cap \theta = \{x_0, x\}$ and let θ' denote the s-arc $\hat{\theta} \setminus \{x\}$. Since any s-arc in $PG(2, s)$ has 2 tangents at each of its points and the sum in $C_1(2, s)$ of all its tangents is

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zero, the sum f_1+f_2 of the tangents to θ' at x_0 is equal to the sum of the tangents to θ' at its points $\neq x_0$. Therefore (a) holds in this case.

Now we consider the case when $t \geqslant 2$ and assume that (a) holds for smaller values of t. First consider the case when $f_1 \cap f_2$ is an l-flat, $l>0$. Let H be an $(l-1)$-flat in $f_1 \cap f_2$ with $x_0 \notin H$ and let $\bar{R}$ denote the image of an r-flat R of $PG(n, s)$ containing H in the quotient space $PG(n, s)/H$ (see [3, p. 25]). Now $\bar{f}_1 \cap \bar{f}_2$ is a point and the induction hypothesis applied to $PG(n, s)/H \simeq PG(n-l, s)$ implies the existence of an even number of t-flats $\{P_\alpha : \alpha \in I\}$ in $PG(n, s)$ such that $x_0 \notin P_\alpha$ and $H \subset P_\alpha$ for each $\alpha \in I$, and $\bar{f}_1+\bar{f}_2 = \sum\{\bar{P}_\alpha : \alpha \in I\}$. Since $x \in \sum\{P_\alpha : \alpha \in I\}$ if and only if x lies in an odd number of P_α's and so $x \notin H$, it follows that $f_1+f_2 = \sum\{P_\alpha : \alpha \in I\}$.

Now we consider the case when $f_1 \cap f_2 = \{x_0\}$. Let $x_0 \neq x_i \in f_i$ $(i=1,2)$ and f_0 be a t-flat of $PG(n, s)$ containing $\{x_0, x_1, x_2\}$. Since $f_i \cap f_0$ is an l-flat containing x_0 for some $l>0$ and $f_1+f_2 = (f_1+f_0)+(f_0+f_2)$, the conclusion of the preceeding paragraph applied to f_i+f_0 implies (a) in this case.

(b) Since any expression of w as a sum of t-flats contains an even number of t-flats containing x, (b) follows from (a). $\square$

Lemma 2. *In $C_{n-1}(n, s)$,*

(a) *the weight of a sum of an odd (respectively even) number of hyperplanes is odd (respectively even), and*

(b) *any line of $PG(n, s)$ meets a word of $C_{n-1}(n, s)$ of odd (respectively even) weight in an odd (respectively even) number of points.*

Proof. If $w_1, w_2 \in C_{n-1}(n, s)$, H a hyperplane, l a line of $PG(N, s)$ and $w_1 = w_2 + H$, then $|w_1| = |w_2| + |H| - 2|H \cap w_2|$ and $|l \cap w_1| = |l \cap w_2| + |l \cap H| - 2|l \cap H \cap w_2|$. Since $|H|$ and $|l \cap H|$ are odd, $|w_1|$ (respectively $|l \cap w_1|$) is odd if and only if $|w_2|$ (respectively $|l \cap w_2|$) is even. This together with an easy induction on the number of summands yields both (a) and (b). $\square$

Proof of the theorem. The proof is by induction on n $(>t)$ for each fixed value of t. Let $0 \neq w \in C_t(n, s)$. Clearly, we can assume that $w \neq PG(n, s)$.

First consider the case when $n = t+1$. If $|w|$ is even and $x \in w$, then, by Lemma 2(b), each line incident with x meets w again and so $|w| \geqslant 1 + (s^n - 1)/(s-1)$. If $|w|$ is odd and $x \in PG(n, s) \setminus w$, then, by Lemma 2(b), $|l \cap w| \geqslant 1$ for each line incident with x and so $|w| \geqslant (s^n-1)/(s-1)$, with equality if and only if $|l \cap w| = 1$ for each l not contained in w. This implies that w contains the line joining any two of its points and so is a flat. Now, since $|w| = (s^n-1)/(s-1)$, w is necessarily a hyperplane and the theorem follows in this case.

