Cohomology of certain moduli spaces of vector bundles

V BALAJI
The Institute of Mathematical Sciences, Madras 600 113, India

MS received 26 October 1987; revised 20 December 1987

Abstract. Let $X$ be a smooth irreducible projective curve of genus $g$ over the field of complex numbers. Let $M_g$ be the moduli space of semi-stable vector bundles on $X$ of rank two and trivial determinant. A canonical desingularization $N_0$ of $M_g$ has been constructed by Seshadri [17]. In this paper we compute the third and fourth cohomology groups of $N_0$. In particular we give a different proof of the theorem due to Nitsure [12], that the third cohomology group of $N_0$ is torsion-free.

Keywords. Stable bundles; semi-stable bundles; parabolic bundles; conic bundles; Gysin map; Hecke correspondence; Brauer group.

1. Introduction

Let $X$ be a smooth irreducible projective curve of genus $g$ over the field of complex numbers. Non-singular models of the moduli space of semi-stable vector bundles on $X$ of rank two and degree zero have been constructed by Narasimhan–Ramanan [7] and Seshadri [17]. In this paper, we propose to compute some of the Betti numbers of the non-singular model due to Seshadri. In particular we prove the following theorems.

Theorem (A). The third cohomology group of the non-singular model $N_0$ of [17] is torsion-free, $g \geq 2$.

Theorem (B). Let $B_i$ denote the Betti numbers of $N_0$. Then we have:

$$B_3 = 2g, \quad B_4 = \binom{2g}{2} + 4, \quad g \geq 4.$$

Theorem (A) is due to Nitsure [11]. He proved this for the non-singular model of [7]. By Artin–Mumford [1], the torsion subgroup of the third cohomology group of a smooth projective variety is a birational invariant. Therefore any non-singular model has torsion-free third cohomology.

We present here a considerably simpler proof of Theorem (A) using the model of [17]; in fact, this was the initial motivation for this work. However we should point out that the general line of attack is as in Nitsure [11]. An extension of the ideas involved in the proof also yields Theorem (B). For computing $B_4$ and $B_5$ we make use of the results of Kirwan [5].

Nitsure showed independently that $B_3 = 2g$ for the model of [7] (cf [12]).
In Appendix 1 we present a proof of Theorem (A) due to Coliot-Thélène which is independent of the non-singular model chosen.

Theorems (A) and (B) are of interest in understanding the rationality of these non-singular models of the moduli space of vector bundles.

The layout of the paper is as follows. Section 2 of this paper gives various properties of the non-singular model constructed in [17]. In §3 we construct a canonical generalized conic bundle on the non-singular model $N_0$. In §4 by using a result of [9], we prove Theorem (A) and show how to compute the Betti numbers of the open subset $Z$ of $N_0$ lying over the stable bundles and the bundles in the non-nodal part of the Kummer variety. In this section, we also give a description of the Hecke correspondence in terms of parabolic bundles as mentioned in (*). This facilitates the computation of the Betti numbers. In §5 we compute explicitly the codimension of the complement of $Z$ in $N_0$ and thereby compute its Betti numbers.

The author is grateful to Prof C S Seshadri for suggesting this approach and for many fruitful discussions. He thanks A J Parameshwaran for many an interesting discussion. He also thanks Prof. J Coliot Thélène for communicating his proof.

2. Preliminaries

In this section we shall recall very briefly the definitions and terminologies of [17]. The proofs of most of the statements made in this section can be found in [17] or [18]. We state at the outset that for us the ground field of all our varieties is the field of complex numbers.

(i) $X$ is a smooth irreducible projective curve of genus $g \geq 3$.

(ii) Let $V$ be a vector bundle on $X$. A parabolic structure at a point $P \in X$ gives

(a) a quasi-parabolic structure i.e. a flag $V_p = F^1 V_p \supseteq F^2 V_p \supseteq \ldots \supseteq F^r V_p.$

(b) weights $\alpha_1, \ldots, \alpha_r$ attached to $F^1 V_p, \ldots, F^r V_p$ such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_r < 1.$

Call $k_i = \dim F^i V_p - \dim F^2 V_p, \ldots, k_r = \dim F^r V_p$.

the multiplicities of $\alpha_1, \alpha_2, \ldots, \alpha_r$.

The parabolic degree of $V$ is defined by

$$\text{par deg } V = \deg V + \sum_i k_i \alpha_i$$

and write $\text{par } \mu(V) = \text{par deg } V / \text{rk } V$.

If $W$ is a subbundle of $V$, it acquires, in an obvious way, a quasi-parabolic structure. To make it a parabolic subbundle, we attach weights as follows:

Given $i_0, F^{i_0} W \subset F^i V$ for some; let $j_0$ be such that $F^{j_0} W \subset F^i V$ and $F^{j_0} W \not\subset F^{j_0 + 1} V$; then the weight of $F^{j_0} V = F^{j_0} W$. Define $V$ to be parabolic stable (resp. semistable) if for every proper subbundle $W$ of $V$, one has $\text{par } \mu(W) < \text{par } \mu(V)$ (resp. $\leq$).

Cohomology of certain moduli spaces of vector bundles

If \( V \) be the category of semistable vector bundles on \( X \) of rank \( n \) and degree 0, then we denote by \( PV \) the category of parabolic semistable vector bundles at a fixed point \( P \in X \) and fixed parabolic structure. Recall that, one can choose the weights \((\alpha)\) small enough so as to have the condition 'parabolic semistable' equivalent to 'parabolic stable'.

(iii) \( N \) is the set of isomorphism classes of \((V, \Delta) \in PV\) (\( \Delta \) a parabolic structure), such that \( \text{End} \ V \) is a 'specialization' of \( \mathcal{M}_2 \) — the \( 2 \times 2 \) matrix algebra.

