Combinatorial triangulations of homology spheres

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Abstract

Let M be an n-vertex combinatorial triangulation of a $\mathbb{Z}_2$-homology d-sphere. In this paper we prove that if $n \leq d+8$ then M must be a combinatorial sphere. Further, if $n = d+9$ and M is not a combinatorial sphere then M can not admit any proper bistellar move. Existence of a 12-vertex triangulation of the lens space $L(3,1)$ shows that the first result is sharp in dimension three.

In the course of the proof we also show that any $\mathbb{Z}_2$-acyclic simplicial complex on ≤ 7 vertices is necessarily collapsible. This result is best possible since there exist 8-vertex triangulations of the Dunce Hat which are not collapsible.

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1 Introduction and results

All the simplicial complexes considered in this paper are finite. We say that a simplicial complex K *triangulates* a topological space X (or K is a *triangulation* of X) if X is homeomorphic to the geometric carrier $|K|$ of K.

The vertex-set of a simplicial complex K is denoted by $V(K)$. If K, L are two simplicial complexes, then a *simplicial isomorphism* from K to L is a bijection $\pi : V(K) \to V(L)$ such that for $\sigma \subseteq V(K)$, σ is a face of K if and only if $\pi(\sigma)$ is a face of L. The complexes K, L are called (simplicially) *isomorphic* when such an isomorphism exists. We identify two simplicial complexes if they are isomorphic.

A simplicial complex K is called *pure* if all the maximal faces of K have the same dimension. A maximal face in a pure simplicial complex is also called a *facet*.

If σ is a face of a simplicial complex K then the *link* of σ in K, denoted by $\mathrm{Lk}_K(\sigma)$ (or simply by $\mathrm{Lk}(\sigma)$), is by definition the simplicial complex whose faces are the faces τ of K such that τ is disjoint from σ and $\sigma \cup \tau$ is a face of K.

A subcomplex L of a simplicial complex K is called an *induced* (or *full*) subcomplex of K if $\sigma \in K$ and $\sigma \subseteq V(L)$ imply $\sigma \in L$. The induced subcomplex of K on the vertex set U is denoted by $K[U]$.

For a commutative ring R, a simplicial complex K is called R-acyclic if $|K|$ is R-acyclic, i.e., $\widetilde{H}_q(|K|, R) = 0$ for all $q \geq 0$ (where $\widetilde{H}^q(|K|, R)$ denotes the reduced homology).

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By a *subdivision* of a simplicial complex K we mean a simplicial complex K' together with a homeomorphism from $|K'|$ onto $|K|$ which is facewise linear. Two simplicial complexes K and L are called *combinatorially equivalent* (denoted by $K \approx L$) if they have isomorphic subdivisions. So, $K \approx L$ if and only if $|K|$ and $|L|$ are piecewise-linear (pl) homeomorphic (see [11]).

For a set U with $d+1$ elements, let K be the simplicial complex whose faces are all the non-empty subsets of U. Then K triangulates the d-dimensional closed unit ball. This complex is called the *standard d-ball* and is denoted by $\Delta^d_{d+1}(U)$ or simply by Δ^d_{d+1}. A polyhedron is called a *pl d-ball* if it is pl homeomorphic to $|\Delta^d_{d+1}|$. A simplicial complex X is called a *combinatorial d-ball* if it is combinatorially equivalent to Δ^d_{d+1}. So, X is a combinatorial d-ball if and only if $|X|$ is a pl d-ball.

For a set V with $d+2$ elements, let S be the simplicial complex whose faces are all the non-empty proper subsets of V. Then S triangulates the d-sphere. This complex is called the *standard d-sphere* and is denoted by $S^d_{d+2}(V)$ or simply by S^d_{d+2}. A polyhedron is called a *pl d-sphere* if it is pl homeomorphic to $|S^d_{d+2}|$. A simplicial complex X is called a *combinatorial d-sphere* if it is combinatorially equivalent to S^d_{d+2}. So, X is a combinatorial d-sphere if and only if $|X|$ is a pl d-sphere.

A simplicial complex K is called a *combinatorial d-manifold* if the link of each vertex is a combinatorial $(d-1)$-sphere. A simplicial complex K is a combinatorial d-manifold if and only if $|K|$ is a closed pl d-manifold (see [11]).

If a triangulation K of a space X is a combinatorial manifold then K is called a *combinatorial triangulation* of X. If K is a triangulation of a 3-manifold then the link of a vertex is a triangulation of the 2-sphere and all triangulations of the 2-sphere are combinatorial 2-spheres. So, any triangulation of a 3-manifold is a combinatorial triangulation.

Let $\tau \subset \sigma$ be two faces of a simplicial complex K. We say that τ is a *free face* of σ if σ is the only face of K which properly contains τ. (It follows that $\dim(\sigma) - \dim(\tau) = 1$ and σ is a maximal simplex in K.) If τ is a free face of σ then $K' := K \setminus \{\tau, \sigma\}$ is a simplicial complex. We say that there is an *elementary collapse* of K to K'. We say K *collapses* to L and write $K \overset{s}{\searrow} L$ if there exists a sequence $K = K_0, K_1, \dots, K_n = L$ of simplicial complexes such that there is an elementary collapse of K_{i-1} to K_i for $1 \leq i \leq n$ (see [3]). If L consists of a 0-simplex (a point) we say that K is *collapsible* and write $K \overset{s}{\searrow} 0$. Clearly, if $K \overset{s}{\searrow} L$ then $|K| \searrow |L|$ as polyhedra and hence $|K|$ and $|L|$ have the same homotopy type (see [11]). So, if a simplicial complex K is collapsible then $|K|$ is contractible and hence, in particular, K is $\mathbb{Z}_2$-acyclic. Here we prove:

Theorem 1. *If a $\mathbb{Z}_2$-acyclic simplicial complex has ≤ 7 vertices then it is collapsible.*

As an application of Theorem 1, we prove our main result - a recognition theorem for combinatorial spheres:

Theorem 2. *Let M be an n-vertex combinatorial triangulation of a $\mathbb{Z}_2$-homology d-sphere. Suppose M has an m-vertex combinatorial d-ball as an induced subcomplex, where $n \leq m+7$. Then M is a combinatorial sphere.*

In consequence we get the following.

Corollary 3. *Let M be an n-vertex combinatorial d-manifold. If $|M|$ is a $\mathbb{Z}_2$-homology sphere and $n \leq d+8$ then M is a combinatorial sphere.*

Corollary 4. *Let M be a $(d+9)$-vertex combinatorial triangulation of a $\mathbb{Z}_2$-homology d-sphere. If M is not a combinatorial sphere then M can not admit any bistellar i-move for $i < d$.*

Since by the universal coefficient theorem any integral homology sphere is a $\mathbb{Z}_2$-homology sphere, Theorem 2, Corollary 3 and Corollary 4 remain true if we replace $\mathbb{Z}_2$-homology by integral homology in the hypothesis. In particular, we have:

Corollary 5. *Let M be an n-vertex combinatorial triangulation of an integral homology d-sphere.*

(*a*) *If $n \leq d+8$ then M is a combinatorial sphere.*

(*b*) *If $n = d+9$ and M is not a combinatorial sphere then M can not admit any bistellar i-move for $i < d$.*

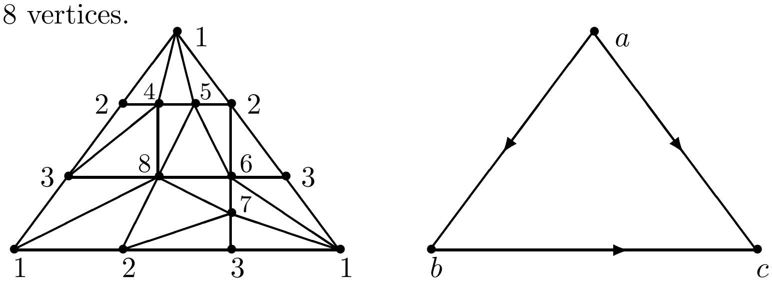
Remark 1. Corollary 3 is clearly trivial for $d \leq 2$. In [5], Brehm and Kühnel proved that any n-vertex combinatorial d-manifold is a combinatorial d-sphere if $n < 3\lceil d/2 \rceil + 3$ and it is either a combinatorial d-sphere or a cohomology projective plane if $n = 3d/2 + 3$. So, Corollary 3 has new content only for $3 \leq d \leq 8$.

