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Cohomology of certain moduli spaces of vector bundles

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Abstract. Let X be a smooth irreducible projective curve of genus g over the field of complex numbers. Let M_0 be the moduli space of semi-stable vector bundles on X of rank two and trivial determinant. A canonical desingularization N_0 of M_0 has been constructed by Seshadri [17]. In this paper we compute the third and fourth cohomology groups of N_0. In particular we give a different proof of the theorem due to Nitsure [12], that the third cohomology group of N_0 is torsion-free.

Keywords. Stable bundles; semi-stable bundles; parabolic bundles; conic bundles; Gysin map; Hecke correspondence; Brauer group.

1. Introduction

Let X be a smooth irreducible projective curve of genus g over the field of complex numbers. Non-singular models of the moduli space of semi-stable vector bundles on X of rank two and degree zero have been constructed by Narasimhan–Ramanan [7] and Seshadri [17]. In this paper, we propose to compute some of the Betti numbers of the non-singular model due to Seshadri. In particular we prove the following theorems.

Theorem (A). *The third cohomology group of the non-singular model N_0 of [17] is torsion-free, $g \geqslant 2$.*

Theorem (B). *Let B_1 denote the Betti numbers of N_0. Then we have*:

$$B_3 = 2g, \quad B_4 = \binom{2g}{2} + 4, \quad g \geqslant 4.$$

Theorem (A) is due to Nitsure [11]. He proved this for the non-singular model of [7]. By Artin–Mumford [1], the torsion subgroup of the third cohomology group of a smooth projective variety is a birational invariant. Therefore any non-singular model has torsion-free third cohomology.

We present here a considerably simpler proof of Theorem (A) using the model of [17]; in fact, this was the initial motivation for this work. However we should point out that the general line of attack is as in Nitsure [11]. An extension of the ideas involved in the proof also yields Theorem (B). For computing B_4 and B_5 we make use of the results of Kirwan [5].

Nitsure showed independently that $B_3 = 2g$ for the model of [7] (cf [12]).

In Appendix 1 we present a proof of Theorem (A) due to Coliot-Thélène which is independent of the non-singular model chosen.

Theorems (A) and (B) are of interest in understanding the rationality of these non-singular models of the moduli space of vector bundles.

The layout of the paper is as follows. Section 2 of this paper gives various properties of the non-singular model constructed in [17]. In § 3 we construct a canonical generalized conic bundle on the non-singular model N_0. In § 4 by using a result of [9], we prove Theorem (A) and show how to compute the Betti numbers of the open subset Z of N_0 lying over the stable bundles and the bundles in the non-nodal part of the Kummer variety. In this section, we also give a description of the Hecke correspondence in terms of parabolic bundles as mentioned in (*). This facilitates the computation of the Betti numbers. In § 5 we compute explicitly the codimension of the complement of Z in N_0 and thereby compute its Betti numbers.

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2. Preliminaries

In this section we shall recall very briefly the definitions and terminologies of [17]. The proofs of most of the statements made in this section can be found in [17] or [18]. We state at the outset that for us the ground field of all our varieties is the field of complex numbers.

(i) X is a smooth irreducible projective curve of genus $g \geqslant 3$.

(ii) Let V be a vector bundle on X. A *parabolic structure* at a point $P \in X$ gives

(a) a *quasi-parabolic structure* i.e. a flag $V_P = F^1 V_P \supsetneqq F^2 V_P \supsetneqq \cdots \supsetneqq F^r V_P$.

(b) weights $\alpha_1, \ldots, \alpha_r$ attached to $F^1 V_P, \ldots, F^r V_P$ such that $0 \leqslant \alpha_1 < \alpha_2 < \cdots \alpha_r < 1$.

$$\text{Call } k_1 = \dim F^1 V_P - \dim F^2 V_P, \ldots, k_r = \dim F^r V_P$$

the multiplicities of $\alpha_1, \alpha_2, \ldots, \alpha_r$.

The *parabolic degree* of V is defined by

$$\operatorname{par} \deg V = \deg V + \sum_i k_i \alpha_i$$

and write $\operatorname{par} \mu(V) = \operatorname{par} \deg V / \operatorname{rk} V$.

If W is a subbundle of V, it acquires, in an obvious way, a quasi-parabolic structure. To make it a *parabolic subbundle*, we attach weights as follows:

Given $i_0, F^{i_0} W \subset F^j V$ for some; let j_0 be such that $F^{i_0} W \subset F^{j_0} V$ and $F^{i_0} W \not\subset F^{j_0+1} V$; then the weight of $F^{j_0} V = F^{i_0} W$. Define V to be parabolic stable (resp. semistable) if for every proper subbundle W of V, one has $\operatorname{par} \mu(W) < \operatorname{par} \mu(V)$ (resp. $\leqslant$).

(*) Mehta V and Seshadri C S *Math. Ann.* **248** (1980) 205–239.

If V_n be the category of semistable vector bundles on X of rank n and degree 0, then we denote by PV_n the category of parabolic semistable vector bundles at a fixed point $P \in X$ and fixed parabolic structure. Recall that, one can choose the weights (α) small enough so as to have the condition 'parabolic semistable' equivalent to 'parabolic stable!

(iii) N is the set of isomorphism classes of $(V, \Delta) \in PV_4$ (Δ a parabolic structure), such that End V is a 'specialization' of $\mathcal{M}_2$—the 2×2 matrix algebra.

In fact, if (V_1, Δ_1) and (V_2, Δ_2) belong to N, they represent the same element of N (i.e. isomorphic in PV_4) if and only if the underlying bundles V_1 and V_2 are isomorphic (cf [17]). *Hence we often simply write* $V \in N$.

(iv) $\mathcal{A}$ is the variety of all algebra structures on a fixed 4-dimensional vector space which are specializations of $\mathcal{M}_2$ and admit a fixed identity element. We have a *canonical group of automorphisms* acting on $\mathcal{A}$, namely the subgroup of $GL(4)$, fixing this identity element.

(v) M denotes the normal projective variety of equivalence classes of semistable vector bundles of rank 2 and degree 0 under the equivalence relation $V \sim V'$ if and only if gr $V =$ gr V'.

(vi) M^s will be the open subset of M consisting of the stable bundles.

It is known that $M - M^s$ is precisely the singular locus of M (cf [6]). The main theorem of [17] is stated below.

Theorem 1. (*Seshadri*) *There is a natural structure of a smooth projective variety on* N *and there exists a canonical morphism* $p: N \to M$, *which is an isomorphism over* M^s. *More precisely, if* $V \in N$, *then* gr $V = D \oplus D$, *with* rk $D = 2$, D *is a direct sum of stable line bundles of degree 0 and the morphism* $p: N \to M$ *is given by* $V \mapsto D$. *Further* $V \in p^{-1}(M^s)$ *if and only if* End $V \simeq \mathcal{M}_2$ *or equivalently (which is easily seen)* $V = W \oplus W$, *where* W *is stable.*

In the course of proving the smoothness of N, Seshadri defined a morphism from a neighbourhood U of a given point of N into A which we shall denote by

$$\varphi^U: U \to A.$$

We shall briefly indicate the construction of φ^U: The functor defining the moduli space N being representable, we have a defining vector bundle E on $X \times N$ of rank 4. Let $f: X \times N \to N$ be the canonical projection and End E the vector bundle associated to the shear of endomorphisms of E. Set

$$B = f_*(\text{End } E).$$

B is the canonical family of specializations of $\mathcal{M}_2$, parametrized by N (see Prop. 5 [17] for details). Consider any given point $u \in N$; then choosing a neighbourhood U of u, which trivialises B, we get a natural morphism

$$\varphi^U: U \to A \quad \text{by} \quad V \mapsto \text{End } V, \quad V \in U.$$

This morphism exists by the so-called versal property of A. Further, let $A_0 = \text{End } V_u$, V_u the vector bundle corresponding to the point $u \in U$, i.e. $A_0 = \varphi^U(u)$. Then, if A_u is the mini-versal deformation space of A_0, the morphism

$$\varphi_1^U : U \to A_u$$

induced by the versality of A_u from φ^U is in fact *smooth*.

Note 1. By an abuse of notation, in the course of this work, we shall suppress U and the mini-versal deformation space corresponding to each point, and simply denote by $\varphi: N \to A$ the *smooth local morphism* defined above. In fact, we will be using it only in this form in this work.

Note further that these φ^U are uniquely determined modulo automorphism coming from the canonical group of automorphisms acting on A.

PROPOSITION 1

The restriction of the local morphism φ to the subvariety N_0 remains smooth.

Proof. Let J denote the Jacobian variety of line bundles of degree zero on X. Then we have a natural morphism

$$\psi: N_0 \times J \to N$$
$$(E, L) \mapsto E \otimes L$$

(that this map is a morphism follows from the universal property of N and the fact that $E \otimes L$ gives a family on X parametrized by $N_0 \times J$).

We claim that ψ is *smooth*. In fact, ψ is *étale*. For, let $\Gamma \subset J$ be the finite subgroup of J consisting of the elements of order 2. Then there is a natural diagonal action of Γ on $N_0 \times J$ which is obviously fixed point free. It is not difficult to see that N is actually the quotient of $N_0 \times J$ by Γ and $\psi: N_0 \times J \to N$ the quotient morphism (note that our ground field is $\mathbb{C}$ and if A and B are smooth complex manifolds and G a finite group acting on A such that B is the set theoretic quotient of A by G, then B is A/G).