In fact, if \((V_1, \Delta_1) \) and \((V_2, \Delta_2) \) belong to \( N \), they represent the same element of \( N \) (i.e. isomorphic in \( PV \)) if and only if the underlying bundles \( V_1 \) and \( V_2 \) are isomorphic (cf [17]). Hence we often simply write \( V \in N \).

(iv) \( \mathcal{A} \) is the variety of all algebra structures on a fixed 4-dimensional vector space which are specializations of \( \mathcal{M}_2 \) and admit a fixed identity element. We have a canonical group of automorphisms acting on \( \mathcal{A} \), namely the subgroup of \( GL(4) \), fixing this identity element.

(v) \( M \) denotes the normal projective variety of equivalence classes of semistable vector bundles of rank 2 and degree 0 under the equivalence relation \( V \sim V' \) if and only if \( \text{gr} \ V = \text{gr} \ V' \).

(vi) \( M^s \) will be the open subset of \( M \) consisting of the stable bundles.

It is known that \( M - M^s \) is precisely the singular locus of \( M \) (cf [6]). The main theorem of [17] is stated below.

**Theorem 1.** (Seshadri) There is a natural structure of a smooth projective variety on \( N \) and there exists a canonical morphism \( p : N \to M \), which is an isomorphism over \( M^s \). More precisely, if \( V \in N \), then \( \text{gr} \ V = D \oplus D \), with \( \text{rk} \ D = 2 \), \( D \) is a direct sum of stable line bundles of degree 0 and the morphism \( p : N \to M \) is given by \( V \mapsto D \). Further \( V \in p^{-1}(M^s) \) if and only if \( \text{End} \ V \cong \mathcal{M}_2 \) or equivalently (which is easily seen) \( V = W \oplus W \), where \( W \) is stable.

In the course of proving the smoothness of \( N \), Seshadri defined a morphism from a neighbourhood \( U \) of a given point of \( N \) into \( A \) which we shall denote by

\[ \phi^U : U \to A. \]

We shall briefly indicate the construction of \( \phi^U \). The functor defining the moduli space \( N \) being representable, we have a defining vector bundle \( E \) on \( X \times N \) of rank 4. Let \( f : X \times N \to N \) be the canonical projection and \( \text{End} \ E \) the vector bundle associated to the sheaf of endomorphisms of \( E \). Set

\[ B = f_*(\text{End} \ E). \]

\( B \) is the canonical family of specializations of \( \mathcal{M}_2 \), parametrized by \( N \) (see Prop. 5 [17] for details). Consider any given point \( u \in N \); then choosing a neighbourhood \( U \) of \( u \), which trivialises \( B \), we get a natural morphism

\[ \phi^U : U \to A \quad \text{by} \quad V \mapsto \text{End} \ V, \quad V \in U. \]

This morphism exists by the so-called versal property of \( A \). Further, let \( A_u = \text{End} \ V_u \) the vector bundle corresponding to the point \( u \in U \), i.e. \( A_u = \phi^U(u) \). Then, if \( A_u \) is the
mini-versal deformation space of $A_0$, the morphism
\[ \varphi_U^V: U \to A_u \]
induced by the versality of $A_u$ from $\varphi^V$ is in fact smooth.

**Note 1.** By an abuse of notation, in the course of this work, we shall suppress $U$ and the mini-versal deformation space corresponding to each point, and simply denote by $\varphi: N \to A$ the smooth local morphism defined above. In fact, we will be using it only in this form in this work.

Note further that these $\varphi_U^V$ are uniquely determined modulo automorphism coming from the canonical group of automorphisms acting on $A$.

**PROPOSITION 1**

*The restriction of the local morphism $\varphi$ to the subvariety $N_0$ remains smooth.*

**Proof.** Let $J$ denote the Jacobian variety of line bundles of degree zero on $X$. Then we have a natural morphism
\[ \psi: N_0 \times J \to N \]
\[ (E, L) \mapsto E \otimes L \]
(that this map is a morphism follows from the universal property of $N$ and the fact that $E \otimes L$ gives a family on $X$ parametrized by $N_0 \times J$).

We claim that $\psi$ is smooth. In fact, $\psi$ is étale. For, let $\Gamma \subset J$ be the finite subgroup of $J$ consisting of the elements of order 2. Then there is a natural diagonal action of $\Gamma$ on $N_0 \times J$ which is obviously fixed point free. It is not difficult to see that $N$ is actually the quotient of $N_0 \times J$ by $\Gamma$ and $\psi: N_0 \times J \to N$ the quotient morphism (note that our ground field is $\mathbb{C}$ and if $A$ and $B$ are smooth complex manifolds and $G$ a finite group acting on $A$ such that $B$ is the set theoretic quotient of $A$ by $G$, then $B$ is $A/G$).

This $\Gamma$-action being fixed point free, $\psi$ is étale.

For $b \in N_0 \times J$, choosing a neighbourhood $U$ of $\psi(b) = u$ in $N$, we get the following diagram
\[
\begin{array}{ccc}
\psi^{-1}(U) & \xrightarrow{\varphi_U^V} & U \\
\downarrow & & \downarrow \\
A_u & \xrightarrow{\varphi} & A
\end{array}
\]
where $A_u$ is the mini-versal deformation space of the algebra $A_0 = \varphi^V(u)$ in $A$.

Since $\varphi_U^V, \psi$ are smooth, so is $\varphi_U^V \psi$. In other words the local morphism (again by abuse of notation)
\[ \varphi \circ \psi: N_0 \times J \to A \]
is smooth. If $L \in J$, then $\text{End}(E \otimes J) = \text{End} E$ and hence $\varphi \circ \psi$ clearly factors through $N_0$ to give the smoothness of the restriction of $\varphi$ from $N_0 \to A$. Q.E.D.
Remark 1. Because of Prop. 1, by the same arguments as in [17], we see that \( N_0 \) is a smooth-projective variety. We then get an obvious generalization of Theorem 1 namely that \( p: N_0 \rightarrow M_0 \) which is a desingularization of \( M_0 \), and that it is an isomorphism over \( M_0^* \) etc.