Remark 2. Another result in [5] says that any n-vertex combinatorial d-manifold is simply connected for $n \leq 2d+2$. Since a simply connected integral homology sphere is a sphere for $d \neq 3$, and since for $d \neq 4$ all combinatorial triangulations of d-spheres are combinatorial spheres, this result implies that all combinatorial triangulations of integral homology d-spheres ($d \neq 3, 4$) with $\leq 2d+2$ vertices are combinatorial spheres. This is stronger than Corollary 5 (*a*) for $d \geq 6$. Thus Corollary 5 (*a*) has new content only for $d = 3, 4, 5$.

Remark 3. In [8, p. 35], Lutz presented a 12-vertex combinatorial triangulation of the lens space $L(3,1)$. (It is mentioned in [7, p. 79] that Brehm obtained a 12-vertex combinatorial triangulation of $L(3,1)$ earlier.) Since $L(3,1)$ is a $\mathbb{Z}_2$-homology 3-sphere ($H_1(L(3,1),\mathbb{Z}) = \mathbb{Z}_3$, $H_2(L(3,1),\mathbb{Z}) = 0$), Corollary 3 is sharp for $d = 3$.

It follows from Corollary 3 that 12 is the least number of vertices required to triangulate $L(3,1)$. It follows from Corollary 4 that a 12-vertex combinatorial triangulation of $L(3,1)$ can not admit any bistellar i-move for $0 \leq i \leq 2$.

Remark 4. Recall that the Dunce Hat is the topological space obtained from the solid triangle abc by identifying the oriented edges $\vec{ab}$, $\vec{bc}$ and $\vec{ac}$. The following is a triangulation of the Dunce Hat using 8 vertices.



Since this example is contractible but not collapsible, it follows that the bound 7 in Theorem 1 is best possible.

Remark 5 . Let H^3 be the non-orientable 3-manifold obtained from $S^2 \times [0,1]$ by identifying $(x,0)$ with $(-x,1)$. It follows from works of Walkup [14, Theorems 3, 4] that *if* K *is a combinatorial* 3*-manifold and* $|K|$ *is not homeomorphic to* S^3*,* $S^2 \times S^1$ *or* H^3 *then* $f_1(K) \geq 4f_0(K) + 8$ *and hence* $f_0(K) \geq 11$. Thus if M $(\neq S^3)$ is a $\mathbb{Z}_2$-homology 3-sphere then at least 11 vertices are needed for any combinatorial triangulation of M. Now, Corollary 3 implies that at least 12 vertices are needed. In [4], Björner and Lutz have presented a 16-vertex combinatorial triangulation of the Poincaré homology 3-sphere.

In [2], we have shown that all combinatorial triangulations of S^4 with at most 10 vertices are combinatorial 4-spheres. Now, Corollary 3 implies that all combinatorial triangulations of S^4 with at most 12 vertices are combinatorial spheres. So, any combinatorial triangulation (if it exists) of S^4 which is not a combinatorial sphere requires at least 13 vertices.

Remark 6 . The conclusion in Corollary 4 (namely, that certain combinatorial manifolds do not admit any proper bistellar move) appears to be a strong structural restriction. We owe to F. H. Lutz the information that the smallest known combinatorial sphere (other than a standard sphere) not admitting any proper bistellar move is a 16-vertex 3-sphere.

2 Preliminaries and Definitions.

For a simplicial complex K, the maximum k such that K has a k-face is called the *dimension* of K. An 1-dimensional simplicial complex is called a *graph*. A simplicial complex K is called *connected* if $|K|$ is connected.

For $i = 1, 2, 3$, the i-faces of a simplicial complex are also called the *edges*, *triangles* and *tetrahedra* of the complex, respectively. For a face σ in a simplicial complex K, the number of vertices in $\mathrm{Lk}_K(\sigma)$ is called the *degree* of σ in K and is denoted by $\deg_K(\sigma)$.

If the number of i-simplices of a d-dimensional simplicial complex K is $f_i(K)$, then the vector $f = (f_0, \dots, f_d)$ is called the *f-vector* of K and the number $\chi(K) := \sum_{i=0}^{d}(-1)^i f_i(K)$ is called the *Euler characteristic* of K. If $f_{k-1} = \binom{f_0}{k}$ then K is called *k-neighbourly*.

For two simplicial complexes K, L with disjoint vertex sets, the *join* $K * L$ is the simplicial complex $K \cup L \cup \{\sigma \cup \tau \ : \ \sigma \in K, \tau \in L\}$.

If K is a d-dimensional simplicial complex then define the *pure part* of K as the simplicial complex whose simplices are the sub-simplices of the d-simplices of K.

A d-dimensional pure simplicial complex K is called a *weak pseudomanifold* if each $(d-1)$-face is contained in exactly two facets of K. A d-dimensional weak pseudomanifold K is called a *pseudomanifold* if for any pair τ, σ of facets, there exists a sequence $\tau = \tau_0, \dots, \tau_n = \sigma$ of facets of K, such that $\tau_{i-1} \cap \tau_i$ is a $(d-1)$-simplex of K for $1 \leq i \leq n$. In other words, a weak pseudomanifold is a pseudomanifold if and only if it does not have any weak pseudomanifold of the same dimension as a proper subcomplex. Clearly, any connected combinatorial manifold is a pseudomanifold.

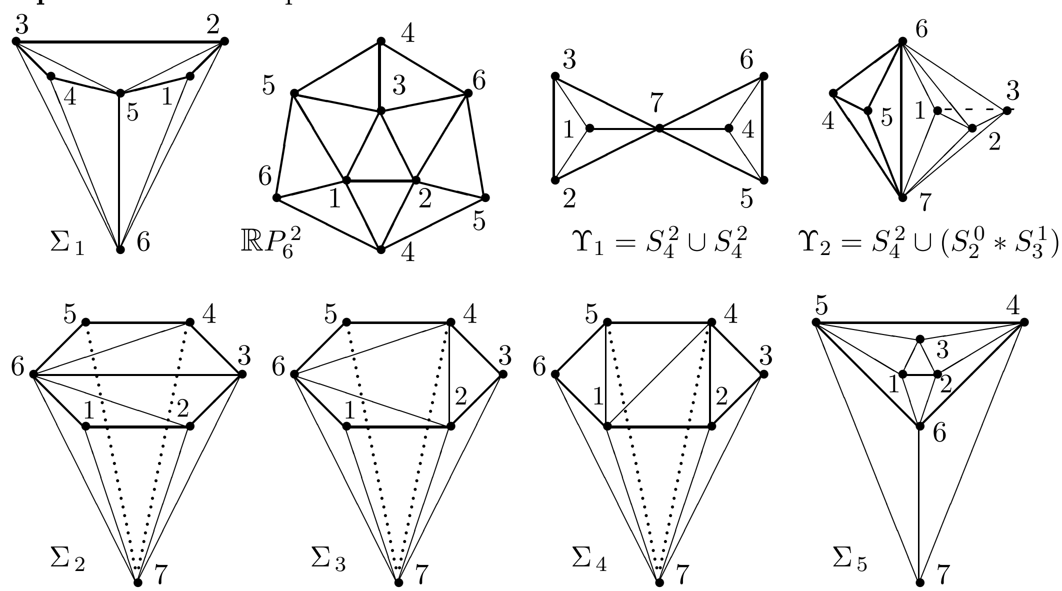
For $n \geq 3$, the n-vertex combinatorial 1-sphere (*n-cycle*) is the unique n-vertex 1-dimensional pseudomanifold and is denoted by S^1_n.

A d-dimensional pure simplicial complex K is called a *weak pseudomanifold with boundary* if each $(d-1)$-face is contained in 1 or 2 facets of K and there exists a $(d-1)$-face of degree 1. The boundary ∂K of K is by definition the pure simplicial complex whose facets are the degree one $(d-1)$-faces of K.

A simplicial complex K is called a *combinatorial d-manifold with boundary* if the link of each vertex is either a combinatorial $(d-1)$-sphere or a combinatorial $(d-1)$-ball and there exists a vertex whose link is a combinatorial $(d-1)$-ball. A simplicial complex K

is a combinatorial d-manifold with boundary if and only if $|K|$ is a compact pl d-manifold with non-empty boundary. Clearly, if K is a combinatorial d-manifold with boundary then $\partial K \neq \emptyset$ and $\mathrm{Lk}_{\partial K}(v) = \partial(\mathrm{Lk}_K(v))$, for $v \in V(\partial K)$. Therefore, ∂K is a combinatorial $(d-1)$-manifold. Clearly, if K is a combinatorial d-ball ($d > 0$) then K is a combinatorial d-manifold with boundary and ∂K is a combinatorial $(d-1)$-sphere.