This Γ-action being fixed point free, ψ is *étale*.

For $b \in N_0 \times J$, choosing a neighbourhood U of $\psi(b) = u$ in N, we get the following diagram

$$\begin{array}{ccc} \psi^{-1}(U) & \longrightarrow & U \\ & \searrow^{\varphi_1^U \circ \psi} & \downarrow \varphi^U \\ & & A_u \end{array}$$

where A_u is the mini-versal deformation space of the algebra $A_0 = \varphi^U(u)$ in A.

Since φ_1^U, ψ are smooth, so is $\varphi_1^U \circ \psi$. In other words the *local morphism* (again by abuse of notation)

$$\varphi \circ \psi : N_0 \times J \to A$$

is smooth. If $L \in J$, then $\operatorname{End}(E \otimes J) = \operatorname{End} E$ and hence $\varphi \circ \psi$ clearly factors through N_0 to give the smoothness of the restriction of φ from $N_0 \to A$. Q.E.D.

Remark 1. Because of Prop. 1, by the same arguments as in [17], we see that N_0 is a smooth-projective variety. We then get an obvious generalization of Theorem 1 namely that $p: N_0 \to M_0$ which is a desingularization of M_0^s, and that it is an isomorphism over M_0^s etc.

3. Conic bundles

DEFINITION 1

Let S be a variety. A *generalized conic bundle* $\mathscr{C}$ on S gives

(a) a vector bundle V on S of rank 3 and
(b) a closed subscheme $\mathscr{C}$ of $\mathbb{P}(V)$ over S, such that, given $s \in S$, there exists a neighbourhood U of s, where $\mathscr{C} \cap p^{-1}(U)$ is defined by $q = 0$, $q \in \Gamma(p^{-1}(U), H^2)$, H being the tautological line bundle for $\mathbb{P}(V) \xrightarrow{p} S$; i.e. $p_*(H) \simeq V^*$ and therefore $p_*(H^2) = S^2 V^*$, etc.

By definition, $\mathscr{C}$ is an effective Cartier divisor and is therefore defined by a section of a line bundle θ on $\mathbb{P}(V)$. Now locally over S, θ and H^2 coincide and therefore by the "see-saw" theorem (cf. Mumford's *Abelian varieties*) there exists a line bundle L on S such that $\theta = H^2 \otimes p^*(L)$. Since $p_*(\theta) = p_*(H^2) \otimes L = S^2(V^*) \otimes L$, *the condition* (b) *above is equivalent* to an element q of $\Gamma(S^2(V^*) \otimes L)$ or that is to say a *quadratic form*

$$q: V \to L.$$

The *discriminant* Δ of q can be defined as a section of $L^3 \otimes (\Lambda^3(V^*))^2$ and locally as the usual discriminant of a quadratic form. The equation $\Delta = 0$ gives locally the *degeneracy locus* of $\mathscr{C}$.

We now introduce *subschemes* on S, namely for $i = 1, 2, 3$, set

$$S_i = \{s \in S \mid q \text{ restricted to } V_s, \text{ the fibre at } s, \text{ has rank} \leqslant 3 - i\}.$$

Then $S_3 \subset S_2 \subset S_1 \subset S = S_0$. If $g: \mathscr{C} \to S$ be the projection, let $\mathscr{C}_i = g^{-1}(S_i)$, $i = 1, 2, 3$. Then we have S_1 to be the degeneracy locus of $\mathscr{C}$, i.e. given by $\Delta = 0$, and $S_2 \subset S_1$ is the singular locus of S_1. The space $\mathscr{C}$ can be described as follows: $\mathscr{C} - \mathscr{C}_1$ consists of non-degenerate conics; $\mathscr{C}_1 - \mathscr{C}_2$ of pairs of lines intersecting transversally; $\mathscr{C}_2 - \mathscr{C}_3$ of repeated lines and $\mathscr{C}_3$ of the whole plane. We call S_i the *canonical subschemes associated to the degenerate loci of the conic bundle* $\mathscr{C}$ *on* S. Accordingly we make the following.

DEFINITION 2

A *generalized conic bundle* $\mathscr{C}$ is of type I if $\mathscr{C}_1 = \phi$; of type II if $\mathscr{C}_2 = \phi$ and of type III if $\mathscr{C}_3 = \phi$.

DEFINITION 3 (cf p. 164 [17])

Let T be an algebraic scheme and $\{G_t\}_{t \in T}$ a family of algebras parametrized by T and defined by a locally free $\mathcal{O}_T$-module G of rank 4. We say that this is a *family of specializations* of $\mathscr{M}_2$ if, given $t \in T$, there is a neighbourhood T_1 of t and a morphism

$f: T_1 \to \mathscr{A}$, such that $\{G_t\}_{t \in T_1}$ is the base change of $\{A_Y\}_{Y \in \mathscr{A}}$ by f, where A_Y is the algebra structure corresponding to $y \in \mathscr{A}$.

Remark 2. By Remark 3 [17], the above definition has an equivalent formulation as follows: Let $T = \operatorname{Spec} R$, and G be an R-algebra with identity e_0 such that the underlying R-module is free of rank 4. Let $J = G/Re_0$. Consider the canonical Lie algebra structure on J induced by the associative algebra structure on G. This gives a canonical skew-symmetric bilinear map $J \times J \to J$ or equivalently (in our case) an element of $J \otimes J$. Then we say the algebra G gives a *family of specializations* of $\mathscr{M}_2$ parametrized by T, if this Lie algebra structure is defined by a *symmetric element of* $J \otimes J$. Further, the algebra G is isomorphic to C_q^+, q being the corresponding quadratic form. This definition generalizes, in an obvious way, when T is any scheme, and G a vector bundle of rank 4 on T; however, the quadratic form q on J takes values in a line bundle on T.

Note 2. We shall use this reformulation in the course of this work.

Remark 3

(i) Restrict the canonical family B of specialization of $\mathscr{M}_2$ parametrized by N to the subvariety N_0. *Call this family* B_0.
(ii) For $y \in \mathscr{A}$, let A_y be the corresponding algebra structure; then $\{A_y\}_{y \in \mathscr{A}}$ gives an obvious family of specializations of $\mathscr{M}_2$.
(iii) Let $T = \operatorname{Spec} R$ and G an R-algebra giving a family of specializations of $\mathscr{M}_2$. Then by Remark 2, we get a symmetric element of $J \otimes J = G/Re_0$. This symmetric element naturally gives rise to a symmetric bilinear form on J^* (the R-dual of J) and therefore a quadratic form on J^*. Now J^* being a projective R-module of rank 3, it defines a vector bundle of rank 3 on T. More generally, if we are given an algebraic scheme T, a family $\{G_t\}_{t \in T}$ of specializations of $\mathscr{M}_2$, then we have a canonical vector bundle. V of rank 3 on T together with a $\mathcal{O}_T$-valued quadratic form $q: V \to \mathcal{O}_T$, and thus a conic bundle on T.
(iv) The families B_0 on N_0 and $\{A_y\}_{y \in \mathscr{A}}$ on $\mathscr{A}$ give generalized conic bundles on N_0 and $\mathscr{A}$ respectively.

Notation 1. Denote these conic bundles by P on N_0 and Q on $\mathscr{A}$.

PROPOSITION 2

The conic bundle P on N_0 *is locally the base change of Q on* $\mathscr{A}$ *by the local morphism* $\varphi: N_0 \to \mathscr{A}$ *of* § 2.

Proof. This is an immediate consequence of the definitions of φ, B_0 and $\{A_y\}_{y \in \mathscr{A}}$.

Remark 4. Following § 3, we introduce the *canonical subschemes*

$$\mathscr{A}_3 \subset \mathscr{A}_2 \subset \mathscr{A}_1 \subset \mathscr{A} \quad \text{and}$$
$$N_3 \subset N_2 \subset N_1 \subset N_0$$

associated to the degeneracy locus of Q and P respectively. Then, by Prop. 2. φ

locally maps $N_0 - N_2$ into $\mathscr{A} - \mathscr{A}_2$ in such a way that $N_1 - N_2 \to \mathscr{A}_1 - \mathscr{A}_2$, $N_0 - N_1 \to \mathscr{A} - \mathscr{A}_1$.

Remark 5. By Theorem 1 [17] we know that $\mathscr{A} \simeq \Phi \times \Lambda$, where Λ is the 3-dimensional affine space and Φ the 6-dimensional affine space whose points are identified with the set of quadratic forms on a fixed 3-dimensional vector space (or algebras of the form C_q^+—the even degree elements of the Clifford algebra associated to the quadratic form q). Therefore we have for $i = 1, 2, 3$

$$\mathscr{A}_i = \{q \in \Phi \mid \text{rank } q \leqslant 3 - i\} \times \mathbb{A}^3.$$

Note that

$$\mathscr{A}_0 - \mathscr{A}_1 = \{q \mid q \in \Phi, C_q^+ \simeq \mathscr{M}_2\} \times \mathbb{A}^3 \quad \text{or equivalently}$$

$$\mathscr{A}_0 - \mathscr{A}_1 = \{y \mid A_y \simeq \mathscr{M}_2\}.$$

Notation 2. We denote the subsets $N_0 - N_2$ and $N_1 - N_2$ of N_0 by Z and Y respectively.