3. Conic bundles

DEFINITION 1

Let \( S \) be a variety. A generalized conic bundle \( \mathcal{C} \) on \( S \) gives

(a) a vector bundle \( V \) on \( S \) of rank 3 and
(b) a closed subscheme \( \mathcal{C} \) of \( \mathbb{P}(V) \) over \( S \), such that, given \( s \in S \), there exists a
neighbourhood \( U \) of \( s \), where \( \mathcal{C} \cap p^{-1}(U) \) is defined by \( \theta = 0 \), \( \theta \in \Gamma(p^{-1}(U), H^{2}, H) \) being the tautological line bundle for \( \mathbb{P}(V) \rightarrow S \); i.e. \( p_*(H) \cong V^* \) and therefore \( p_*(V^*) = S^2 V^* \), etc.

By definition, \( \mathcal{C} \) is an effective Cartier divisor and is therefore defined by a section of a line bundle \( \theta \) on \( \mathbb{P}(V) \). Now locally over \( S \), \( \theta \) and \( H^2 \) coincide and therefore by the "see-saw" theorem (cf. Mumford's Abelian varieties) there exists a line bundle \( L \) on \( S \) such that \( \theta = H^2 \otimes p^*(L) \). Since \( p_*(\mathcal{C}) = p_*(H^2) \otimes L = S^2(V^*) \otimes L \), the condition (b) above is equivalent to an element \( q \) of \( \Gamma(S^2(V^*) \otimes L) \) or that is to say a quadratic form

\[ q: V \rightarrow \mathbb{L}. \]

The discriminant \( \Delta \) of \( q \) can be defined as a section of \( L^2 \otimes (\Lambda^3(V^*))^2 \) and locally as the usual discriminant of a quadratic form. The equation \( \Delta = 0 \) gives locally the degeneracy locus of \( \mathcal{C} \).

We now introduce subschemes on \( S \), namely for \( i = 1, 2, 3 \), set

\[ S_i = \{ s \in S | q \text{ restricted to } V_s, \text{ the fibre at } s, \text{ has rank } \leq 3 - i \}. \]

Then \( S_3 \subset S_2 \subset S_1 \subset S = S_0 \). If \( g: \mathcal{C} \rightarrow S \) be the projection, let \( \mathcal{C}_i = g^{-1}(S_i), i = 1, 2, 3 \). Then we have \( S_1 \) to be the degeneracy locus of \( \mathcal{C} \), i.e. given by \( \Delta = 0 \), and \( S_2 \subset S_1 \) is the singular locus of \( S_1 \). The space \( \mathcal{C} \) can be described as follows: \( \mathcal{C} \setminus \mathcal{C}_1 \) consists of non-degenerate conics; \( \mathcal{C}_1 \setminus \mathcal{C}_2 \) of pairs of lines intersecting transversally; \( \mathcal{C}_2 \setminus \mathcal{C}_3 \) of repeated lines and \( \mathcal{C}_3 \) of the whole plane. We call \( S_1 \) the canonical subschemes associated to the degenerate loci of the conic bundle \( \mathcal{C} \) on \( S \). Accordingly we make the following.

DEFINITION 2

A generalized conic bundle \( \mathcal{C} \) is of type I if \( \mathcal{C}_1 = \phi \); of type II if \( \mathcal{C}_2 = \phi \) and of type III if \( \mathcal{C}_3 = \phi \).

DEFINITION 3 (cf p. 164 [17])

Let \( T \) be an algebraic scheme and \( \{ G_t \}_{t \in T} \) a family of algebras parametrized by \( T \) and defined by a locally free \( \mathcal{O}_T \)-module \( G \) of rank 4. We say that this is a family of specializations of \( \mathcal{M}_2 \) if, given \( t \in T \), there is a neighbourhood \( T_t \) of \( t \) and a morphism
Let $f: T \to \mathcal{A}$, such that $\{G_i\}_{i \in T}$ is the base change of $\{A_y\}_{y \in \mathcal{A}}$ by $f$, where $A_y$ is the algebra structure corresponding to $y \in \mathcal{A}$.

**Remark 2.** By Remark 3 [17], the above definition has an equivalent formulation as follows: Let $T = \text{Spec } R$, and $G$ be an $R$-algebra with identity $e_0$ such that the underlying $R$-module is free of rank 4. Let $J = G/R e_0$. Consider the canonical Lie algebra structure on $J$ induced by the associative algebra structure on $G$. This gives a canonical skew-symmetric bilinear map $J \times J \to J$ or equivalently (in our case) an element of $J \otimes J$. Then we say the algebra $G$ gives a family of specializations of $\mathcal{M}_2$ parametrized by $T$, if this Lie algebra structure is defined by a symmetric element of $J \otimes J$. Further, the algebra $G$ is isomorphic to $C^*_q$, $q$ being the corresponding quadratic form. This definition generalizes, in an obvious way, when $T$ is any scheme, and $G$ a vector bundle of rank 4 on $T$; however, the quadratic form $q$ on $J$ takes values in a line bundle on $T$.

**Note 2.** We shall use this reformulation in the course of this work.

**Remark 3**

(i) Restrict the canonical family $B$ of specialization of $\mathcal{M}_2$ parametrized by $N$ to the subvariety $N_0$. Call this family $B_0$.

(ii) For $y \in \mathcal{A}$, let $A_y$ be the corresponding algebra structure; then $\{A_y\}_{y \in \mathcal{A}}$ gives an obvious family of specializations of $\mathcal{M}_2$.