Example 1. Some weak pseudomanifolds on 6 or 7 vertices.



$\Sigma_1, \dots, \Sigma_5$ are combinatorial spheres. $\mathbb{R}P^2_6$ triangulates the real projective plane. Υ_1, Υ_2 are the smallest examples of weak pseudomanifolds which are not pseudomanifolds.

The following results (which we need later) follow from the classification of all 2-dimensional weak pseudomanifolds on ≤ 7 vertices (e.g., see [1, 6]).

Proposition 2.1. *Let K be an n-vertex 2-dimensional weak pseudomanifold. If $n \leq 6$ then K is isomorphic to S^2_4, $S^1_3 * S^0_2$, $S^0_2 * S^0_2 * S^0_2$, $\mathbb{R}P^2_6$ or Σ_1 above.*

Proposition 2.2. *Let K be a 7-vertex 2-dimensional weak pseudomanifold. If the number of facets of K is ≤ 10 then K is isomorphic to $S^1_5 * S^0_2$, $\Sigma_2, \dots, \Sigma_5$, Υ_1 or Υ_2 above.*

Let X be a pure simplicial complex of dimension $d \geq 1$. Let A be a set of size $d+2$ such that A contains at least one and at most $d+1$ facets of X. (It follows that all except at most one element of A are vertices of X.) Define the pure d-dimensional simplicial complex $\kappa_A(X)$ as follows. The facets of $\kappa_A(X)$ are (i) the facets of X not contained in A and (ii) the $(d+1)$-subsets of A which are not facets of X. κ_A is said to be a *generalized bistellar move*. Clearly $\kappa_A(\kappa_A(X)) = X$. Let $\beta = \{x \in A : A \setminus \{x\} \in X\}$ and $\alpha = A \setminus \beta$. Then $\alpha \in X$ and $\beta \in \kappa_A(X)$. The set β is called the *core* of A. If α is an i-simplex of X then κ_A is also called a generalized bistellar i-*move*. Observe that if d is even and κ_A is a generalized bistellar $(d/2)$-move then $f_d(\kappa_A(X)) = f_d(X)$.

Now suppose X is a weak pseudomanifold, and A, α and β are as above. Notice that (a) either α is a d-simplex in X or $V(\mathrm{Lk}_X(\alpha)) \supseteq \beta$ and (b) if $\beta \in X$ then $\mathrm{Lk}_{\kappa_A(X)}(\beta) = \mathrm{Lk}_X(\beta) \cup S^{i-1}_{i+1}(\alpha) \neq S^{i-1}_{i+1}(\alpha)$ (and therefore $\kappa_A(X)$ is not a combinatorial manifold even if X is so). We shall say that κ_A is a *bistellar* move if (bs1) $\beta \not\in X$ and (bs2) either α is a d-simplex in X or $V(\mathrm{Lk}_X(\alpha)) = \beta$ (and hence $\mathrm{Lk}_X(\alpha)$ is the standard sphere on the vertex set β). If $1 \leq i \leq d-1$ then a bistellar i-move is called a *proper* bistellar move. Observe that if X is a combinatorial d-manifold then (bs2) holds for any $(d+2)$-subset A. If a generalized bistellar move is not a bistellar move then it is called *singular*.

Two weak pseudomanifolds are called *bistellar equivalent* if there exists a finite sequence of bistellar moves leading from one to the other. Let κ_A be a bistellar move on X. If X_1 is obtained from X by starring ([1]) a new vertex in α and X_2 is obtained from $\kappa_A(X)$ by starring a new vertex in β then X_1 and X_2 are isomorphic. Thus if X and Y are bistellar equivalent then $X \approx Y$. In [10], Pachner proved the following: *Two combinatorial manifolds are bistellar equivalent if and only if they are combinatorially equivalent.*

Example 2. Let the notations be as in Example 1.

(*a*) Let $A = \{1,2,5,6\} \subset V(\mathbb{R}P^2_6)$. Put $R = \kappa_A(\mathbb{R}P^2_6)$. Then R is not a weak pseudomanifold. Observe that (bs1) is not satisfied here and hence κ_A is a singular bistellar move. Note that the automorphism group A_5 of $\mathbb{R}P^2_6$ is transitive on the 4-subsets of its vertex set. In consequence, all singular bistellar 1-moves on $\mathbb{R}P^2_6$ yield isomorphic simplicial complexes.

(*b*) Let $B = \{2,3,6,7\} \subseteq V(\Sigma_2)$. Then $\kappa_B(\Sigma_2)$ is the union of two spheres with one common edge 67. Here (bs1) is not satisfied.

(*c*) Let $C = \{1,2,3,6\} \subseteq V(\Upsilon_1)$. Then $\kappa_C(\Upsilon_1) = \Upsilon_2$. Here also (bs1) is not satisfied and $\kappa_C(\Upsilon_1) \not\approx \Upsilon_1$ but $\kappa_C(\Upsilon_1)$ is a weak pseudomanifold.

(*d*) Let $D = \{1,2,3,6\} \subseteq V(\Upsilon_2)$. Then $\kappa_D(\Upsilon_2) = \Upsilon_1$. Here (bs2) is not satisfied.

(*e*) If $E = \{2,3,4,6\} \subseteq V(\Sigma_4)$ then $\kappa_E(\Sigma_4)$ is a 7-vertex pseudomanifold with 12 facets. In this case, (bs1) is not satisfied.

(*f*) Let $F = \{2,3,4,6\} \subseteq V(\Sigma_2)$. Then κ_F is a bistellar move and $\kappa_F(\Sigma_2) = \Sigma_3$.

Let $L \subseteq K$ be simplicial complexes. The *simplicial neighbourhood* of L in K is the subcomplex $N(L,K)$ of K whose maximal simplices are those maximal simplices of K which intersect $V(L)$. Clearly, $N(L,K)$ is the smallest subcomplex of K whose geometric carrier is a topological neighbourhood of $|L|$ in $|K|$. The induced subcomplex $C(L,K)$ on the vertex-set $V(K) \setminus V(L)$ is called the *simplicial complement* of L in K.

Suppose $P' \subseteq P$ are polyhedra and $P = P' \cup B$, where B is a pl k-ball (for some $k \geq 1$). If $P' \cap B$ is a pl $(k-1)$-ball then we say that there is an *elementary collapse* of P to P'. We say that P collapses to Q and write $P \searrow Q$ if there exists a sequence $P = P_0, P_1, \dots, P_n = Q$ of polyhedra such that there is an elementary collapse of P_{i-1} to P_i for $1 \leq i \leq n$. If Q is a point we say that P is collapsible and write $P \searrow 0$. For two simplicial complexes K and L, if $K \overset{s}{\searrow} L$ then clearly $|K| \searrow |L|$. A *regular neighbourhood* of a polyhedron P in a pl d-manifold M is a d-dimensional submanifold W with boundary such that $W \searrow P$ and W is a neighbourhood of P in M. The following is a direct consequence of the Simplicial Neighbourhood Theorem ([11, Theorem 3.11]).

Proposition 2.3. *Let K be a combinatorial d-manifold with boundary. Suppose ∂K is an induced subcomplex of K. Let L be the simplicial complement of ∂K in K. Then $|K| \searrow |L|$.*

Proof. Let M be a pl d-manifold such that $|K|$ is in the interior of M (we can always find such M, e.g., one such M can be obtained from $|K| \sqcup (|\partial K| \times [0,1])$ by identifying $(x,0)$ with $x \in |\partial K|$).

Since $L = C(\partial K, K)$, $|L| \subseteq |K| \setminus |\partial K|$ and hence $|K|$ is a neighbourhood of $|L|$ in $\mathrm{int}(M)$. Again, since L is the simplicial complement of ∂K in K and ∂K is an induced subcomplex of K, $C(L,K) = \partial K$. Finally, since ∂K is an induced subcomplex of dimension $d-1$, each d-simplex of K intersects $V(L)$. This implies that $N(L,K) = K$.

Let $P = |L|$, $A = |K|$ and $J = \partial K$. Then $\partial A = |\partial K|$ and $\dot{N}(L,K) := N(L,K) \cap C(L,K) = J$. Thus (i) P is a compact polyhedron in the interior of the pl manifold M, (ii) A is a neighbourhood of P in $\mathrm{int}(M)$, (iii) A is a compact pl manifold with boundary and (iv) (K,L,J) are triangulations of $(A,P,\partial A)$ where L is an induced subcomplex of K, $K = N(L,K)$ and $J = \dot{N}(L,K)$. Then, by the Simplicial Neighbourhood Theorem, A is a regular neighbourhood of P. Hence $A \searrow P$. $\Box$

We need the following well-known results (see [11, Lemma 1.10, Corollaries 3.13, 3.28]) later.