Let $K = M_0 - M_0^s$, be the singular locus of M_0. The bundles here are of the form $L \oplus L^{-1}$, where L is a line bundle of degree 0. Let K_0 be the 'nodes' of K (i.e. consisting of bundles of the type $L \oplus L$ with L^2 trivial). Then

$$K - K_0 = L \oplus L^{-1}, \quad L \in J - \Gamma,$$

J and Γ as in §2. It may be noted that K *is a Kummer variety* of $\dim g$ (cf [6])

PROPOSITION 3

The subsets Z and Y of N_0 are precisely $N_0 - p^{-1}(K_0)$ and $p^{-1}(K - K_0)$ respectively, where $p: N_0 \to M_0$ is the desingularization morphism. In particular, $Z - Y = p^{-1}(M_0^s)$.

Proof. By Remark 3, it is enough to show that the subsets $p^{-1}(M_0^s)$ and $p^{-1}(K - K_0)$ of N_0 are mapped locally by φ into the subsets $\mathscr{A}_0 - \mathscr{A}_1$ and $\mathscr{A}_1 - \mathscr{A}_2$ of $\mathscr{A}_1$ respectively. We know that $V \in p^{-1}(M_0^s)$ if and only if $\operatorname{End} V \simeq \mathscr{M}_2$, which shows $p^{-1}(M_0^s)$ maps to $\mathscr{A}_0 - \mathscr{A}_1$.

Therefore it is enough to show that, for $E \in p^{-1}(K - K_0)$, $\operatorname{End} E$ has the same defining relations as that of the algebra C_q^+, for a quadratic form q of rank 2 on a 3-dimensional vector space.

By definition of the desingularization, the endomorphism algebras of any two points in a fibre $p^{-1}(L \oplus L^{-1})$ are isomorphic. So we consider a point E in $p^{-1}(L \oplus L^{-1})$ where $E = V \oplus W$, $V \in \operatorname{Ext}(L, L^{-1})$ $W \in \operatorname{Ext}(L^{-1}, L)$, $L \in J - \Gamma$. i.e.

$$\begin{aligned} &0 \to L \to V \to L^{-1} \to 0, \\ &0 \to L^{-1} \to W \to L \to 0. \end{aligned} \tag{1}$$

It is clear that points of this type are actually in $p^{-1}(K - K_0)$. Using (1), it is easy to see that $\operatorname{End}(V \oplus W)$ has four generators, which in terms of block matrices can be described as

$$x=\begin{pmatrix}0 & 0\\ \gamma_2 & 0\end{pmatrix}\quad w=\begin{pmatrix}0 & \gamma_1\\ 0 & 0\end{pmatrix}\quad u=\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}\quad v=\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix},$$

where $I=2\times 2$ identity matrix, and γ_1,γ_2 coming from identification of the line bundles in the exact sequence (1). The defining relations can be given as

$$\begin{aligned}&u^2=u,\quad v^2=v,\quad uv=0,\quad u+v=I,\\ &w^2=x^2=wx=0,\quad uw=w,\quad wu=0,\\ &ux=0,\quad xu=x,\quad vw=0,\quad wv=w\\ &vx=x,\quad xv=0.\end{aligned}\tag{2}$$

If q is a quadratic form of rank 2 on a 3-dimensional vector space over an algebraically closed field k then it is easily seen that C_q^+ = the even degree elements of the Clifford algebra of q is a 4-dimensional k-algebra with

$$\begin{aligned}&C_q^+=k+k\alpha+k\beta+k\gamma\quad\text{such that}\\ &\alpha^2=-1,\quad \alpha\beta=-\gamma,\quad \alpha\gamma=\beta\\ &\beta\alpha=\gamma,\quad \gamma\alpha=-\beta.\end{aligned}$$

Now put $a=\frac{1}{2}(1+i\alpha)$, $b=\frac{1}{2}(1-i\alpha)$, $c=i\beta+\gamma$, $d=i\beta-\gamma$, where $i=\sqrt{-1}\in k$. Then a,b,c,d are new generators of C_q^+ with the following defining relations

$$\begin{aligned}&a^2=a,\quad b^2=b,\quad ab=0,\quad a+b=1,\\ &c^2=d^2=cd=0,\quad ac=c,\quad ca=0,\\ &ad=0,\quad da=d,\quad bc=0,\quad cb=c,\\ &bd=d,\quad db=0.\end{aligned}\tag{3}$$

A glance at (2) and (3) proves our claim.

Q.E.D.

COROLLARY 1

$Y\xrightarrow{p}K-K_0$ is a $\mathbb{P}^{g-2}\times\mathbb{P}^{g-2}$ fibration associated to a vector bundle on $K-K_0$.

Proof. Indeed; we claim that, if $E\in Y=p^{-1}(K-K_0)$ then $E=V\oplus W$, for some $V\in\mathbb{P}(\mathrm{Ext}(L,L^{-1}))$, $W\in\mathbb{P}(\mathrm{Ext}(L^{-1},L))$ $L\in J-\Gamma$.

Let $E\in p^{-1}(K-K_0)$; then, End W has four generators x,w,u,v with defining relations (2) as in Prop. 3. Consider $u\in\mathrm{End}\,E$, and let $V=\ker u$. Then V is a subbundle of E and we have an exact sequence

$$0\to V\to E\to W\to 0.$$

It is clear then that W is in fact $\ker v$, $v\in\mathrm{End}\,E$ and therefore we get a splitting of the exact sequence, implying $E=V\oplus W$.

Now using Prop. 1 of [17], V and W cannot be of the type $L\oplus L$ or $L^{-1}\oplus L^{-1}$. For the same reason, since $E\in PV_4$, we rule out $V=L\oplus L^{-1}$, $W=L^{-1}\oplus L$. Hence we are left with $V\in\mathbb{P}$ (Ext(L,L^{-1})), $W\in\mathbb{P}$ (Ext(L^{-1},L)) or vice versa.

Note that for $L\in K-K_0$, $\mathrm{Ext}(L,L^{-1})=H^1(X,L^{-2})$ has dimension $g-1$ and therefore Y is a $\mathbb{P}^{g-2}\times\mathbb{P}^{g-2}$ fibration over $K-K_0$. The vector bundle to which this is associated has fibre at any $L\in K-K_0$ to be $\mathrm{Ext}(L,L^{-1})\oplus\mathrm{Ext}(L^{-1},L)$.

COROLLARY 2

The fibration $Y\xrightarrow{p}K-K_0$ is locally trivial in the Zariski topology.

Proof. This follows from Cor. 1 and Serre (cf [15]).

PROPOSITION 4

Let $P-P_2$ be the restriction of the conic bundle P over points of N_0-N_2 (i.e. Z). Then the total space of $P-P_2$ is smooth.

Proof. By Prop. 2, $P-P_2$ is locally the base change of $Q-Q_2$ (the restriction of Q over points of $\mathscr{A}-\mathscr{A}_2$). Since $\varphi:N_0\to\mathscr{A}$ is a smooth local morphism, the total space of $P-P_2$ is smooth if and only if the total space of $Q-Q_2$ is so.

Consider any point $(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{A}^6$. This defines a quadratic form

$$q=a_1X^2+a_2XY+a_3Y^2+a_4XZ+a_5YZ+a_6Z^2.$$

We therefore have a conic bundle C over $\mathbb{A}^6$ by considering the conics defined by the quadratic forms. By Remark 4 it is clear that the conic bundle Q on $\mathscr{A}$ is 'essentially' the conic bundle C. Thus we would have proved our claim if we show that the total space of $C\to\mathbb{A}^6-S^1$ is smooth, where S is the degeneracy locus of C and $S'\subset S$ its singular locus. We have in fact more.

Lemma 1. Let $\theta:C\to\mathbb{A}^6$ be the canonical morphism. Then $\theta^{-1}(\mathbb{A}^6-(0))$ is smooth.

Proof. Let $P\in C$ be any point. Then P can be given by $(a_1,a_2,a_3,a_4,a_5,a_6,X,Y,Z)$ where not all $a_i=0$ and not all $X,Y,Z=0$, P lying on the conic defined by $q=a_1X^2+a_2XY+a_3Y^2+a_4XZ+a_5YZ+a_6Z^2$. Taking partial derivatives of q with respect to a_i, $i=1,\dots,6$, we have

$$\partial q/\partial a_i=0,\quad i=1,\dots,6\Rightarrow X=Y=Z=0.$$

Q.E.D.

4. Cohomology computations

4.1 The Gysin map

Let W be a conic bundle of type I (cf Def. 2) on a variety S. This gives rise to a topological Brauer class b_W in $H^3(S,\mathbb{Z})_{\mathrm{tors}}$ (i.e. the torsion subgroup of $H^3(S,\mathbb{Z})$).