(iii) Let $T = \text{Spec } R$ and $G$ an $R$-algebra giving a family of specializations of $\mathcal{M}_2$. Then by Remark 2, we get a symmetric element of $J \otimes J = G/R e_0$. This symmetric element naturally gives rise to a symmetric bilinear form on $J^*$ (the $R$-dual of $J$) and therefore a quadratic form on $J^*$. Now $J^*$ being a projective $R$-module of rank 3, it defines a vector bundle of rank 3 on $T$. More generally, if we are given an algebraic scheme $T$, a family $\{G_i\}_{i \in T}$ of specializations of $\mathcal{M}_2$, then we have a canonical vector bundle $V$ of rank 3 on $T$ together with a $C^*_T$-valued quadratic form $q: V \to C^*_T$, and thus a conic bundle on $T$.

(iv) The families $B_0$ on $N_0$ and $\{A_y\}_{y \in \mathcal{A}}$ on $\mathcal{A}$ give generalized conic bundles on $N_0$ and $\mathcal{A}$ respectively.

**Notation 1.** Denote these conic bundles by $P$ on $N_0$ and $Q$ on $\mathcal{A}$.

**PROPOSITION 2**

The conic bundle $P$ on $N_0$ is locally the base change of $Q$ on $\mathcal{A}$ by the local morphism $\varphi: N_0 \to \mathcal{A}$ of §2.

**Proof.** This is an immediate consequence of the definitions of $\varphi$, $B_0$ and $\{A_y\}_{y \in \mathcal{A}}$.

**Remark 4.** Following §3, we introduce the canonical subschemes

$$\mathcal{A}_3 \subset \mathcal{A}_2 \subset \mathcal{A}_1 \subset \mathcal{A} \quad \text{and} \quad N_3 \subset N_2 \subset N_1 \subset N_0$$

associated to the degeneracy locus of $Q$ and $P$ respectively. Then, by Prop. 2, $\varphi$
Cohomology of certain moduli spaces of vector bundles

locally maps $N_0 - N_2$ into $\mathcal{A} - \mathcal{A}_2$ in such a way that $N_1 - N_2 \to \mathcal{A}_1 - \mathcal{A}_2$, $N_0 - N_1 \to \mathcal{A} - \mathcal{A}_1$.

Remark 5. By Theorem 1 [17] we know that $\mathcal{A} \cong \Phi \times \Lambda$, where $\Lambda$ is the 3-dimensional affine space and $\Phi$ the 6-dimensional affine space whose points are identified with the set of quadratic forms on a fixed 3-dimensional vector space (or algebras of the form $C_q^+\text{—the even degree elements of the Clifford algebra associated to the quadratic form } q$). Therefore we have for $i = 1, 2, 3$

$$\mathcal{A}_i = \{ q \in \Phi | \text{rank } q \leq 3 - i \} \times \Lambda^3.$$  

Note that

$$\mathcal{A}_0 - \mathcal{A}_1 = \{ q | q \in \Phi, C_q^+ \cong \mathcal{M}_2 \} \times \Lambda^3$$

or equivalently

$$\mathcal{A}_0 - \mathcal{A}_1 = \{ y | A_y \cong \mathcal{M}_2 \}.$$

Notation 2. We denote the subsets $N_0 - N_2$ and $N_1 - N_2$ of $N_0$ by $Z$ and $Y$ respectively.

Let $K = M_5 - M_0$, be the singular locus of $M_0$. The bundles here are of the form $L \oplus L^{-1}$, where $L$ is a line bundle of degree 0. Let $K_0$ be the 'nodes' of $K$ (i.e. consisting of bundles of the type $L \oplus L$ with $L^2$ trivial). Then

$$K - K_0 = L \oplus L^{-1}, \quad \text{let } J - \Gamma,$$

$J$ and $\Gamma$ as in §2. It may be noted that $K$ is a Kummer variety of dim $q$ (cf [6]).

PROPOSITION 3

The subsets $Z$ and $Y$ of $N_0$ are precisely $N_0 - p^{-1}(K_0)$ and $p^{-1}(K - K_0)$ respectively, where $p: N_0 \to M_0$ is the desingularization morphism. In particular, $Z - Y = p^{-1}(M_0^*)$.

Proof. By Remark 3, it is enough to show that the subsets $p^{-1}(M_0^*)$ and $p^{-1}(K - K_0)$ of $N_0$ are mapped locally by $\varphi$ into the subsets $\mathcal{A}_0 - \mathcal{A}_1$ and $\mathcal{A}_1 - \mathcal{A}_2$ of $\mathcal{A}_1$ respectively. We know that $V \in p^{-1}(M_0^*)$ if and only if $\text{End } V \cong \mathcal{M}_2$, which shows $p^{-1}(M_0^*)$ maps to $\mathcal{A}_0 - \mathcal{A}_1$.

Therefore it is enough to show that, for $E \in p^{-1}(K - K_0)$, $\text{End } E$ has the same defining relations as that of the algebra $C_q^+$, for a quadratic form $q$ of rank 2 on a 3-dimensional vector space.