Proposition 2.4 . *Let B, D be pl d-balls and $h\colon \partial B \to \partial D$ a pl homeomorphism. Then h extends to a pl homeomorphism $h_1\colon B \to D$.*

Proposition 2.5 . *Let S be a pl d-sphere. If $B \subseteq S$ is a pl d-ball then the closure of $S \setminus B$ is a pl d-ball.*

Proposition 2.6 . *A collapsible pl manifold with boundary is a pl ball.*

Question . Is it true that under the hypothesis of Proposition 2.3, we have $K \overset{s}{\searrow} L$?

3 $\mathbb{Z}_2$-acyclic simplicial complexes.

In this section we prove Theorem 1.

Lemma 3.1 . *Let X be a 7-vertex simplicial complex. Suppose (a) X is $\mathbb{Z}_2$-acyclic, (b) X is not collapsible, and (c) X is minimal subject to (a) and (b) (i.e., X has no proper subcomplex satisfying (a) and (b)). Then X is pure of dimension $d = 2$ or 3 and each $(d-1)$-face of X occurs in at least two facets.*

Proof. Notice that, because of the minimality assumption, X has no free face. Clearly, $\dim(X) \leq 5$, since otherwise X is a combinatorial ball. Suppose $\dim(X) = 5$. By minimality, each 4-face of X is in 0 or ≥ 2 facets. Since X has 7 vertices, it follows that each 4-face is in 0 or 2 facets. Therefore the pure part Y of X is a 7-vertex 5-dimensional weak pseudomanifold and hence $Y = S^5_7 \subseteq X$. Then $H_5(X, \mathbb{Z}_2) \neq 0$, a contradiction. Thus $\dim(X) \leq 4$.

Suppose, if possible, $\dim(X) = 4$. Let Y be the pure part of X. Then, each 3-face of Y occurs in at least two facets. If $\#(V(Y)) \leq 6$, then $Y = S^4_6$ and hence $H_4(X, \mathbb{Z}_2) \neq 0$, a contradiction. Thus $V(Y) = V(X)$ has size 7. Define a binary relation $\sim$ on $V(Y)$ by $y_1 \sim y_2$ if $V(Y) \setminus \{y_1, y_2\}$ is not a facet of Y. Since each 3-face of Y is in at least two facets, it follows that $\sim$ is an equivalence relation with at least two equivalence classes. Therefore either there is an equivalence class W of size 6 or else we can write $V(Y) = V_1 \sqcup V_2$, where

V_1, V_2 are unions of $\sim$-classes and $\#(V_1) \geq 2$, $\#(V_2) \geq 2$. In consequence Y (and hence X) contains a 4-sphere as a subcomplex: the standard sphere on W or the join of the standard spheres on V_1 and V_2. Therefore $H_4(X, \mathbb{Z}_2) \neq 0$, a contradiction. Thus $\dim(X) \leq 3$.

If $\dim(X) = 1$ then X is a $\mathbb{Z}_2$-acyclic connected graph and hence is a tree. But any tree has end vertices and hence is collapsible, a contradiction. So, $\dim(X) = 2$ or 3.

Since $\widetilde{H}_0(X, \mathbb{Z}_2) = 0$, X is connected. Since X has no free vertex, it follows that each vertex of X is in at least two edges.

Next we show that X has no maximal edge. Suppose, on the contrary, X has a maximal edge e. Then $Y := X \setminus \{e\}$ is a subcomplex of X. We claim that Y is disconnected. If not, then there is a subcomplex $K = S^1_n$ of X containing the edge e. The formal sum of the edges in K is an 1-cycle over $\mathbb{Z}_2$ which is not a boundary since it involves the maximal edge e. Hence $H_1(X, \mathbb{Z}_2) \neq 0$, a contradiction. So, Y is disconnected. Since each vertex of X is in at least two edges, it follows that each component of Y has ≥ 3 vertices. Since X has seven vertices, it follows that some component of Y has exactly three vertices and contains an S^1_3. If these three vertices span a 2-face then its edges are free in X, contradicting minimality. In the remaining case X has an induced S^1_3 whose edges are maximal, contradicting $\mathbb{Z}_2$-acyclicity of X.

In case $\dim(X) = 2$, this shows that X is pure. In case $\dim(X) = 3$, we proceed to show that X has no maximal 2-face, proving that it is pure in that case too.

Suppose, on the contrary, that $\dim(X) = 3$ and X has a maximal 2-face $\Delta = abc$. Let's say that an edge of X is *good* if it is in a tetrahedron of X, and call it *bad* otherwise. First suppose that all three edges in Δ are good. Since X has no free triangle, each vertex in the link of an edge has degree 0 or ≥ 2 and hence there are at least three vertices of degree ≥ 2 in the link of a good edge. Since Δ is maximal, it follows that the link of each of the three edges in Δ has ≥ 3 vertices outside Δ. Since, there are only four vertices outside Δ, it follows from the pigeonhole principle that there is a common vertex x outside Δ which occurs in the link of all three edges in Δ. Hence $S^2_4(\Delta \cup \{x\})$ is a subcomplex of X. The sum of the four triangles in this S^2_4 is a 2-cycle (with $\mathbb{Z}_2$ coefficients) which can not be the boundary of a 3-chain since one of these triangles is maximal. Therefore $H_2(X, \mathbb{Z}_2) \neq 0$, a contradiction. Thus Δ contains at least one bad edge.

We claim that Δ can't have more than one bad edges. Suppose, on the contrary, that ab and ac are bad edges in X. Notice that (arguing as in the proof of the case $\dim(X) = 4$), if a 3-dimensional simplicial complex on ≤ 6 vertices has ≥ 2 tetrahedra through each triangle then it contains a combinatorial S^3. Therefore the pure part Y of X must have seven vertices. In particular $a \in Y$. Since ab and ac are bad edges, $b, c \notin \mathrm{Lk}_Y(a)$ and hence $\deg_Y(a) \leq 4$. Therefore $\mathrm{Lk}_Y(a) = S^2_4$. Hence we can apply an improper bistellar move to Y to remove the vertex a, yielding a 6-vertex 3-dimensional simplicial complex $\widetilde{Y}$ with ≥ 2 tetrahedra through each triangle. Hence $\widetilde{Y}$ has an S^3 as a subcomplex, so that $H_3(Y, \mathbb{Z}_2) = H_3(\widetilde{Y}, \mathbb{Z}_2) \neq 0$. Therefore $H_3(X, \mathbb{Z}_2) \neq 0$, a contradiction. Thus Δ contains exactly one bad edge, say ab. Hence ac and bc are good edges.

Since X has no free edge, there is a second triangle, say abd, through ab. Since ab is a bad edge, abd is maximal. By the above argument, ad and bd are good edges. If both acd and bcd are triangles of X then X has $S^2_4(a, b, c, d)$ as a subcomplex, and at least one of the triangles of this S^2_4 is maximal in X, yielding the contradiction $H_2(X, \mathbb{Z}_2) \neq 0$ as before. Therefore, without loss of generality, we may assume $bcd \notin X$. Note that a is an isolated vertex in $\mathrm{Lk}_X(bc)$ and d does not occur in $\mathrm{Lk}_X(bc)$. Since bc is a good edge, it follows that all three vertices outside $\{a, b, c, d\}$ (say x, y and z) occur in $\mathrm{Lk}_X(bc)$. Similarly, $x, y, z \in \mathrm{Lk}_X(bd)$. Again, the good edges ac and ad have at most one non-isolated vertex from $\{a, b, c, d\}$ in their links, hence each of them has at least two of x, y, z in their links. Therefore, there is one vertex, say x, which occurs in the link of all the four edges ac, bc, ad, bd. Hence $S^0_2(c, d) * S^1_3(a, b, x)$ is a subcomplex of X. Since one of the triangles in this 2-sphere is maximal, it follows that $H_2(X, \mathbb{Z}_2) \neq 0$, a contradiction. Thus X has no maximal triangles nor maximal edges, so X is pure.

Finally, the last assertion follows from purity and minimality of X. □

Lemma 3.2 . *Let X be a 7-vertex 2-dimensional $\mathbb{Z}_2$-acyclic simplicial complex. Then X is collapsible.*

Proof. Let X be a minimal counter example. Let f_i, $0 \leq i \leq 2$, be the number of i-faces in X. Since X is $\mathbb{Z}_2$-acyclic, $\chi(X) = 1$. Thus, $f_0 = 7$ and $f_1 = f_2 + 6$.