Let W be a conic bundle of type II (cf Def. 2). Then if W degenerates to a pair of lines over an irreducible divisor $S_1 \subset S$, the restriction W_1 of W over S, gives rise in a natural way to a double cover of S_1 (cf Lemma on p. 29 of [8]) and $W - W_1$ is a conic bundle of type I over $S - S_1$. We shall denote by 'α' the element in $H^2(S_1, \mathbb{Z})$ coming from this double cover. Consider the part of the Gysin sequence for $S_1 \subset S$ which involves $H^3(S, \mathbb{Z})$, i.e.

$$H^1(S_1, \mathbb{Z}) \to H^3(S, \mathbb{Z}) \to H^3(S - S_1, \mathbb{Z}) \xrightarrow{g} H^2(S_1, \mathbb{Z}).$$

Then we have here the

Theorem 2. (*Nitsure* [9], [11]) *Let W be a conic bundle of type II on S. If the total space of W is smooth, then the image of $b_{W-W_1} \in H^3(S - S_1, \mathbb{Z})_{\text{tors}}$ under the Gysin map g, is precisely $\alpha \in H^2(S_1, \mathbb{Z})$. In particular if $\alpha \neq 0$, then $b_{W-W_1} \neq 0$.*

PROPOSITION 5

Let W be a conic bundle of type I over S where $H^1(S, \mathbb{Z}) = 0$ and with $b_W \neq 0$ in $H^3(S, \mathbb{Z})_{\text{tors}}$. Suppose that there exists another topological $\mathbb{P}^1$ – bundle $U \to S$ with the property that $H^3(U, \mathbb{Z})_{\text{tors}} = (0)$. Then $b_W = \pm b_U$ and $H^3(S, \mathbb{Z})_{\text{tors}}$ is generated by b_W.

Proof. To prove this proposition, we shall appeal to the following well-known (cf [11]).

Lemma 2. Let $U \to S$ be a $\mathbb{P}^1$ – bundle over a path connected space S with $H^1(S) = 0$. Then the kernel of the induced homomorphism $H^3(S, \mathbb{Z}) \to H^3(U, \mathbb{Z})$ is generated by b_U.

We now apply the lemma to $U \to S$. Since we have $H^3(U, \mathbb{Z})_{\text{tors}} = (0)$, we get $H^3(S, \mathbb{Z})_{\text{tors}}$ to be generated by b_U, which is a 2-torsion element. Also b_W lies in $H^3(S, \mathbb{Z})_{\text{tors}}$, and $b_W \neq 0$ which implies $b_W = \pm b_U$. This proves Prop. 5.

The next step is to construct explicitly a $\mathbb{P}^1$ – bundle on the subspace $Z - Y$ which satisfies the property of Prop. 5. For this purpose, we elaborate in some detail, what is called the 'Hecke correspondence' of [7], in terms of parabolic bundles as remarked in (*).

Let V be a vector bundle on X of rank 2 and degree 0. Suppose we are given a parabolic structure at a point $x \in X$, defined by a 1-dimensional subspace

$$F^2 V_x \subset F^1 V_x = V_x \quad \text{and weights } (\alpha_1, \alpha_2) \text{ such that}$$

(i) parabolic stable = parabolic semi-stable,
(ii) parabolic stable ⇒ underlying bundle is semi-stable, and
(iii) underlying bundle stable ⇒ any parabolic structure is stable.

Let T be the torsion $\mathcal{O}_x$-module given by

$$T_x = V_x / F^2 V_x, \quad T_y = 0, \quad x \neq y.$$

(*) Mehta V and Seshadri C S, Moduli of vector bundles on curves with parabolic structures. *Math. Ann.* **248** (1980) 205–239.

Then we have a homomorphism of V onto T (as $\mathcal{O}_X$-modules). If W is the kernel of this map, we have $0 \to W \to V \to T \to 0$ and W is locally free of rank 2 and degree -1.

Let $\tilde{M}$ be the moduli space of parabolic stable bundles of rank 2, degree 0 on X and M_{-1} the moduli space of stable bundles of rank 2, degree -1, $f: \tilde{M} \to M$, the canonical morphism; and $\tilde{M}_0 = f^{-1}(M_0)$.

PROPOSITION 6

If $V \in \tilde{M}$ then W defined above, is in M_{-1} and the map $\psi: \tilde{M} \to M_{-1}$, $V \mapsto W$ is a $\mathbb{P}^1$-bundle, locally trivial in the Zariski topology. In fact it is the dual projective Poincaré bundle on M_{-1}.

Proof. We first claim that if V is parabolic stable then W is stable. To see this, let $F \subset W$ be a line subbundle. We need to show that $\deg F < 0$. Suppose this is not the case i.e. $\deg F \geqslant 0$.

Let G be the line subbundle of V generated by the image of F in V. Then $\deg F \leqslant \deg G$. Since the underlying bundle of V is certainly semi-stable, we have $\deg G \leqslant 0$. By our assumption $\deg F \geqslant 0$ and hence we have $\deg F = \deg G = 0$. This implies that the canonical homomorphism $F \to G$ is an isomorphism. We also see that by the definition of T

$$G_x \subset F^2 V_x,$$

but V being parabolic stable with weights $0 < \alpha_1 < \alpha_2$ we get

$$\operatorname{par\,deg} G = \alpha_2 < \tfrac{1}{2}(\alpha_1 + \alpha_2) = \operatorname{par\,deg} V / \operatorname{rk} V$$

which leads to a contradiction. Hence W is stable. Conversely, we claim that $\tilde{M}$ is isomorphic to the dual projective Poincaré bundle of M_{-1} restricted to M_{-1}. To see this, we start with a $W \in M_{-1}$. Then, given a point in $\mathbb{P}(W_x^*)$, $x \in X$, one can easily obtain a vector bundle V of rank 2 and degree 0 and an injection $W \to V$ as $\mathcal{O}_x$-modules. The cokernel then gives a 1-dimensional subspace $F^2 V_x$ of V_x and therefore a 'quasi-parabolic structure'. The stability of W together with an argument as above, makes V parabolic stable. That this map is an *isomorphism* is a consequence of the universal property of the moduli space of parabolic stable bundles.

That $\tilde{M} \to M_{-1}$ is locally trivial in the Zariski topology, now follows from Serre [15].

Q.E.D.

PROPOSITION 7

Consider the canonical morphism $f: \tilde{M}_0 \to M_0$. Then f is a $\mathbb{P}^1$-fibration over M_0^s and $f^{-1}(K)$ has codimension $g-1$ in $\tilde{M}_0$.

Proof. That f is a $\mathbb{P}^1$-fibration over M_0^s is immediate by the property (3) mentioned before Prop. 6. Let $L \oplus L^{-1} \in K - K_0$. Then the points of $\tilde{M}_0$ lying over $L \oplus L^{-1}$ are of the following form:

Case 1. V is a non-trivial extension of L^{-1} by L (or L by L^{-1})

We claim that a parabolic structure on V which is equivalent to giving a subspace $F^2 V_P$ of V_P of dimension one, is stable if and only if $L_P \not\subset F^2 V_P$. This is necessary to ensure parabolic stability, for otherwise if $L_P \not\subset F^2 V_P$, then per deg $L = \deg L + \alpha_2 = \alpha_2$ and $\alpha_2 \nless$ par deg $V/\text{rk } V = \frac{1}{2}(\alpha_1 + \alpha_2)$, since $\alpha_1 < \alpha_2$.

Case 2. $V = L \oplus L^{-1}$

We claim that a parabolic structure $F^2 V_P$ such that $F^2 V_P \neq L_P$ or L_P^{-1} is stable. This is easily checked as above. In fact we see by an argument as in Prop. 1 of [17] all the parabolic structures of Case 2 are isomorphic and hence give one point of M. Hence the total dimension of the fibre at $L \oplus L^{-1} = \dim \text{Ext}(L, L^{-1}) + 1 = g - 1$. Therefore, $\dim f^{-1}(K - K_0) = 2g - 1$.

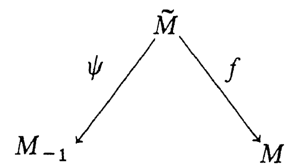
In fact, it is not difficult to see that for $x \in K - K_0, f^{-1}(x)$ is the union of two projective spaces of dimension $g - 1$ meeting at a point.

Finally, let $V \in M_0$ be such that gr $V = L \oplus L$. (L of order two). Then the following can easily be checked.

(i) V has a parabolic stable structure if and only if V is a non-trivial extension of L by L.
(ii) A parabolic structure given by $F^2 V_P$ is stable iff $F^2 V_P \neq L_P$ (where L is the unique line subbundle of V).

Once again by an argument as in Prop. 1 [17] we see that all the parabolic structures on a non-trivial extension V of L by L are isomorphic. Hence the fibre of f over $L \oplus L$ is isomorphic to $\mathbb{P}(H^1(X, \mathcal{O}_X))$ which has dimension $g - 1$, implying codim $(f^{-1}(K), \tilde{M}_0) = g - 1$.

Remark 6. Thus we have the following diagram



which gives a correspondence between M_{-1} and M.

PROPOSITION 8

The fibration $Y \xrightarrow{p} K - K_0$ with fibre $F = \mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ satisfies the conditions of the Leray–Hirsch theorem and consequently we have

$$H^*(Y, \mathbb{R}) \simeq H^*(K - K_0, \mathbb{R}) \otimes H^*(F, \mathbb{R}).$$

Proof. The following form of the Leray–Hirsch theorem will suit our purposes.