By definition of the desingularization, the endomorphism algebras of any two points in a fibre $p^{-1}(L \oplus L^{-1})$ are isomorphic. So we consider a point $E$ in $p^{-1}(L \oplus L^{-1})$ where $E = V \oplus W, V \in \text{Ext}(L, L^{-1})W \in \text{Ext}(L^{-1}, L), L \in J - \Gamma$. i.e.

$$0 \to L \to V \to L^{-1} \to 0,$$

$$0 \to L^{-1} \to W \to L \to 0.$$  

(1)

It is clear that points of this type are actually in $p^{-1}(K - K_0)$. Using (1), it is easy to see that $\text{End}(V \oplus W)$ has four generators, which in terms of block matrices can
be described as

\[ x = \begin{pmatrix} 0 & 0 \\ \gamma_2 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & \gamma_1 \\ 0 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \]

where \( I = 2 \times 2 \) identity matrix, and \( \gamma_1, \gamma_2 \) coming from identification of the line bundles in the exact sequence (1). The defining relations can be given as

\[ u^2 = u, \quad v^2 = v, \quad uv = 0, \quad u + v = I, \]
\[ w^3 = wx = 0, \quad uv = w, \quad wu = 0, \]
\[ u = 0, \quad xu = x, \quad vw = 0, \quad wv = w \]
\[ w = x, \quad xv = 0. \]

(2)

If \( q \) is a quadratic form of rank 2 on a 3-dimensional vector space over an algebraically closed field \( k \) then it is easily seen that \( C_q^+ \) is the even degree elements of the Clifford algebra of \( q \) is a 4-dimensional \( k \)-algebra with

\[ C_q^+ = k + k\alpha + k\beta + k\gamma \quad \text{such that} \]
\[ \alpha^2 = -1, \quad \alpha\beta = -\gamma, \quad \alpha\gamma = \beta \]
\[ \beta\alpha = \gamma, \quad \gamma\alpha = -\beta. \]

Now put \( a = \frac{1}{2}(1 + i\alpha), \ b = \frac{1}{2}(1 - i\alpha), \ c = i\beta + \gamma, \ d = i\beta - \gamma \), where \( i = \sqrt{-1} \in k \). Then \( a, b, c, d \) are new generators of \( C_q^+ \) with the following defining relations

\[ a^2 = a, \quad b^2 = b, \quad ab = 0, \quad a + b = 1, \]
\[ c^2 = d^2 = cd = 0, \quad ac = c, \quad ca = 0, \]
\[ ad = 0, \quad da = d, \quad bc = 0, \quad cb = c, \]
\[ bd = d, \quad db = 0. \]

(3)

A glance at (2) and (3) proves our claim.

Q.E.D.

COROLLARY 1

\( Y \overset{p}{\rightarrow} K - K_0 \) is a \( \mathbb{P}^{s-2} \times \mathbb{P}^{s-2} \) fibration associated to a vector bundle on \( K - K_0 \).

Proof. Indeed; we claim that, if \( E \in Y = p^{-1}(K - K_0) \) then \( E = V \oplus W \), for some \( V \in \mathbb{P}(\text{Ext}(L, L^{-1})), \ W \in \mathbb{P}(\text{Ext}(L^{-1}, L)) \ L \in I - \Gamma \).

Let \( E \in p^{-1}(K - K_0) \); then, \( \text{End } W \) has four generators \( x, w, u, v \) with defining relations (2) as in Prop. 3. Consider \( u \in \text{End } E \), and let \( V = \ker u \). Then \( V \) is a subbundle of \( E \) and we have an exact sequence

\[ 0 \rightarrow V \rightarrow E \rightarrow W \rightarrow 0. \]

It is clear then that \( W \) is in fact \( \ker v, v \in \text{End } E \) and therefore we get a splitting of the exact sequence, implying \( E = V \oplus W \).
Now using Prop. 1 of [17], V and W cannot be of the type \( L \oplus L \) or \( L^{-1} \oplus L^{-1} \). For the same reason, since \( E \in P(V) \), we rule out \( V = L \oplus L^{-1}, W = L^{-1} \oplus L \). Hence we are left with \( V \in \mathcal{P}(\text{Ext}(L, L^{-1})), W \in \mathcal{P}(\text{Ext}(L^{-1}, L)) \) or vice versa.

Note that for \( L \to K - K_0, \text{Ext}(L, L^{-1}) = H^1(X, L^{-2}) \) has dimension \( g - 1 \) and therefore \( Y \) is a \( \mathbb{P}^{g-2} \times \mathbb{P}^{g-2} \) fibration over \( K - K_0 \). The vector bundle to which this is associated has fibre at any \( L \in K - K_0 \) to be \( \text{Ext}(L, L^{-1}) \oplus \text{Ext}(L^{-1}, L) \).

**COROLLARY 2**

The fibration \( Y \to K - K_0 \) is locally trivial in the Zariski topology.

**Proof.** This follows from Cor. 1 and Serre (cf [15]).

**PROPOSITION 4**

Let \( P - P_2 \) be the restriction of the conic bundle \( P \) over points of \( N_0 - N_2 \) (i.e. \( Z \)). Then the total space of \( P - P_2 \) is smooth.

**Proof.** By Prop. 2, \( P - P_2 \) is locally the base change of \( Q - Q_2 \) (the restriction of \( Q \) over points of \( \mathcal{A} - \mathcal{A}_2 \)). Since \( \varphi: N_0 \to \mathcal{A} \) is a smooth local morphism, the total space of \( P - P_2 \) is smooth if and only if the total space of \( Q - Q_2 \) is so.

Consider any point \( (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{A}^6 \). This defines a quadratic form

\[
q = a_1 X^2 + a_2 XY + a_3 Y^2 + a_4 XZ + a_5 YZ + a_6 Z^2.
\]

We therefore have a conic bundle \( C \) over \( \mathbb{A}^6 \) by considering the conics defined by the quadratic forms. By Remark 4 it is clear that the conic bundle \( Q \) on \( \mathcal{A} \) is 'essentially' the conic bundle \( C \). Thus we would have proved our claim if we show that the total space of \( C \to \mathbb{A}^6 - S^1 \) is smooth, where \( S \) is the degeneracy locus of \( C \) and \( S' \subset S \) its singular locus. We have in fact more.

**Lemma 1.** Let \( \theta: C \to \mathbb{A}^6 \) be the canonical morphism. Then \( \theta^{-1}(\mathbb{A}^6 - \{0\}) \) is smooth.