For $i \geq 0$, let e_i be the number of edges of degree i in X. By Lemma 3.1, $e_i = 0$ for $i \leq 1$. Two-way counting yields

$$\sum_{i=2}^{5} e_i = f_1 = f_2 + 6, \quad \sum_{i=2}^{5} i\, e_i = 3f_2.$$

Hence

$$e_3 + 3e_5 \leq e_3 + 2e_4 + 3e_5 = f_2 - 12. \tag{1}$$

Let's say that an edge of X is *odd* (respectively *even*) if it lies in an odd (respectively even) number of triangles. Note that each graph has an even number of vertices of odd degree. Applying this trivial observation to the vertex links of X, we conclude that each vertex of X is in an even number of odd edges. Thus the total number $e_3 + e_5$ of odd edges is $= 0$ or ≥ 3. If there is no odd edge then the sum of all the triangles gives a non-zero element of $H_2(X, \mathbb{Z}_2)$, a contradiction. So, $e_3 + e_5 \geq 3$. Combining this with (1), we get $f_2 \geq 15$ and hence $f_1 \geq 21 = \binom{7}{2}$. Hence $f_1 = 21$, $f_2 = 15$, $e_3 = 3$, $e_4 = e_5 = 0$.

Since each vertex is in an even number of odd edges, it follows that the three odd edges form a triangle Δ, which may or may not be in X.

If Δ is in X, then the sum of the remaining triangles gives a non-zero element of $H_2(X, \mathbb{Z}_2)$, a contradiction. If Δ is not in X then (as each of the three edges in Δ has three vertices in its link and there are four vertices outside Δ) by the pigeonhole principle there is a vertex $x \notin \Delta$ such that x occurs in the link of each of the three edges in Δ. Then the sum of all the triangles excepting the three triangles in $\Delta \cup \{x\}$ gives a non-zero element of $H_2(X, \mathbb{Z}_2)$, a contradiction. □

Lemma 3.3 . *Let U be a 2-dimensional pure simplicial complex on ≤ 7 vertices. Suppose the number of triangles in U is ≤ 10 and each edge of U is in an even number of triangles. Then either U is the union of two combinatorial spheres (on 4 or 5 vertices) with no common triangle, or U is isomorphic to one of S^2_4, $S^1_3 * S^0_2$, $S^0_2 * S^0_2 * S^0_2$, $S^1_5 * S^0_2$, $\mathbb{R}P^2_6$, $\Sigma_1, \ldots, \Sigma_5$ or R (of Example 1 and Example 2 (a)).*

Proof. Let $\mathcal{S}$ be the list of simplicial complexes in the statement of this lemma. We find by inspection that $\mathcal{S}$ is closed under generalized bistellar 1-moves.

If $f_0(U) \leq 5$ then U is a weak pseudomanifold and hence, by Proposition 2.1, $U \in \mathcal{S}$. So assume $f_0(U) = 6$ or 7. The proof is by induction on the number $n(U)$ of degree 4 edges in U. If $n(U) = 0$ then U is a weak pseudomanifold and hence, by Propositions 2.1 and 2.2, $U \in \mathcal{S}$. So let $n(U) > 0$ and suppose that we have the result for all smaller values of $n(U)$.

By the assumption, all the edges of U are of degree 2 or 4. Therefore, a two-way counting yields $4n(U) + 2(f_1(U) - n(U)) = 3f_2(U) \leq 30$. Thus, $n(U) + f_1(U) \leq 15$. Therefore,

$$f_1(U) < 15, \tag{2}$$

showing that U has at least one non-edge. Fix an edge ab of degree 4 in U. Let W be the link of ab. If each pair of vertices in W formed an edge in U then $f_1(U)$ would be ≥ 15, contradicting (2). So, there exist $c, d \in W$ such that cd is a non-edge in U.

Let $A = \{a, b, c, d\}$. Then κ_A is a generalized bistellar 1-move and hence $\kappa_A(U)$ also satisfies the hypothesis of the lemma, and $n(\kappa_A(U)) = n(U) - 1$. Therefore, by the induction hypothesis, $\kappa_A(U) \in \mathcal{S}$. Since $\mathcal{S}$ is closed under generalized bistellar 1-moves, $U = \kappa_A(\kappa_A(U)) \in \mathcal{S}$. □

Lemma 3.4. *Let X be a 7-vertex 3-dimensional simplicial complex. Suppose (a) X is $\mathbb{Z}_2$-acyclic, (b) X is not collapsible, and (c) X is minimal subject to (a) and (b). Then the f-vector of X is* $(7,20,30,16)$, $(7,21,32,17)$, $(7,21,33,18)$, $(7,21,34,19)$ *or* $(7,21,35,20)$.

Proof. For $0 \leq i \leq 3$, let f_i be the number of i-faces of X. For $i \geq 0$, let t_i be the number of triangles of degree i in X. By Lemma 3.1, we have $t_i = 0$ for $i \leq 1$. Two way counting yields

$$\sum_{i=2}^{4} t_i = f_2, \quad \sum_{i=2}^{4} i t_i = 4f_3$$

and hence

$$t_3 \leq t_3 + 2t_4 = 4f_3 - 2f_2. \tag{3}$$

Say that a triangle of X is *odd* (respectively *even*) if it is in an odd (respectively even) number of tetrahedra of X. By the same argument as in Lemma 3.2, each edge is in an even number of odd triangles, so that the number t_3 of odd triangles is 0 or ≥ 4.

If there is no odd triangle then the sum of all the tetrahedra gives a non-zero element of $H_3(X, \mathbb{Z}_2)$, a contradiction. So, $t_3 \geq 4$. Combining this with (3) we get

$$2f_3 - f_2 \geq 2. \tag{4}$$

Since X is $\mathbb{Z}_2$-acyclic, by a result of Stanley ([13]), X has a 2-dimensional subcomplex Y such that the f-vector of X equals the f-vector of a cone over Y. (In [13], the author uses the vanishing of the reduced cohomology groups as his definition of acyclicity, while we have used the homology definition. However, since the coefficient ring used is a field, these two definitions coincide.) Let (g_0, g_1, g_2) be the f-vector of Y. Thus, $g_0 = 6$ and

$$f_1 = g_1 + 6, \quad f_2 = g_1 + g_2, \quad f_3 = g_2. \tag{5}$$

Hence (4) yields

$$g_2 \geq g_1 + 2. \tag{6}$$

Let $m = \binom{6}{2} - g_1$, $n = \binom{6}{3} - g_2$ be the number of non-edges and non-triangles of Y, respectively. Since each non-edge is in exactly four non-triangles and any two non-edges are shared by at most one non-triangle, we have $n \geq 4m - \binom{m}{2}$. Also, from (6) we get $n \leq m + 3$. Hence $m + 3 \geq 4m - \binom{m}{2}$ or $(m-1)(m-6) \geq 0$. So, either $m \leq 1$ or $m \geq 6$.

First suppose $m \geq 6$, i.e., $g_1 \leq 9$. If each edge of Y was in ≤ 3 triangles then we would have $g_2 \leq g_1$, contradicting (6). So, there is an edge of Y contained in four triangles, together covering all the nine edges of Y. But, apart from the four triangles already seen, no three of these nine edges form a triangle of Y. Thus $g_2 = 4$, $g_1 = 9$ – contradicting (6). So, $m \leq 1$, i.e., $g_1 = 14$ or 15.

If $g_1 = 14$ then the four triangles through the missing edge are missing from Y, so that $g_2 \leq 16$. Thus, by (6), $(g_1, g_2) = (14, 16)$, $(15, 17)$, $(15, 18)$, $(15, 19)$ or $(15, 20)$. The lemma now follows from (5). $\square$

Lemma 3.5 . *Let X be a 7-vertex 3-dimensional $\mathbb{Z}_2$-acyclic simplicial complex. Then X is collapsible.*

Proof. Let X be a minimal counter example. As before, each edge is in an even number of odd triangles. Let f_i's and t_j's be as in the proof of Lemma 3.4. Then, by Lemma 3.4, $t_3 + 2t_4 = 4f_3 - 2f_2 \leq 10$ and hence the number t_3 of odd triangles is ≤ 10.

Let U denote the pure 2-dimensional simplicial complex whose facets are the odd triangles of X. Then each edge of U is in an even number of triangles of U. Therefore, by Lemma 3.3, we get the following cases:

Case 1: U is the union of two combinatorial spheres with no common triangle (on 4 or 5 vertices), say on vertex sets A and B.