Leray–Hirsch. Let E be a fibre bundle over B and compact fibre F. Suppose B has a finite good cover. If there are global cohomology classes $e_1, \ldots, e_r$ on E which, when restricted to each fibre freely, generate the cohomology of the fibre, then $H^*(E, \mathbb{R})$ is a free-module over $H^*(B, \mathbb{R})$ with basis $e_1, \ldots, e_r$; or more precisely, if the canonical map $j: H^*(E, \mathbb{R}) \to H^*(F, \mathbb{R})$, is surjective, then for any subspace W of $H^*(E, \mathbb{R})$ such that $j | W: W \to H^*(F, \mathbb{R})$ is an isomorphism, one has

$$H^*(E, \mathbb{R}) = H^*(B, \mathbb{R}) \otimes W.$$

Since F in our case is $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}, H^*(F, \mathbb{R})$ is generated by line bundles on F. Therefore it is enough to check that any line bundle on F can be extended to a line bundle on Y.

By Cor. 2, $Y \to K - K_0$ is locally trivial in the Zariski topology. Let L be a line bundle on F, and $U \subset K - K_0$ be the trivializing Zariski open subset. Then L can be obviously extended to a line bundle on $U \times F$, which we continue to denote by L. Since Y is *smooth*, the bundle L on the open subset $U \times F$ of Y can be extended to a line bundle on Y.

Q.E.D.

PROPOSITION 9

The element $\alpha \in H^2(Y, \mathbb{Z})$, associated to the double cover on Y arising from the conic bundle P is non-zero.

Proof. We claim that this double cover on Y is in fact the pull-back of the covering

$$J - \Gamma \to K - K_0.$$

J being the Jacobian of line bundles of deg 0 on X [for notations cf. §2].

Since this covering is non-split, it follows from Prop. 3, that the double cover on Y is non-split and the covering element in $H^1(Y, \mathbb{Z}/(2))$ is non-zero.

By Prop. 8 and Spanier [19], $H^1(Y, \mathbb{Z}) = 0$. Hence if we consider the cohomology exact sequence for

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/(2) \to 0$$

we get

$$H^1(Y, \mathbb{Z}/(2)) \hookrightarrow H^2(Y, \mathbb{Z}).$$

Since $\alpha \in H^2(Y, \mathbb{Z})$ is the image of the covering element in $H^1(Y, \mathbb{Z}/(2))$, it is non-zero.

Thus to complete the proof of Prop. 9, it is enough to prove the claim.

Fix $t_0 \in X$. Then if $E \in N_0$, one can easily see that E_{t_0} can be identified with right regular representation of $A = \operatorname{End} E$ (see for e.g. Prop. 5 [17]).

Let $E = V \oplus W$ be an element of Y as in Prop. 3. It is easy to see that the scalars in A do not meet V_{t_0} and W_{t_0} under the above identification. So if we consider the projective space $\mathbb{P}(A')$, $A' = A/(\text{scalars})$, then V_{t_0} and W_{t_0} give a pair of lines in $\mathbb{P}(A')$. By Prop. 3, identifying the algebra A with a C_q^+ corresponding to a quadratic form q in Φ, it is clear that this pair of lines is indeed the ones in the conic bundle over Y.

Then the one-dimensional subspaces L_{t_0} and $L_{t_0}^{-1}$ give a pair of points $\bar{L}_{t_0}$ and $\bar{L}_{t_0}$ in $\mathbb{P}(A')$. Then the correspondence

$$E \mapsto (\bar{L}_{t_0}, \bar{L}_{t_0}^{-1})$$

gives a double covering on Y since we have a defining family of vector bundles $E_y = \{V_y \oplus W_y\}_{y \in Y}$. Obviously, this is the canonical double cover associated to the conic bundle on Y.

Note that $\{L_y \oplus L_y^{-1}\}_{y \in Y}$ gives a family on Y which is clearly the pull-back $p^*\{L_u \oplus L_u^{-1}\}_{u \in K - K_0}$, under $p: Y \to K - K_0$.

The double cover of Y given above is therefore the pull-back of the double cover of $K - K_0$ given by $J - \Gamma \to K - K_0$.

Q.E.D.

PROPOSITION 10

(a) *Let Z and Y be as in* §3. *Then there exists a topological* $\mathbb{P}^1$ *- bundle D on* $Z-Y$ *with* $H^*(D,\mathbb{Z})$ *torsion free. In fact* $D=f^{-1}(M_0^s)$.
(b) *The topological Brauer class* $b_D \neq 0$.

Proof. (a) By Prop. 7, $f^{-1}(K)$ has codimension $g-1$ in $\tilde{M}_0$ and $D=\tilde{M}_0-f^{-1}(K)$.

Consider $\psi:\tilde{M}_0\to M_{-1,x}, M_{-1,x}$ being bundles in M_{-1} with determinant L_x. Since the $\mathbb{P}^1$ fibration ψ is locally trivial in the Zariski topology, a line bundle L on the fibre $\mathbb{P}^1$ can be extended obviously to $\mathbb{P}^1\times U$, where U is a Zariski open subset of $M_{-1,x}$. Since $\tilde{M}_0$ is smooth, the closure of L in $\tilde{M}_0$ gives a line bundle on $\tilde{M}_0$. Now, the cohomology of $\mathbb{P}^1$ is generated by line bundles and therefore we can apply Leray–Hirsch theorem to conclude that the cohomology groups of $\tilde{M}_0$ are those of $\mathbb{P}^1\times M_{-1,x}$.

By Atiyah–Bott [2], all the cohomology groups of $M_{-1,x}$ are torsion-free and therefore all the cohomology groups of $\tilde{M}_0$ are also torsion-free.

Since $g\geqslant 3$, the complex codimension of $f^{-1}(k)$ in $\tilde{M}_0=g-1\geqslant 2$. This implies $\mathrm{Codim}_{\mathbb{R}} f^{-1}(K)$ in $\tilde{M}_0\geqslant 4=g-1\geqslant 2$.

Consider the homology sequence of the pair $(\tilde{M}_0,D)$

$$H_k(\tilde{M}_0,D,\mathbb{Z})\to H_{k-1}(D,\mathbb{Z})\to H_{k-1}(\tilde{M}_0,\mathbb{Z})\to H_{k-1}(\tilde{M}_0,D,\mathbb{Z})$$

$\tilde{M}_0$ is a compact complex manifold and so we can apply Alexander duality to the pair $(\tilde{M}_0,D)$ to get

$$\begin{aligned} H_k(\tilde{M}_0,D,\mathbb{Z}) &\simeq H^{n-k}(\tilde{M}_0-D;\mathbb{Z})\\ &= H^{n-k}(f^{-1}(K),\mathbb{Z})\\ n &= \dim_{\mathbb{R}}\tilde{M}_0. \end{aligned}$$

Since $\dim_{\mathbb{R}} f^{-1}(K)\leqslant n-4$, we therefore get

$$H_2(\tilde{M}_0,D;\mathbb{Z})=H^{n-2}(f^{-1}(K),\mathbb{Z})=0$$

and similarly $H_3(\tilde{M}_0,D,\mathbb{Z})=0$.

$$H_2(D,\mathbb{Z})=H_2(\tilde{M}_0,\mathbb{Z}).$$

By the 'universal coefficient theorem' one has torsion subgroup of $H_k(T,\mathbb{Z})$ to be that of $H^{k+1}(T,\mathbb{Z})$, T any topological space, and therefore we conclude that

$$H^3(D,\mathbb{Z})_{\text{tors}}=H^3(\tilde{M}_0,\mathbb{Z})_{\text{tors}}=(0).$$

Note that $Z-Y=M_0^s$ and this completes the proof.

Q.E.D.

The claim (b) is due to Ramanan (p. 52 [18])

Theorem 3. $H^3(Z,\mathbb{Z})$ *is torsion free.*

Proof. Consider the Gysin sequence for $(Z, Z - Y)$,

$$H^1(Y, \mathbb{Z}) \to H^3(Z, \mathbb{Z}) \to H^3(Z - Y, \mathbb{Z}) \xrightarrow{g} H^2(Y, \mathbb{Z})$$

Now by Cor. 2, Y is a $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ fibration over $K - K_0$ and by ([19] p. 159) $H^1(K - K_0, \mathbb{Z}) = 0$ implying by standard arguments $H^1(Y, \mathbb{Z}) = 0$ (note that $H^1(Y, \mathbb{Z})$ is torsion-free by the universal coefficient theorem).

Thus we have from the Gysin sequence an injection

$$H^3(Z, \mathbb{Z}) \hookrightarrow H^3(Z - Y, \mathbb{Z}). \qquad (*)$$

Now note that $H^1(Z - Y, \mathbb{Z}) = 0$. (This follows for example from the Gysin sequence. For, note that $H^1(Z - Y, \mathbb{Z}) \simeq H^1(Z, \mathbb{Z}.)$ Also we will be seeing in § 5 that the codimension of $N_0 - Z$ in N_0 is actually 6. But N_0 is unirational and is therefore simply connected, being smooth projective (cf. Serre [16]). Hence $H^1(N_0, \mathbb{Z}) = 0$ implying $H^1(Z, \mathbb{Z}) = 0 = H^1(Z - Y, \mathbb{Z})$.