**Proof.** Let \( P \in C \) be any point. Then \( P \) can be given by \( (a_1, a_2, a_3, a_4, a_5, a_6, X, Y, Z) \) where not all \( a_i = 0 \) and not all \( X, Y, Z = 0 \). \( P \) lying on the conic defined by \( q = a_1 X^2 + a_2 XY + a_3 Y^2 + a_4 XZ + a_5 YZ + a_6 Z^2 \). Taking partial derivatives of \( q \) with respect to \( a_i, i = 1, \ldots, 6 \), we have

\[
\frac{\partial q}{\partial a_i} = 0, \quad i = 1, \ldots, 6 \Rightarrow X = Y = Z = 0.
\]

Q.E.D.

4. Cohomology computations

4.1 The Gysin map

Let \( W \) be a conic bundle of type I (cf Def. 2) on a variety \( S \). This gives rise to a topological Brauer class \( b_W \in H^3(S, \mathbb{Z})_{\text{tors}} \) (i.e. the torsion subgroup of \( H^3(S, \mathbb{Z}) \)).
Let $W$ be a conic bundle of type II (cf Def. 2). Then if $W$ degenerates to a pair of lines over an irreducible divisor $S_1 \subset S$, the restriction $W_1$ of $W$ over $S_1$ gives rise in a natural way to a double cover of $S_1$ (cf Lemma on p. 29 of [8]) and $W-W_1$ is a conic bundle of type I over $S-S_1$. We shall denote by $\alpha$ the element in $H^3(S_1, \mathbb{Z})$ coming from this double cover. Consider the part of the Gysin sequence for $S_1 \subset S$ which involves $H^3(S, \mathbb{Z})$, i.e.

$$H^1(S_1, \mathbb{Z}) \to H^3(S, \mathbb{Z}) \to H^3(S-S_1, \mathbb{Z}) \xrightarrow{\alpha} H^2(S_1, \mathbb{Z}).$$

Then we have here the

**Theorem 2.** (Nitsure [9],[11]) Let $W$ be a conic bundle of type II on $S$. If the total space of $W$ is smooth, then the image of $b_{w-W_1} \in H^3(S-S_1, \mathbb{Z}_{\text{tors}})$ under the Gysin map $g$, is precisely $\alpha \in H^2(S_1, \mathbb{Z})$. In particular if $\alpha \neq 0$, then $b_{W-W_1} \neq 0$.

**PROPOSITION 5**

Let $W$ be a conic bundle of type I over $S$ where $H^1(S, \mathbb{Z}) = 0$ and with $b_W \neq 0$ in $H^3(S, \mathbb{Z}_{\text{tors}})$. Suppose that there exists another topological $\mathbb{P}^1 - \text{bundle } U \to S$ with the property that $H^2(U, \mathbb{Z}_{\text{tors}}) = 0$. Then $b_W = \pm b_U$ and $H^3(S, \mathbb{Z}_{\text{tors}})$ is generated by $b_W$.

**Proof.** To prove this proposition, we shall appeal to the following well-known (cf[11]).

**Lemma 2.** Let $U \to S$ be a $\mathbb{P}^1$-bundle over a path connected space $S$ with $H^1(S) = 0$. Then the kernel of the induced homomorphism $H^1(S, \mathbb{Z}) \to H^1(U, \mathbb{Z})$ is generated by $b_U$.

We now apply the lemma to $U \to S$. Since we have $H^2(U, \mathbb{Z}_{\text{tors}}) = 0$, we get $H^3(S, \mathbb{Z}_{\text{tors}})$ to be generated by $b_U$, which is a 2-torsion element. Also $b_W$ lies in $H^3(S, \mathbb{Z}_{\text{tors}})$, and $b_W \neq 0$ which implies $b_W = \pm b_U$. This proves Prop. 5.

The next step is to construct explicitly a $\mathbb{P}^1$-bundle on the subspace $Z - Y$ which satisfies the property of Prop. 5. For this purpose, we elaborate in some detail, what is called the 'Hecke correspondence' of [7], in terms of parabolic bundles as remarked in (*).

Let $V$ be a vector bundle on $X$ of rank 2 and degree 0. Suppose we are given a parabolic structure at a point $x \in X$, defined by a 1-dimensional subspace

$$F^2V_x \subset F^1V_x = V_x$$

and weights $(\alpha_1, \alpha_2)$ such that

(i) parabolic stable = parabolic semi-stable,
(ii) parabolic stable $\Rightarrow$ underlying bundle is semi-stable, and
(iii) underlying bundle stable $\Rightarrow$ any parabolic structure is stable.

Let $T$ be the torsion $\mathcal{O}_x$-module given by

$$T_x = V_x/F^2V_x, \quad T_y = 0, \quad x \neq y.$$

Then we have a homomorphism of $V$ onto $T$ (as $O_X$-modules). If $W$ is the kernel of this map, we have $0 \to W \to V \to T \to 0$ and $W$ is locally free of rank 2 and degree $-1$.

Let $\tilde{M}$ be the moduli space of parabolic stable bundles of rank 2, degree 0 on $X$ and $M_{-1}$ the moduli space of stable bundles of rank 2, degree $-1$, $f: \tilde{M} \to M$, the canonical morphism, and $M_0 = f^{-1}(M_0)$.

**Proposition 6**

If $V \in \tilde{M}$ then $W$ defined above, is in $M_{-1}$ and the map $\psi: \tilde{M} \to M_{-1}, V \mapsto W$ is a $\mathbb{P}^1$-bundle, locally trivial in the Zariski topology. In fact it is the dual projective Poincaré bundle on $M_{-1}$.