Next, let $n > t+1$ and assume that the theorem holds for smaller values of n. If every line l with $l \cap w \neq \emptyset$ meets w in at least two points, then the argument in the preceeding paragraph implies that $w = 0$ or $|w| > (s^n-1)/(s-1) > (s^{t+1}-1)/(s-1)$. So, we may assume that there is a line l with $|l \cap w| = 1$. (We only wish to ensure that x is a point outside w which is incident with at least one line l with $|l \cap w|$ odd. If $|w|$ is odd, then we can choose x to be an arbitrary point outside w.) Fix a point $x \in l \setminus w$. By Lemma 1(b), there exist t-flats $\{f_i\colon i \in I\}$ with $x \notin f_i$ for each i and $w = \sum \{f_i\colon i \in I\}$. Let H be a hyperplane with $x \notin H$ and let $\pi\colon PG(n, s) \setminus x \to H$ be the projection onto H with center at x. Then, $\pi(f_i)$ is a t-flat in H and $\sum \{\pi(f_i)\colon i \in I\}$ is in the code $C_t(n-1, s)$ associated with H. Now, each f_i meets l in at most one point, otherwise $l \subset f_i$ and $x \in f_i$, a contradiction. Therefore $|\{i \in I : l \cap f_i \neq \varphi\}| = \sum_{y \in l} |\{i \in I : y \in f_i\}|$. Since a point $y \in w$ if and only if $|\{f_i : y \in f_i\}|$ is odd and since $|l \cap w|$ is odd, it follows that $\sum \{\pi(f_i)\colon i \in I\} \neq 0$. Now

$$|w| \geqslant |\pi(w)| = |\pi(\sum \{f_i\colon i \in I\})| \geqslant |\sum \{\pi(f_i)\colon i \in I\}| \geqslant (s^{t+1} - 1)/(s-1).$$

Here, the first inequality is trivial, the second holds because $\pi(\sum f_i) \supseteq \sum \pi(f_i)$ and the third is a consequence of the induction hypothesis.

If $|w| = (s^{t+1} - 1)/(s-1)$, then, by the induction hypothesis, $\pi(w) = \sum \{\pi(f_i)\colon i \in I\}$ is a t-flat and so the restriction of π to w is a bijection for each choice of x and the hyperplane H with $x \notin w \cup H$. This implies that w is a flat because if a line m containing distinct points x_1 and x_2 of w is not contained in w and $x \in m \setminus w$, then $\pi(x_1) = \pi(x_2)$ for the projection $\pi\colon PG(n, s) \setminus \{x\} \to H$ with center at x, a contradiction. Now, since $|w| = (s^{t+1} - 1)/(s-1)$, w is necessarily a t-flat. This completes the proof of the theorem. $\square$

Remark. Though the theorem holds in greater generality, our methods do not seem to extend to the case of odd characteristic. However, our proof is elementary and geometric whereas the original proofs are algebraic. It is not true in general that the words of $C_1(n, s)$ of weight $(s^{t+1} - 1)/(s-1)$ are necessarily t-flats as, for example, the weight enumerator

$$A(Z) = 2^{-n}(1+Z)^{2^n-1} + 2^{-n}(2^n - 1)(1+Z)^{(2^{n-1}-1)}(1+Z)^{2^{n-1}}$$

[1, p. 48] of $C_1(n, 2)$ the binary $(2^n - 1, 2^n - 1 - n, 2)$-Hamming code [1, corolllary to Theorem 7.2, p. 185] shows. Finally, this theorem may be useful in the study of the minimum weights of the codes associated with the incidence systems embedded in projective geometries, for example see [4].

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