Thus we can now apply Prop. 5 and Prop. 10 to see that $H^3(Z - Y, \mathbb{Z})_{\text{tors}}$ is generated by b_{P-P_1}, the Brauer element coming from the conic bundle $P - P_1$ over $N_0 - N_1 = Z - Y$. By Prop. 4 the total space of $P - P_2$ is smooth and hence the theorem due to Nitsure mentioned in § 4·1 is applicable. Thus we have

$$g(b_{P-P_1}) = \alpha \neq 0 \quad (\alpha \neq 0 \text{ by Prop. 9}).$$

This together with (*) and the exactness of the Gysin sequence gives $H^3(Z, \mathbb{Z})_{\text{tors}} = (0)$ Q.E.D.

Lemma 3. Pic Z *is generated by* Pic$(Z - Y)$ *and the element* $[Y]$ *coming from the irreducible divisor* $Y \subset Z$.

Proof. This follows from the following general fact:

If X is a smooth variety, $U \subset X$ open with $Y = X - U$ an irreducible divisor, then

$$\operatorname{Pic} X \to \operatorname{Pic} U$$

is a surjection and the kernel of this homomorphism is generated by $[Y]$.

Lemma 4. Let $N_1 \subset N_0$ *be as in* § 3. *Then* Pic N_0 *is generated by* Pic M_0 *and* $[N_1]$ *over* $\mathbb{Q}$(*).

Proof. Firstly, we remark that N_1 is precisely $\bar{Y}$ in N_0. Actually, we will be showing in § 5 that $Y \subset N_1$ is precisely the set of non-singular points of N_1. Let us assume this. Suppose N_1 is not irreducible and let A, B be subvarieties such that $N_1 = A \cup B$. Then $A \cap B \subset N_1 - Y$ and hence $A \cap Y$ and $B \cap Y$ will disconnect Y which

(*) In fact, over $\mathbb{Z}$ (see Remark in Appendix 2).

is false since Y is connected. Thus N_1 is irreducible. Also since Y is irreducible it follows that $\bar{Y}=N_1$.

An application of Lemma 4 and the result of Appendix 2 yields our result.

Remark 7. Thus by the above lemma, any $L\in \mathrm{Pic}\, N_0$ can be expressed as $L=aL_1+bL_2$, $L_1=[N_1]$ and $L_2\in \mathrm{Pic}\, M_0$, $a, b\in \mathbb{Q}$.

In particular, let L be chosen ample. Then if F is the fibre of $Y\to K-K_0$, L when restricted to F is $(aL_1+bL_2)|F$. But since $L_2\in \mathrm{Pic}\, M_0$, which is trivial on F, we have

$$L|F=(aL_1)|F$$

F is $\mathbb{P}^{g-2}\times\mathbb{P}^{g-2}$ and L is ample, therefore we have the restriction of L_1 to each $\mathbb{P}^{g-2}$ to be either ample or negatively ample.

Let $e\in H^2(Y,\mathbb{R})$ be the Euler class of the irreducible divisor Y in Z. Then by the 'adjunction formula', we have

$$e=[Y]|_Y,$$

where $[Y]$ is the class of $Y\subset Z$. Now $L_1=[N_1]$ and $N_1=\bar{Y}$, hence it follows from the above reasoning that *the Euler class e when restricted to the factors of F is ample or negatively ample.*

PROPOSITION 11

Let E be the normal bundle of Y in Z and E_0 be the compliment of the zero section. Consider the Gysin sequence for the 2-plane bundle (E, E_0)

$$H^k(Y,\mathbb{R})\to H^{k+2}(Y,\mathbb{R})\to H^{k+2}(E_0,\mathbb{R})\to H^{k+1}(Y,\mathbb{R})\to H^{k+3}(Y,\mathbb{R}).$$

Then the Gysin homomorphism

$$h: H^k(Y,\mathbb{R})\to H^{k+2}(Y,\mathbb{R}),$$

given by 'wedging' with the Euler class $e\in H^2(Y,\mathbb{R})$ is an injection for $k\leqslant \dim_{\mathbb{R}}\mathbb{P}^{g-2}-2=2g-6$.

Proof. By Prop. 8 we have

$$H^k(Y)\simeq \sum_{l+m=k} H^l(K-K_0)\otimes H^m(F)$$

or using the subspace W of $H^*(Y)$ as in Prop. 3.8, we have, any $u\in H^k(Y)$ $u\neq 0$ and $k\leqslant \dim_{\mathbb{R}} F$, to be expressible as

$$v=\sum_i u_i\otimes w_i,\quad u_i\in H^*(K-K_0),\quad w_i\in W,$$

where not all $w_i=0$ (this is so since $k\leqslant \dim_{\mathbb{R}} F$). Without loss of generality, the u_i's can be chosen linearly independent.

Now consider $u\otimes e$, e the Euler class in $H^2(Y,\mathbb{R})$

$$u\otimes e=\sum_i u_i\otimes(w_i\otimes e).$$

Consider the class $w_i \otimes e$. This when restricted to the fibre F is non-zero, since by Remark 7, the class e restricted to the factors of F is ample or negatively ample and w_i by definition lies in W and so $w_i \wedge e$ is non-zero on F for $w_i \in H^k(F, \mathbb{R})$, $k \leqslant \dim_{\mathbb{R}} \mathbb{P}^{g-2} - 2$. Hence by the linear independence of the u_i's we get

$$u \otimes e = \sum_i u_i \otimes (w_i \otimes e) \neq 0$$

$\Rightarrow \quad h: H^k(Y, \mathbb{R}) \to H^{k+2}(Y, \mathbb{R})$ is an injection for $k \leqslant \dim_{\mathbb{R}} \mathbb{P}^{g-2} - 2 = 2g - 6$.

COROLLARY 3

The Gysin map considered in Theorem 3 i.e.

$$h^1: H^k(Y, \mathbb{R}) \to H^{k+2}(Z, \mathbb{R})$$

is also an injection for $k \leqslant 2g - 6$.

Proof. In fact, the Gysin sequences for (E, E_0) and $(Z, Z - Y)$ are related as follows.

$$\begin{array}{ccc} H^k(Y, \mathbb{R}) & \xrightarrow{\quad h \quad} & H^{k+2}(Y, \mathbb{R}) \\ & {\scriptstyle h'} \searrow \qquad \nearrow {\scriptstyle \mathrm{Res}} & \\ & H^{k+2}(Z, \mathbb{R}) & \end{array}$$

and therefore, since h is an injection by Prop. 11, so is h^1..

COROLLARY 4

$H^k(Z, \mathbb{R}) = H^{k-2}(Y, \mathbb{R}) \oplus H^k(Z - Y, \mathbb{R})$ $k \leqslant 2g - 4$.

Proof. Consider the Gysin sequence for $(Z, Z - Y)$.

$$\to H^{k-2}(Y, \mathbb{R}) \to H^k(Z, \mathbb{R}) \to H^k(Z - Y, \mathbb{R}) \to H^{k-1}(Y, \mathbb{R}) \to H^{k+1}(Z, \mathbb{R}) \to$$

Since h' is an injection for $k \leqslant 2g - 6$, we get

$$0 \to H^{k-2}(Y, \mathbb{R}) \to H^k(Z, \mathbb{R}) \to H^k(Z - Y, \mathbb{R}) \to 0$$

for $k \leqslant 2g - 4$ and this proves the corollary.

Remark 8. By Kirwan [5], the Betti numbers of M_0^s are known if genus $g \geqslant 4$, for $i < 2g - 3$. This together with Prop. 8, Cor. 4 and Spanier [19], yields the Betti numbers of Z for $i < 2g - 3$.

Remark 9. Let us assume $g \geqslant 4$ and recall from Prop. 10, we had a topological $\mathbb{P}^1$-bundle D on $Z - Y$. By the proof of Prop. 10 we see that if $g \geqslant 4$, then $\mathrm{codim}_{\mathbb{R}} f^{-1}(K)$ in $M_0 \geqslant 6$ and hence

$$H_k(D, \mathbb{Z}) = H_k(\tilde{M}_0, \mathbb{Z}) \quad \text{for } k \leqslant 4.$$

The homology groups of $\tilde{M}_0$ are known by [10] or by using Atiyah–Bott [2] for $M_{-1,x}$. In particular, rank of $H_3(\tilde{M}_0, \mathbb{Z})$ is $2g$ and hence rank of $H_3(D, \mathbb{Z})$ is $2g$.

We have already seen that $H^1(Z - Y, \mathbb{R}) = 0$. Now D is a $\mathbb{P}^1$-fibration over $Z - Y$ and $H^1(\mathbb{P}^1, \mathbb{R}) = 0, H^1(Z - Y, \mathbb{R}) = 0$. Therefore by the Serre sequence of this fibration (see for example Spanier *Algebraic topology* pp. 519) we get an exact sequence

$$H_3(\mathbb{P}^1,\mathbb{R}) \to H_3(D,\mathbb{R}) \to H_3(Z-Y,\mathbb{R}) \to H_2(\mathbb{P}^1,\mathbb{R}) \to H_2(D,\mathbb{R}) \to H_2(Z-Y,\mathbb{R}) \to H_1(\mathbb{P}^1,\mathbb{R}).$$

Now, $H_3(\mathbb{P}^1, \mathbb{R}) = H_1(\mathbb{P}^1, \mathbb{R}) = 0, H_2(\mathbb{P}^1, \mathbb{R}) \simeq \mathbb{R}$. Thus we have

$$0 \to H_3(D, \mathbb{R}) \to H_3(Z - Y, \mathbb{R}) \to H_2(\mathbb{P}^1, \mathbb{R}) \to H_2(D, \mathbb{R})$$

$H_2(Z - Y, \mathbb{R}) \to 0$. By the Picard group computations it follows that, $H_2(D, \mathbb{R}) = \mathbb{R}^2$ and $H_2(Z - Y, \mathbb{R}) = \mathbb{R}$, and therefore we have

$$\text{rank of } H_3(Z - Y, \mathbb{R}) = \text{rank}\, H_3(D, \mathbb{R}) = 2g.$$

Thus the rank of $H_3(Z - Y, \mathbb{R}) = 2g$ and hence the rank of $H^3(Z - Y, \mathbb{R})$ is $2g$.