**Proof.** We first claim that if $V$ is parabolic stable then $W$ is stable. To see this, let $F \subseteq W$ be a line subbundle. We need to show that $\deg F < 0$. Suppose this is not the case i.e. $\deg F \geq 0$.

Let $G$ be the line subbundle of $V$ generated by the image of $F$ in $V$. Then $\deg G \leq \deg F$. Since the underlying bundle of $V$ is certainly semi-stable, we have $\deg G \leq 0$. By our assumption $\deg F \geq 0$ and hence we have $\deg F = \deg G = 0$. This implies that the canonical homomorphism $F \to G$ is an isomorphism. We also see that by the definition of $T$

$$G_\xi = F^2 V_\xi,$$

but $V$ being parabolic stable with weights $0 < x_1 < x_2$ we get

$$\text{par } \deg G = x_2 < \frac{1}{2}(x_1 + x_2) = \text{par } \deg V/\text{rk } V$$

which leads to a contradiction. Hence $W$ is stable. Conversely, we claim that $\tilde{M}$ is isomorphic to the dual projective Poincaré bundle of $M_{-1}$ restricted to $M_{-1}$. To see this, we start with a $W \in M_{-1}$, then, given a point in $P(W^*_x), x \in X$, one can easily obtain a vector bundle $V$ of rank 2 and degree 0 and an injection $W \to V$ as $O_\xi$-modules. The cokernel then gives a 1-dimensional subspace $F^2 V_\xi$ of $V_x$ and therefore a 'quasi-parabolic structure'. The stability of $W$ together with an argument as above, makes $V$ parabolic stable. That this map is an isomorphism is a consequence of the universal property of the moduli space of parabolic stable bundles.

That $\tilde{M} \to M_{-1}$ is locally trivial in the Zariski topology, now follows from Serre [15].

Q.E.D.

**Proposition 7**

Consider the canonical morphism $f: \tilde{M}_0 \to M_0$. Then $f$ is a $\mathbb{P}^1$-fibration over $M_0$ and $f^{-1}(K)$ has codimension $q-1$ in $\tilde{M}_0$.

**Proof.** That $f$ is a $\mathbb{P}^1$-fibration over $M_0$ is immediate by the property (3) mentioned before Prop. 6. Let $L \oplus L^{-1} \subseteq K - K_0$. Then the points of $\tilde{M}_0$ lying over $L \oplus L^{-1}$ are of the following form:

**Case 1.** $V$ is a non-trivial extension of $L^{-1}$ by $L$ (or $L$ by $L^{-1}$)

We claim that a parabolic structure on $V$ which is equivalent to giving a subspace $F^2 V_\xi$ of $V_\xi$ of dimension one, is stable if and only if $L_\phi \not\subseteq F^2 V_\phi$. This is necessary to
ensure parabolic stability, for otherwise if \( L_p \neq F^2 V_p \), then per deg \( L = \deg L + \alpha_2 = \alpha_2 \) and \( \alpha_2 \neq \text{par deg} V/rk V = \frac{1}{2}(\alpha_1 + \alpha_2) \), since \( \alpha_1 < \alpha_2 \).

**Case 2.** \( V = L \oplus L^{-1} \)

We claim that a parabolic structure \( F^2 V_p \) such that \( F^2 V_p \neq L_p \) or \( L^{-1} \) is stable. This is easily checked as above. In fact we see by an argument as in Prop. 1 of [17] all the parabolic structures of Case 2 are isomorphic and hence give one point of \( M \). Hence the total dimension of the fibre at \( L \oplus L^{-1} = \dim \text{Ext}(L, L^{-1}) + 1 = g - 1 \). Therefore, \( \dim f^{-1}(K - K_0) = 2g - 1 \).

In fact, it is not difficult to see that for \( x \in K - K_0 \), \( f^{-1}(x) \) is the union of two projective spaces of dimension \( g - 1 \) meeting at a point.

Finally, let \( V \in M_0 \) be such that \( \text{gr} V = L \oplus L \) (\( L \) of order two). Then the following can easily be checked.

(i) \( V \) has a parabolic stable structure if and only if \( V \) is a non-trivial extension of \( L \) by \( L \).

(ii) A parabolic structure given by \( F^2 V_p \) is stable iff \( F^2 V_p \neq L_p \) (where \( L \) is the unique line subbundle of \( V \)).

Once again by an argument as in Prop. 1 [17] we see that all the parabolic structures on a non-trivial extension \( V \) of \( L \) by \( L \) are isomorphic. Hence the fibre of \( f \) over \( L \oplus L \) is isomorphic to \( \mathbb{P}(H^1(X, E)) \) which has dimension \( g - 1 \), implying \( \text{codim} (f^{-1}(K), M_0) = g - 1 \).

**Remark 6.** Thus we have the following diagram

\[ \begin{array}{ccc}
\bar{M} & \xleftarrow{\psi} & M \\
\downarrow f & & \downarrow \\
M_{-1} & \xleftarrow{\beta} & M
\end{array} \]

which gives a correspondence between \( M_{-1} \) and \( M \).

**PROPOSITION 8**

The fibration \( Y \to K - K_0 \) with fibre \( F = \mathbb{P}^{g-2} \times \mathbb{P}^{g-2} \) satisfies the conditions of the Leray–Hirsch theorem and consequently we have

\[ H^*(Y, \mathbb{R}) \simeq H^*(K - K_0, \mathbb{R}) \otimes H^*(F, \mathbb{R}) \].