First suppose $\#(A) = \#(B) = 4$. If both A and B are 3-faces in X then the pure simplicial complex $\widetilde{X}$ whose facets are those of X other than A, B is a 3-dimensional weak pseudomanifold. This implies that the sum of all the tetrahedra, excepting A and B, gives a non-zero element of $H_3(X, \mathbb{Z}_2)$, a contradiction. So, without loss of generality $A \not\in X$.

Since each of the four triangles inside A is of degree 3 in X, the three vertices (say x, y, z) outside A occur in the link of all the four triangles. Then the 3-sphere $S^2_4(A) * S^0_2(x, y)$ occurs as a subcomplex of X, forcing $H_3(X, \mathbb{Z}_2) \neq 0$, a contradiction.

In the remaining case $\#(A) = 4$, $\#(B) = 5$ (since U has at most 10 triangles, the case $\#(A) = \#(B) = 5$ does not arise). Write $B = \{b_1, b_2, b_3, x, y\}$ and $U = S^2_4(A) \cup (S^1_3(b_1, b_2, b_3) * S^0_2(x, y))$. As above, we must have $A \in X$.

If both $b_1b_2b_3x$ and $b_1b_2b_3y$ are in X, then the sum of the 3-faces other than A, $b_1b_2b_3x$ and $b_1b_2b_3y$ gives a non-zero element of $H_3(X, \mathbb{Z}_2)$, a contradiction. So, without loss of generality, $b_1b_2b_3x \not\in X$. Since the triangles of $S^1_3(b_1, b_2, b_3) * S^0_2(x, y))$ are degree 3 triangles in X, it follows that b_1b_2xy, b_1b_3xy, $b_2b_3xy \in X$. Then the sum of the tetrahedra other than A and these three tetrahedra gives a non-zero element of $H_3(X, \mathbb{Z}_2)$, a contradiction.

Case 2: $U = S^2_4$. We get a contradiction as in Case 1.

Case 3: $U = S^1_3 * S^0_2$. We get a contradiction as in Case 1.

Observation 1 : As $t_3 \geq 8$ in the remaining cases, we have $2f_3 - f_2 \geq 4$ and hence only the following two possibilities survive for the f-vector of X: $(7, 21, 34, 19)$ and $(7, 21, 35, 20)$. Therefore X has at most one missing triangle and at most one triangle of degree 4, and these two cases are exclusive. It follows that, if x is a vertex not covered by the odd triangles, then $\mathrm{Lk}_X(x)$ is a 6-vertex 2-dimensional neighbourly weak pseudomanifold. But, from Proposition 2.1, we see that $\mathbb{R}P^2_6$ is the only possibility. Thus, $\mathrm{Lk}_X(x) = \mathbb{R}P^2_6$. This implies that if $V_1 \subseteq V(U)$ is a 3-set then exactly one of V_1 and $V(U) \setminus V_1$ is a simplex in $\mathrm{Lk}_X(x)$. In particular, any two triangles in $\mathrm{Lk}_X(x)$ intersect.

Case 4: $U = S^0_2(a_1, a_2) * S^0_2(b_1, b_2) * S^0_2(c_1, c_2)$. Then the odd triangles of X are $a_ib_jc_k$, $1 \leq i, j, k \leq 2$. If $\{a_1a_2b_jc_k : 1 \leq j, k \leq 2\} \subseteq X$, then the sum of the remaining tetrahedra gives a non-zero element of $H_3(X, \mathbb{Z}_2)$, a contradiction. So, without loss of generality, $a_1a_2b_1c_1 \notin X$. As $a_1b_1c_1$, $a_2b_1c_1$ are degree 3 triangles, it follows that $a_1b_1b_2c_1$, $a_2b_1b_2c_1 \in X$. If both $a_1b_1b_2c_2$ and $a_2b_1b_2c_2$ are in X then $X \supseteq \{a_ib_1b_2c_k : 1 \le i, k \le 2\}$, hence we get a contradiction as before. So, without loss of generality, $a_2b_1b_2c_2 \notin X$.

Since $a_1a_2b_1c_1$, $a_2b_1b_2c_2 \notin X$ and $a_1b_1c_1$, $a_2b_2c_2$ are degree 3 triangles, it follows that these two disjoint triangles occur in the link of x. But this contradicts Observation 1.

Case 5: $U = \Sigma_1$ of Example 1. Thus, the odd triangles are 125, 126, 156, 235, 236, 345, 346 and 456. If 1256, 3456 $\notin X$ then, since 125 and 346 are degree 3 triangles, they are disjoint triangles in $\mathrm{Lk}_X(x)$, contradicting Observation 1. So, without loss of generality, $1256 \in X$.

If $3456 \notin X$ then, since 345, 346, 456 are degree 3 triangles, 2345, 2346, $2456 \in X$. Then the sum of all the tetrahedra, excepting 1256, 2345, 2346, 2456, gives a non-zero element of $H_3(X, \mathbb{Z}_2)$. So, $3456 \in X$.

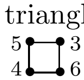
If $2356 \in X$, then the sum of all the tetrahedra, excepting 1256, 2356, 3456, gives a non-zero element of $H_3(X, \mathbb{Z}_2)$. Therefore $2356 \notin X$.

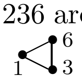
Since 235 and 236 are degree 3 triangles, 2345, $2346 \in X$. First suppose that at least one of 1356, 2456 is in X. Without loss, say $2456 \in X$. Then the sum of all the tetrahedra, excepting 1256, 2456, 2345, 2346, gives a non-zero element of $H_3(X, \mathbb{Z}_2)$. Thus 1356, $2456 \notin X$. Then, since 156, 456 are degree 3 triangles, $156x$, $456x \in X$.

Since $2356, 2456 \notin X$, $x \in \mathrm{Lk}_X(256)$, i.e., $256x \in X$. Similarly, looking at 356, we conclude that $356x \in X$. Thus, $56x$ is a degree 4 triangle in X. But this is not possible since, by Observation 1, $\mathrm{Lk}_X(x)$ is $\mathbb{R}P^2_6$.

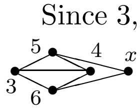
Observation 2 : In the remaining cases, $t_3 = 10$ and hence the f-vector of X is $(7, 21, 35, 20)$. In consequence, $t_4 = 0$. Thus all triangles are of degree 2 or 3. Since $f_3 = \binom{7}{3}$, each edge in X has degree 5. Thus if e is an edge outside U then the link of e is a pentagon (S^1_5).

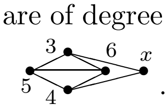
Case 6: $U = \mathbb{R}P^2_6$. In this case, all the 4-sets of vertices not containing x contain exactly two odd triangles each. In particular, all the tetrahedra of X not containing x contain exactly two odd triangles each. Trivially, each tetrahedron through x contains at most one odd triangle. Thus, letting α_i, $i \ge 0$, denote the number of tetrahedra of X containing exactly i odd triangles, we have $\alpha_2 = 20 - 10 = 10$ and $\alpha_0 + \alpha_1 = 10$. But two way counting yields $\alpha_1 + 2\alpha_2 = 10 \times 3 = 30$. Hence $\alpha_1 = 10$, $\alpha_0 = 0$. Thus x occurs in the link of each odd triangle and hence $\mathrm{Lk}_X(x) = U$. Therefore the 10 tetrahedra of X not passing through x add up to a non-zero element of $H_3(X, \mathbb{Z}_2)$, a contradiction.

Case 7: $U = R$ of Example 2 (a). Thus, the odd triangles are 123, 124, 125, 126, 135, 146, 236, 245, 345 and 346. We claim that $\mathrm{Lk}_X(12) \supseteq$ 

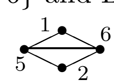
. If, for instance, $1236 \notin X$ then, as 123, 126, 236 are degree 3 triangles, x belongs to the link of each of these triangles. Then $\mathrm{Lk}_X(2x) \supseteq$ 

, contradicting Observation 2. This proves the claim.

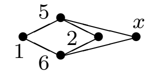
Since 3, 4, 5, 6 are of degree 3 and x is of degree 2 in $\mathrm{Lk}_X(12)$, it follows that $\mathrm{Lk}_X(12) =$ 

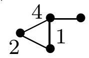
or = 

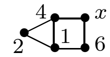
.