Theorem 4. $H^3(Z, \mathbb{Z}) = \mathbb{Z}^{2g}$, *when* $g \geqslant 4$.

Proof. By Theorem 3 $H^3(Z, \mathbb{Z})$ is torsion-free. By Cor. 4

$$H^3(Z, \mathbb{R}) = H^1(Y, \mathbb{R}) \oplus H^3(Z - Y, \mathbb{R})$$

Since $H^1(Y, \mathbb{R}) = 0$, using Remark 9 we conclude that $H^3(Z, \mathbb{Z}) = \mathbb{Z}^{2g}$.

5. The main theorem

Consider the stratification of N_0 in terms of the degeneracy locus as in § 3, $N_3 \subset N_2 \subset N_1 \subset N_0$.

PROPOSITION 12

The subvariety N_2 has codimension 3 in N_0.

Proof. Consider the local morphism

$$\varphi: N_0 \to \mathscr{A}$$

of § 2. We have already seen that $\varphi: N_1 \to \mathscr{A}_1$ and $\varphi: N_2 \to \mathscr{A}_2$. Moreover, φ being a smooth local morphism, its fibres are equidimensional. Hence the codimension of N_2 in N_0 equals the codimension of $\mathscr{A}_2$ in $\mathscr{A}$. We have also seen that $\mathscr{A}_1 \subset \mathscr{A}$ is a hypersurface given by $\Delta = 0$ and $\mathscr{A}_2 \subset \mathscr{A}_1$ is precisely the singular locus of $\mathscr{A}_1$. So we would like to show that

$$\text{codim of } \mathscr{A}_2 \text{ in } \mathscr{A}_1 = 2.$$

Consider the natural conic bundle C on $\mathbb{A}^6$ as in Lemma 1. Let S be the hypersurface of $\mathbb{A}^6$ given by $\Delta = 0$ and let $S^1 \subset S$ be its singular locus. Then by Remark 5, it is enough to show that

codim of S^1 in $S = 2$.

By definition, if

$$q = aX^2 + bY^2 + cZ^2 + fYZ + gXZ + hXY,$$

then Δ is given by

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Thus, if $\mathrm{Sym}(\mathcal{M}_3)$ is all (3×3)-symmetric matrices

$$S = \{A \in \mathrm{Sym}(\mathcal{M}_3) | \text{rank } A \leqslant 2\}.$$

The conditions $\partial\Delta/\partial a = \partial\Delta/\partial b = \partial\Delta/\partial c = \partial\Delta/\partial f = \partial\Delta/\partial g = \partial\Delta/\partial h = 0$, gives

$$bc = f^2, \quad ac = g^2, \quad ab = h^2, \quad af = hg, \quad fh = bg, \quad ch = fg.$$

$$a/h = h/b = g/f \quad \text{and} \quad a/g = h/f = g/c$$

i.e. $\quad S^1 = \{A \in \mathrm{Sym}(\mathcal{M}_3) | \text{rank } A \leqslant 1\}.$

From which we obtain the codim of S^1 in S.

Q.E.D.

COROLLARY 5

$H_k(N_0, \mathbb{Z}) = H_k(Z, \mathbb{Z}), k \leqslant 4.$

Proof. Consider the homology sequence of the pair (N_0, Z)

$$H_{k+1}(N_0, Z; \mathbb{Z}) \to H_k(Z, \mathbb{Z}) \to H_k(N_0, \mathbb{Z}) \to H_k(N_0, Z; \mathbb{Z}).$$

Since N_0 is a compact complex manifold, the Alexander duality as in Theorem 3, gives

$$H_k(N_0, Z, \mathbb{Z}) \simeq H^{n-k}(N_0 - Z, \mathbb{Z}) = H^{n-k}(N_2, \mathbb{Z}).$$

$$n = \dim_{\mathbb{R}} N_0.$$

By Prop. 12, $\dim_{\mathbb{R}} N_2 = n - 6$ since $\mathrm{codim}_{\mathbb{C}}(N_2, N_0) = 3$. Hence $H^{n-k}(N_2, \mathbb{Z}) = 0$ for $k < 6$.

$$\Rightarrow \quad H_k(N_0, Z, \mathbb{Z}) = 0 \quad k < 6$$

$$\Rightarrow \quad H_k(N_0, \mathbb{Z}) = H_k(Z, \mathbb{Z}), \quad k \leqslant 4.$$

Theorem 5. $H^3(N_0, \mathbb{Z}) = \mathbb{Z}^{2g}$.

Proof. Firstly, $H^3(N_0, \mathbb{Z})$ is torsion-free. For, by Cor. 5, $H_2(N_0, \mathbb{Z}) = H_2(Z, \mathbb{Z})$ and therefore by the universal coefficient theorem, since

$$H^3(N_0, \mathbb{Z})_{\text{tors}} = H_2(N_0, \mathbb{Z})_{\text{tors}},$$

we have

$$H^3(N_0, \mathbb{Z})_{\text{tors}} = H^3(Z, \mathbb{Z})_{\text{tors}} = (0) \quad \text{by Theorem 3.6.}$$

Now using Theorem 4 and for Cor. 5 we get

$$H^3(N_0, \mathbb{Z}) = \mathbb{Z}^{2g}.$$

Q.E.D.

Theorem 6. *The Betti number B_4 of N_0 is* $B_4(N_0) = \binom{2g}{2} + 4$.

Proof. To see this, we use Prop. 5.9 and Remark 5.11 of Kirwan [5] to get the Betti numbers of M_0^s as $B_0 = 1$, $B_1 = 0$, $B_2 = 1$, $B_3 = 2g$, $B_4 = 2$, etc.

By Cor. 4,

$$B_4(Z) = B_2(Y) + B_4(Z - Y).$$

Now, by Prop. 8, $B_2(Y) = B_2(K - K_0) + B_2(\mathbb{P}^{g-2} \times \mathbb{P}^{g-2})$. Hence, by Spanier [19]

$$B_2(Y) = \binom{2g}{2} + 2.$$

Also, $B_3(Y) = 0$, since the odd Betti numbers of $K - K_0$ and $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ are zero (cf. [19] again). Combining this with (*), we get

$$B_4(Z) = \binom{2g}{2} + 4.$$

Hence by Cor. 5 we get

$$B_4(N_0) = \binom{2g}{2} + 4.$$

Q.E.D.

Appendix 1

We present here a proof due to Coliot–Thélène of Theorem (A) mentioned in the introduction. We shall make a few remarks before going into the proof.

Let X be a smooth variety over $\mathbb{C}$. For the notations and properties of most of the facts mentioned below (cf. Grothendieck [4] and Saltman [13], [14]).

Define $\mathrm{Br}(X)$ to be the Brauer group of Azumaya algebras on X. Let $\mathrm{Br}'(X)$ be the 'cohomological Brauer group' of X defined to be $H^2_{\text{et}}(X, \mathbb{G}_m)_{\text{tor}}$. Then the following facts are well known:

(i) $\mathrm{Br}\, X$ is contained in $\mathrm{Br}'(X)$.
(ii) If X is a unirational smooth proper variety, then $\mathrm{Br}'(X) = H^3(X(\mathbb{C}), \mathbb{Z})_{\text{tor}}$.
(iii) Define $\mathrm{Br}_{nr}(X)$, the unramified Brauer group of X to be $\mathrm{Br}_{nr}(X) = \mathrm{Br}'(\bar{X})$, $\bar{X}$ any

smooth compactification of X. Then it is know that $\mathrm{Br}_{nr}(X)$ is independent of the choice of $\bar{X}$ since we are in characteristic 0.

(iv) Another way of defining $\mathrm{Br}_{nr}(X)$ is as follows: Let $\mathbb{C}(X)$ be the function field of X. Then for every discrete valuation ring A, with $\mathbb{C} \subset A \subset \mathbb{C}(X)$, and quotient field of $A = \mathbb{C}(X)$, there exists a natural homomorphism

$$\partial_A : \mathrm{Br}\,\mathbb{C}(X) \rightarrow H^1(\mathbb{K}_A, \mathbb{Q}/\mathbb{Z}).$$

$\mathbb{K}_A$–the residue class field of A.

Define

$$\mathrm{Br}_{nr}\,\mathbb{C}(X) = \bigcap_{\text{all such } A} (\mathrm{Ker}\,\partial_A) \quad \text{and} \quad \mathrm{Br}_{nr}\,X = \mathrm{Br}_{nr}\,\mathbb{C}(X).$$

(v) Let k be a field and C a conic over k, i.e. a conic bundle coming from a quaternion algebra over k. Then there is a canonical homomorphism

$$\mathrm{Br}'(k) \rightarrow \mathrm{Br}'(C)$$

and the kernel of this homomorphism is the 2-torsion element coming from the quaternion algebra over k associated to C.

Note that for a field k, $\mathrm{Br}'(k) = \mathrm{Br}(k)$.

PROPOSITION 13

Let C be a conic bundle on X with $\mathrm{Br}'(C) = 0$, and let η be the generic point of X. Let C_η be the restriction of C over $\mathbb{C}(\eta)$. To C_η we associate an element $\alpha_\eta \in \mathrm{Br}\,\mathbb{C}(\eta)$. Suppose that for the conic bundle C on X, there exists a discrete valuation ring A, with quotient field of $A = \mathbb{C}(X)$, $\mathbb{C} \subset A \subset \mathbb{C}(X)$, such that $\partial_A(\alpha_\eta) \neq 0$. Then $\mathrm{Br}_{nr}(X) = 0$.

Proof. Suppose that $\mathrm{Br}_{nr}(X) \neq 0$ and let $\alpha \in \mathrm{Br}_{nr}(X) = \mathrm{Br}_{nr}(\mathbb{C}(X))$ be a non-zero element. Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Br}_{nr}\,X & \longrightarrow & \mathrm{Br}_{nr}\,C \\ \downarrow & & \downarrow \\ \mathrm{Br}'\,\mathbb{C}(\eta) & \longrightarrow & \mathrm{Br}'\,C_\eta \end{array}$$

where the map $\mathrm{Br}_{nr}(X) \rightarrow \mathrm{Br}_{nr}(C)$ is the canonical map induced from $C \rightarrow X$ and the vertical maps are

$$\mathrm{Br}_{nr}\,C \hookrightarrow \mathrm{Br}\,C \hookrightarrow \mathrm{Br}'\,C \rightarrow N\,\mathrm{Br}'\,C_\eta$$

$$\mathrm{Br}_{nr}\,X \hookrightarrow \mathrm{Br}\,\mathbb{C}(X) = \mathrm{Br}\,\mathbb{C}(\eta) = \mathrm{Br}'\,\mathbb{C}(\eta).$$

Consider the image of α in $\mathrm{Br}'\,\mathbb{C}(\eta)$, call it α_η. Then since $\mathrm{Br}'\,C = 0$, the above diagram gives

$$\alpha_\eta \in \mathrm{Ker}\,[\mathrm{Br}'\,\mathbb{C}(\eta) \rightarrow \mathrm{Br}'\,C_\eta]$$

and therefore by Remark (v), α_η is the element in $\mathrm{Br}\,\mathbb{C}(\eta)$ associated to the conic C_η. Now by the hypothesis of the proposition, there exists a discrete valuation ring

$A, \mathbb{C} \subset A \subset \mathbb{C}(X)$ with quotient field of $A = \mathbb{C}(X)$, such that

$$\partial_A(\alpha_\eta) \neq 0. \qquad (*)$$

But $\alpha \in \mathrm{Br}_{nr}\, \mathbb{C}(X)$ and $\mathrm{Br}_{nr}\, \mathbb{C}(X)$ is by definition equal to

$$\bigcap_{\text{all such } A} (\mathrm{Ker}\, \partial_A),$$

implying

$$\partial_A(\alpha_\eta) = 0$$

which contradicts (*). Hence the proposition.

Q.E.D.

Now let us consider the variety M_0^s, the moduli space of stable vector bundles of rank 2 and trivial determinant. Then by Prop. 3, there is a conic bundle D on M_0^s with $H^3(D, \mathbb{Z})_{\text{tors}} = (0)$ and therefore $\mathrm{Br}'(D) = 0$.

The existence of an A with the requisite properties of the Prop. 13 is precisely the theorem due to Nitsure [9]. Indeed, in the notation of §4, the irreducible divisor $Y \subset Z$ provides us with the discrete valuation ring A.

Hence by Prop. 13, $\mathrm{Br}_{nr} M_0^s = 0$. This implies by Remark (3), that $\mathrm{Br}'(N_0) = 0$, since N_0 is a smooth compactification of M_0^s. Now N_0 is unirational, smooth-projective and therefore by Remark (ii), $\mathrm{Br}'(N_0) = H^3(N_0, \mathbb{Z})_{\text{tor}} = (0)$.

Appendix 2

Theorem (*C S Seshadri*). *Let M be the moduli space of semi-stable vector bundles of rank 2^n and degree d. Then*

$$\mathrm{Pic}\, M^s \text{ (as well as } \mathrm{Pic}\, M) \simeq \mathbb{Z}.$$

Proof. For simplicity we present the proof only for rank 2 and degree zero. Choose m such that for all stable bundles V of rank two and degree zero, $V(m)$ is generated by the global sections. Then if E denotes the trivial vector bundle of rank $r = \dim H^0(V(m))$, $V(m)$ is canonically a quotient of E and $V(m)$ represents a point of $Q = Q(E/P)$, the Quot scheme of quotients of E with Hilbert polynomial equal to P.

We then have an open subscheme Q^s of Q representing quotient vector bundles W of E such that W is stable and the canonical homomorphism $H^0(E) \to H^0(W)$, is an isomorphism. Thus we have a canonical morphism

$$p: Q^s \to M_1^s,$$

where M_1^s is the moduli space of stable vector bundles of rank 2 and $\det = \mathcal{O}_X(2m)$ and p is a G-principal fibre space with $G = PGL(H^0(E))$. Note that $M_1^s \approx M_0^s$ of §2.

Let $q: B \to M_1^s$ be the fibre space associated to p with fibre the projective space of dimension $(r-1)$. Hence if $W \in M_1^s$, the fibre $q^{-1}(W)$ can be canonically identified with $\mathbb{P}(H^0(W))$.

Let A denote the projective space $\mathbb{P}(\mathrm{Ext}(L, I))$, the 'Atiyah family' on the vector space of all extensions of the form

$$0 \to I \to W \to L \to 0,$$

where I is the trivial vector bundle of rank one and L the line bundle $\mathcal{O}_X(2m)$. Let A^s denote the subset of A defined by

$$A^s = \{0 \to I \to W \to L \to 0 \mid W \text{ is stable}\}.$$

Then A^s is open and we have a canonical surjective morphism

$$\lambda : A^s \to M_1^s$$

which associates to an extension as above the vector bundle W. Observe that giving an extension as above is equivalent to giving a section $s \in H^0(W)$ which is non-vanishing at every point $x \in X$. From this observation we deduce easily that A^s can be identified canonically as an open subset of the projective bundle B over M_1^s; in fact we have a commutative diagram

$$\begin{array}{ccc} A^s & \overset{i}{\hookrightarrow} & B \\ & \underset{\lambda}{\searrow} & \downarrow p \\ & & M_1^s \end{array}$$

Note that $p^{-1}(W) - \lambda^{-1}(W)$ is *irreducible* in $\mathbb{P}(H^0(W))$ for $p^{-1}(W) - \lambda^{-1}(W)$ is the canonical image in $\mathbb{P}(H^0(W))$ of the set $S = \{s \mid s \in H^0(W), s$ vanishes at least at one point of $X\}$ i.e.

$$S = \bigcup_{x \in X} \operatorname{Ker}(H^0(W) \to W_x).$$

Since $\lambda^{-1}(W)$ is the complement of an irreducible closed subset in a projective space, nonvanishing regular functions on $\lambda^{-1}(W)$ reduce to constants. From this, we easily conclude that, if U is an open subset in M_1^s and f a regular nonvanishing function on $\lambda^{-1}(U)$, then f is a pull-back of a regular nonvanishing function on U. From these properties, it follows easily that the canonical homomorphism

$$\lambda^* : \operatorname{Pic} M_1^s \to \operatorname{Pic} A^s$$

is injective. To see this, let $L \in \ker \lambda^*$. Then if L is given by transition functions $\{\theta_{ij}\}$ on $V_i \cap V_j$, we have nonvanishing regular functions φ_i on $\lambda^*(V_i)$ such that $\lambda^*(\theta_{ij}) = \varphi_i \varphi_j^{-1}$.

Now the φ_i are pull-backs of functions θ_i on V_i and the required assertion follows.

Now A^s is an open subset of $\mathbb{P}(\operatorname{Ext}(L, I))$ and therefore $\operatorname{Pic} A^s$ is either $\mathbb{Z}$ or $\mathbb{Z}/(m)$, $m \neq 0$. Consider the normal projective variety M_1. The ample line bundle on M_1 restricted to M_1^s shows that $\operatorname{Pic} M_1^s$ is not torsion. But $\lambda^* : \operatorname{Pic} M_1^s \to \operatorname{Pic} A^s$ is injective implying, $\operatorname{Pic} A^s = \mathbb{Z}$ and also $\operatorname{Pic} M_1^s = \mathbb{Z}$. Since M_1 is normal, it follows that $\operatorname{Pic} M_1 \subset \operatorname{Pic} M_1^s$ and hence $\operatorname{Pic} M_1 = \mathbb{Z}$.

Remark. A priori, $\operatorname{Pic} M_1$ is just a proper subgroup of $\operatorname{Pic} M_1^s$. But if M_1 is locally factorial then $\mathrm{cl}\, M_1 = \mathrm{Pic}\, M_1$ and we would have $\mathrm{Pic}\, M_1 = \mathrm{Pic}\, M_1^s$.

In fact, this is so and has been recently proved by J M Drezet and M S Narasimhan.

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