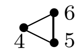
In the first case, $125, 126 \in \mathrm{Lk}_X(x)$. Hence, by Observation 1, 345, $346 \notin \mathrm{Lk}_X(x)$. Since these two are degree 3 triangles, it follows that $\mathrm{Lk}_X(345) = \{1, 2, 6\}$ and $\mathrm{Lk}_X(346) = \{1, 2, 5\}$. Since 1, 2 are of degree 2 in $\mathrm{Lk}_X(34)$, this forces $\mathrm{Lk}_X(34) =$ 

and hence $x \notin \mathrm{Lk}_X(34)$. This is a contradiction since X is 3-neighbourly.

In the second case, $125, 126 \notin \mathrm{Lk}_X(x)$ and hence, by Observation 1, $345, 346 \in \mathrm{Lk}_X(x)$. That is, $5x$, $6x \in \mathrm{Lk}_X(34)$. Also, as $34 \notin \mathrm{Lk}_X(12)$, we have $12 \notin \mathrm{Lk}_X(34)$. Since 5, 6 are of degree 3 and 1, 2, x are of degree 2 in $\mathrm{Lk}_X(34)$, it follows that $\mathrm{Lk}_X(34) =$ 

. Hence $1345, 2345, 345x \in X$. Also, as 123 is a degree 3 triangle and $1234 \notin X$, we have $1235 \in X$. Thus 

$\subseteq \mathrm{Lk}_X(35)$. Since 1, 4 are of degree 3 while 2, 6, x are of degree 2 in this link, it follows that $\mathrm{Lk}_X(35) =$ 

. Hence $356x \in X$. Then 

$\subseteq \mathrm{Lk}_X(3x)$, contradicting Observation 2.

Claim: In the remaining cases, if F is a set of four vertices of U containing at least two odd triangles, then either $F \in X$ or $F \subseteq V(\mathrm{Lk}_U(x))$ for some vertex x.

In these cases, $V(U) = V(X)$. If $F \notin X$ contains two odd triangles, then on the average, a vertex outside F occurs in the links (in X) of $\geq \frac{3\times2+2\times2}{3} > 3$ of the four triangles inside F. Thus there is a vertex x in the link of all these triangles. If $F \not\subseteq V(\mathrm{Lk}_U(x))$ for this x, then choose a vertex $y \in F$ such that $xy \notin U$. Then $\mathrm{Lk}_X(xy) \supseteq S^1_3(F \setminus \{y\})$, contradicting Observation 2. This proves the claim.

Case 8: $U = S^1_5(\mathbb{Z}_5) * S^0_2(u, v)$. In this case, the above claim implies that X contains the five tetrahedra $\{u, v, i, i+1\}$, $i \in \mathbb{Z}_5$. Then the sum of the remaining fifteen tetrahedra gives a non-zero element of $H_3(X, \mathbb{Z}_2)$, a contradiction.

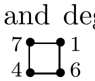
Case 9: $U = \Sigma_2$ of Example 1. Thus, the odd triangles are 126, 127, 167, 236, 237, 346, 347, 456, 457 and 567. By the above claim, $1267, 2367, 3467, 4567 \in X$. Then the sum of the remaining sixteen tetrahedra gives a non-zero element of $H_3(X, \mathbb{Z}_2)$, a contradiction.

Case 10: $U = \Sigma_3$ of Example 1. Thus, the odd triangles are 126, 127, 167, 234, 237, 246, 347, 456, 457 and 567. By the claim, $1267, 2347, 4567 \in X$.

If $2467 \in X$ then the sum of all the tetrahedra, excepting 1267, 2347, 4567, 2467, gives a non-zero element of $H_3(X, \mathbb{Z}_2)$, a contradiction. So, $2467 \notin X$. Then, $\mathrm{Lk}_X(246) = \{1, 3, 5\}$.

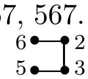
Since $\deg(247) = 2$ and $2347 \in X$, assume without loss of generality, that $2457 \in X$ and $1247 \notin X$. Then $\mathrm{Lk}_X(127) = \{3, 5, 6\}$.

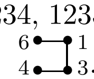
So, $2456, 2457 \in X$ and $\deg(245) = 2$. Hence $2345 \notin X$. Then $\mathrm{Lk}_X(234) = \{1, 6, 7\}$.

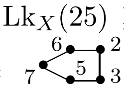
Now, 1234, $1237 \in X$ and $\deg(123) = 2$. Therefore, $1236 \notin X$. Then $\mathrm{Lk}_X(126) = \{4, 5, 7\}$. This implies that 

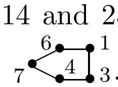
$\subseteq \mathrm{Lk}_X(25)$, a contradiction to Observation 2.

Case 11: $U = \Sigma_4$ of Example 1. Thus, the odd triangles are 124, 127, 145, 156, 167, 234, 237, 347, 457 and 567. By the claim, $1247, 1457, 1567, 2347 \in X$. Then the sum of the remaining sixteen tetrahedra gives a non-zero element of $H_3(X, \mathbb{Z}_2)$, a contradiction.

Case 12: $U = \Sigma_5$ of Example 1. Thus, the odd triangles are 123, 126, 135, 156, 234, 246, 345, 457, 467, 567. By the claim, $1234, 1235, 1246, 1256, 1345, 2345, 3457, 4567 \in X$. Thus $\mathrm{Lk}_X(14) \supseteq$ 

and $\mathrm{Lk}_X(25) \supseteq$ 

. Since 14 and 25 are not in U, Observation 2 implies that $\mathrm{Lk}_X(14) =$ 

and $\mathrm{Lk}_X(25) =$ 

. Thus $1457, 2457 \in X$. Then the triangle 457 is of degree 4 in X, a contradiction. This completes the proof. $\square$

Proof of Theorem 1 . Let Y be a minimal counter example. So, Y is an n-vertex (for some $n \leq 7$) $\mathbb{Z}_2$-acyclic simplicial complex which is not collapsible to any proper subcomplex.

If $n < 7$ then choose a facet α of Y and an element $v \notin V(Y)$. Let $\widetilde{Y}$ be obtained from Y by the bistellar d-move $\kappa_{\alpha\cup\{v\}}$, where d is the dimension of Y. Then $\widetilde{Y}$ is an $(n+1)$-vertex $\mathbb{Z}_2$-acyclic simplicial complex. Since Y has no free face, $\widetilde{Y}$ has no free face and hence $\widetilde{Y}$ is not collapsible to any proper subcomplex. Repeating this construction (if necessary) we get a 7-vertex $\mathbb{Z}_2$-acyclic simplicial complex X which is not collapsible to any proper subcomplex. Then, by Lemma 3.1, X is of dimension 2 or 3. But, this is not possible by Lemmas 3.2 and 3.5. This completes the proof. $\square$

4 Homology spheres.

Lemma 4.1 . *Let Y be a pseudomanifold of dimension d. Let Y_1 be a proper induced subcomplex of Y which is pure of dimension d. Put $L = C(Y_1, Y)$ and $Y_2 = N(L, Y)$. Then* (a) *Y_1, Y_2 are weak pseudomanifolds with boundary,* (b) *∂Y_2 is an induced subcomplex of Y_2 and* (c) *$\partial Y_2 = \partial Y_1 = Y_1 \cap Y_2$.*

Proof . Since Y is a pseudomanifold and $Y_1 \subset Y$ is pure of maximum dimension, Y_1 is a weak pseudomanifold with boundary. Since the maximal simplices of Y_2 are those maximal simplices of Y which intersect $V(L)$, Y_2 is pure of dimension d and each d-simplex of Y is either in Y_1 or in Y_2 but not in both. This implies that Y_2 is a weak pseudomanifold with boundary. This proves (a).

Let $V_1 = V(Y_1)$, $V_2 = V(L)$. Then $V(Y) = V_1 \sqcup V_2$. Now, τ is a facet of $\partial Y_2 \Leftrightarrow$ there exists a unique d-face $\sigma_2 \in Y_2$ containing $\tau \Leftrightarrow$ there exists a unique d-face $\sigma_1 \in Y_1$ containing $\tau \Leftrightarrow \tau$ is a facet of ∂Y_1. Therefore, $\partial Y_2 = \partial Y_1 \subseteq Y_1 \cap Y_2$.

Since $\partial Y_2 = \partial Y_1$, $\partial Y_2 \subseteq Y_2[V_1] = Y_2[V_1 \cap V(Y_2)]$. Conversely, let τ be a maximal face in $Y_2[V_1]$. Since Y_2 is pure, there exists a d-simplex $\sigma_2 \in Y_2$ such that $\tau \subseteq \sigma_2$. Since $Y_1 = Y[V_1]$, $\tau \in Y_1$ and hence there exists a d-simplex $\sigma_1 \in Y_1$ such that $\tau \subseteq \sigma_1$. This implies that $\tau \in \partial Y_1$. Thus $Y_2[V_1] \subseteq \partial Y_1 = \partial Y_2$. So, $Y_2[V_1] = \partial Y_2$. This proves (b).

Since $\tau \in Y_1 \cap Y_2$ implies $\tau \in Y_2[V_1] = \partial Y_2$, $Y_1 \cap Y_2 \subseteq \partial Y_2$. Therefore $Y_1 \cap Y_2 = \partial Y_2$. This completes the proof. $\square$

Lemma 4.2 . *Let X be a connected combinatorial d-manifold. Let X_1 be an induced subcomplex of X which is a combinatorial d-ball. Put $L = C(X_1, X)$ and $X_2 = N(L, X)$. Then*

- (a) *X_2 is a connected combinatorial d-manifold with boundary.*
- (b) $|X_2| \searrow |L|$.
- (c) *If, further, L is collapsible then X is a combinatorial sphere.*

Proof . Let $V_1 = V(X_1)$, $V_2 = V(L)$. Then $V(X) = V_1 \sqcup V_2$. As in the proof of Lemma 4.1, X_2 is pure of dimension d and each d-simplex of X is either in X_1 or in X_2 but not in both.

Let v be a vertex of X_2. Notice that $v \in X_1 \setminus \partial X_1 \Rightarrow \mathrm{Lk}_{X_1}(v) \subseteq \mathrm{Lk}_X(v)$ are $(d-1)$-spheres $\Rightarrow \mathrm{Lk}_{X_1}(v) = \mathrm{Lk}_X(v) \Rightarrow v \notin X_2$, a contradiction. So, either $v \in V_2$ or $v \in \partial X_1$.

If $v \in V_2$ then each d-simplex of X containing v is in X_2 and hence $\mathrm{Lk}_{X_2}(v) = \mathrm{Lk}_X(v)$ is a combinatorial $(d-1)$-sphere.

If $v \in \partial X_1$ then $(Y, Y_1, Y_2) := (\mathrm{Lk}_X(v), \mathrm{Lk}_{X_1}(v), \mathrm{Lk}_{X_2}(v))$ satisfies the hypothesis of Lemma 4.1. Therefore, by Lemma 4.1, $\mathrm{Lk}_{X_1}(v) \cap \mathrm{Lk}_{X_2}(v) = \partial(\mathrm{Lk}_{X_2}(v))$. This implies that the closure of $|\mathrm{Lk}_X(v)| \setminus |\mathrm{Lk}_{X_1}(v)|$ in $|\mathrm{Lk}_X(v)|$ is $|\mathrm{Lk}_{X_2}(v)|$. Since $|\mathrm{Lk}_X(v)|$ is a pl $(d-1)$-sphere and $|\mathrm{Lk}_{X_1}(v)|$ is a pl $(d-1)$-ball, by Proposition 2.5, $|\mathrm{Lk}_{X_2}(v)|$ is a pl $(d-1)$-ball. Thus, $\mathrm{Lk}_{X_2}(v)$ is a combinatorial $(d-1)$-ball.

Thus X_2 is a combinatorial d-manifold with boundary such that ∂X_2 ($= \partial X_1$, by Lemma 4.1) is connected. Therefore, if X_2 were disconnected, it would have a d-dimensional weak pseudomanifold as a component. This is not possible since X is a d-dimensional pseudomanifold. Therefore X_2 is connected. This proves (a).

As $L = X[V_2]$, we have $L \subseteq X_2$ and hence $L = X_2[V_2]$. Since, by Lemma 4.1, ∂X_2 is the induced subcomplex of X_2 on $V_1 \cap V(X_2)$, this implies that L is the simplicial complement of ∂X_2 in X_2. Then, by Proposition 2.3, $|X_2| \searrow |L|$. This proves (b).

Now, if $L \overset{s}{\searrow} 0$ then $|L| \searrow 0$ and hence $|X_2| \searrow 0$. So, by Proposition 2.6, $|X_2|$ is a pl ball.

Let σ be a d-simplex in S^d_{d+2}. Let $B_1 = |\sigma|$ and $B_2 = |S^d_{d+2} \setminus \{\sigma\}|$. Then B_1 and B_2 are pl d-balls. Let $f_2 \colon B_2 \to |X_2|$ be a pl homeomorphism. Let $f = f_2|_{\partial B_2}$. Since $\partial B_1 = \partial B_2$ and $\partial(|X_1|) = |\partial X_1| = |\partial X_2|$, $f \colon \partial B_1 \to \partial(|X_1|)$ is a pl homeomorphism. By Proposition 2.4, there exists a pl homeomorphism $f_1 \colon B_1 \to |X_1|$ such that $f_1|_{\partial B_1} = f = f_2|_{\partial B_2}$. Then $f_1 \cup f_2$ is a pl homeomorphism from $|S^d_{d+2}|$ to $|X|$. This proves (c). $\Box$

Lemma 4.3. *Let X be a combinatorial triangulation of a $\mathbb{Z}_2$-homology d-sphere. Let X_1 be an induced subcomplex of X which is a combinatorial d-ball. Let $L = C(X_1, X)$ and $X_2 = N(L, X)$. Then X_2 is $\mathbb{Z}_2$-acyclic.*

Proof. Let $J = X_1 \cap X_2$. Then, by Lemma 4.1, $J = \partial X_1$. So, J is a combinatorial $(d-1)$-sphere. Therefore, $H_{d-1}(J, \mathbb{Z}_2) = \mathbb{Z}_2$ and $\widetilde{H}_q(J, \mathbb{Z}_2) = 0$ for all $q \neq d-1$. Also $\widetilde{H}_q(X_1, \mathbb{Z}_2) = 0$ for all $q \geq 0$. For $q \geq 1$, we have the following exact Mayer-Vietoris sequence of homology groups with coefficients in $\mathbb{Z}_2$ (see [9, 12]):

$$\cdots \to H_{q+1}(X) \to H_q(J) \to H_q(X_1) \oplus H_q(X_2) \to H_q(X) \to \widetilde{H}_{q-1}(J) \to \cdots \qquad (7)$$

Now, $H_d(X, \mathbb{Z}_2) = \mathbb{Z}_2$ and $\widetilde{H}_q(X, \mathbb{Z}_2) = 0$ for $q \neq d$. By Lemma 4.2, $|X_2|$ is a connected d-manifold with non-trivial boundary. Therefore, $H_d(X_2, \mathbb{Z}_2) = 0$ and $H_0(X_2, \mathbb{Z}_2) = \mathbb{Z}_2$. Then, by (7), $H_q(X_2, \mathbb{Z}_2) = 0$ for $0 < q < d-1$ and for $q = d-1$ we get the following short exact sequence of abelian groups:

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to H_{d-1}(X_2, \mathbb{Z}_2) \to 0.$$

Clearly, this implies $H_{d-1}(X_2, \mathbb{Z}_2) = 0$. Thus, $\widetilde{H}_q(X_2, \mathbb{Z}_2) = 0$ for all $q \geq 0$. $\Box$

Proof of Theorem 2. Let X_1 be an m-vertex induced subcomplex of M which is a combinatorial d-ball. Let $L = C(X_1, M)$ and $X_2 = N(L, M)$. Then, by Part (b) of Lemma 4.2, $|X_2| \searrow |L|$.

Again, by Lemma 4.3, X_2 is $\mathbb{Z}_2$-acyclic and hence L is $\mathbb{Z}_2$-acyclic. Since $n \leq m + 7$, the number of vertices in L is ≤ 7. Therefore, by Theorem 1, L is collapsible. Then, by Part (c) of Lemma 4.2, M is a combinatorial sphere. $\Box$

Proof of Corollary 3. If σ is a d-simplex of M then the induced subcomplex $\Delta^d_{d+1}(\sigma)$ is a $(d+1)$-vertex combinatorial d-ball. Therefore, by Theorem 2, M is a combinatorial sphere. $\Box$

Proof of Corollary 4. Assume, if possible, that M admits a bistellar i-move κ_A for some $i < d$. Let β be the core of A and $\alpha = A \setminus \beta$. Then $M[A] = \Delta^i_{i+1}(\alpha) * S^{d-i-1}_{d-i+1}(\beta)$ is a $(d+2)$-vertex combinatorial d-ball. Therefore, by Theorem 2, M is a combinatorial sphere, a contradiction. This proves the corollary. $\Box$

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