**Proof.** The following form of the Leray–Hirsch theorem will suit our purposes.

**Leray–Hirsch.** Let \( E \) be a fibre bundle over \( B \) and compact fibre \( F \). Suppose \( B \) has a finite good cover. If there are global cohomology classes \( e_1, \ldots, e_s \) on \( E \) which, when restricted to each fibre freely, generate the cohomology of the fibre, then \( H^*(E, \mathbb{R}) \) is a free module over \( H^*(B, \mathbb{R}) \) with basis \( e_1, \ldots, e_s \); or more precisely, if the canonical map \( j: H^*(E, \mathbb{R}) \to H^*(F, \mathbb{R}) \), is surjective, then for any subspace \( W \) of \( H^*(E, \mathbb{R}) \) such that \( j|W: W \to H^*(F, \mathbb{R}) \) is an isomorphism, one has

\[ H^*(E, \mathbb{R}) = H^*(B, \mathbb{R}) \otimes W. \]
Cohomology of certain moduli spaces of vector bundles

Since $F$ in our case is $\mathbb{P}^{s-2} \times \mathbb{P}^{s-2}$, $H^*(F, \mathbb{R})$ is generated by line bundles on $F$. Therefore it is enough to check that any line bundle on $F$ can be extended to a line bundle on $Y$.

By Cor. 2, $Y \to K - K_0$ is locally trivial in the Zariski topology. Let $L$ be a line bundle on $F$, and $U \subset K - K_0$ be the trivializing Zariski open subset. Then $L$ can be obviously extended to a line bundle on $U \times F$, which we continue to denote by $L$. Since $Y$ is smooth, the bundle $L$ on the open subset $U \times F$ of $Y$ can be extended to a line bundle on $Y$.

Q.E.D.

PROPOSITION 9

The element $x \in H^2(Y, \mathbb{Z})$, associated to the double cover on $Y$ arising from the conic bundle $P$ is non-zero.

Proof. We claim that this double cover on $Y$ is in fact the pull-back of the covering

$$J - \Gamma \to K - K_0,$$

$J$ being the Jacobian of line bundles of deg 0 on $X$ [for notations cf. §2]. Since this covering is non-split, it follows from Prop. 3, that the double cover on $Y$ is non-split and the covering element in $H^1(Y, \mathbb{Z}/(2))$ is non-zero.

By Prop. 8 and Spanier [19], $H^1(Y, \mathbb{Z}) = 0$. Hence if we consider the cohomology exact sequence for

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/(2) \to 0$$

we get

$$H^1(Y, \mathbb{Z}/(2)) \subset H^2(Y, \mathbb{Z}).$$

Since $x \in H^2(Y, \mathbb{Z})$ is the image of the covering element in $H^1(Y, \mathbb{Z}/(2))$, it is non-zero. Thus to complete the proof of Prop. 9, it is enough to prove the claim.

Fix $t_0 \in X$. Then if $E \in N_0$, one can easily see that $E_{t_0}$ can be identified with right regular representation of $A = \text{End} E$ (see for e.g. Prop. 5 [17]).

Let $E = V \oplus W$ be an element of $Y$ as in Prop. 3. It is easy to see that the scalars in $A$ do not meet $V_{t_0}$ and $W_{t_0}$ under the above identification. So if we consider the projective space $\mathbb{P}(A')$, $A' = A/\text{(scalars)}$, then $V_{t_0}$ and $W_{t_0}$ give a pair of lines in $\mathbb{P}(A')$. By Prop. 3, identifying the algebra $A$ with a $\mathbb{C}^*_+ \text{ associated to a quadratic form } q \in \Phi$, it is clear that this pair of lines is indeed the ones in the conic bundle over $Y$.

Then the one-dimensional subspaces $L_{t_0}$ and $L_{t_0}^{-1}$ give a pair of points $L_{t_0}$ and $L_{t_0}^{-1}$ in $\mathbb{P}(A')$. Then the correspondence

$$E \mapsto (\bar{L}_{t_0}, \bar{L}_{t_0}^{-1})$$

gives a double covering on $Y$ since we have a defining family of vector bundles $E_y = \{V_y \oplus W_y\}_{y \in Y}$. Obviously, this is the canonical double cover associated to the conic bundle on $Y$.

Note that $\{L_y \oplus L_y^{-1}\}_{y \in Y}$ gives a family on $Y$ which is clearly the pull-back $p^*\{L_{t_0} \oplus L_{t_0}^{-1}\}_{y \in K - K_0}$, under $p: Y \to K - K_0$.

The double cover of $Y$ given above is therefore the pull-back of the double cover of $K - K_0$ given by $J - \Gamma \to K - K_0$.

Q.E.D.
PROPOSITION 10

(a) Let $Z$ and $Y$ be as in §3. Then there exists a topological $\mathbb{P}^1$-bundle $D$ on $Z - Y$ with $H^*(D, \mathbb{Z})$ torsion free. In fact $D = f^{-1}(M_0^*)$.

(b) The topological Brauer class $b_D \neq 0$.

Proof. (a) By Prop. 7, $f^{-1}(K)$ has codimension $g - 1$ in $M_0$ and $D = M_0 - f^{-1}(K)$.

Consider $\psi: \tilde{M}_0 \to M_{-1,x}, M_{-1,x}$ being bundles in $M_{-1}$ with determinant $L_x$. Since the $\mathbb{P}^1$ fibration $\psi$ is locally trivial in the Zariski topology, a line bundle $L$ on the fibre $\mathbb{P}^1$ can be extended obviously to $\mathbb{P}^1 \times U$, where $U$ is a Zariski open subset of $M_{-1,x}$. Since $\tilde{M}_0$ is smooth, the closure of $L$ in $M_0$ gives a line bundle on $\tilde{M}_0$. Now, the cohomology of $\mathbb{P}^1$ is generated by line bundles and therefore we can apply Leray–Hirsch theorem to conclude that the cohomology groups of $\tilde{M}_0$ are those of $\mathbb{P}^1 \times M_{-1,x}$.

By Atiyah–Bott [2], all the cohomology groups of $M_{-1,x}$ are torsion-free and therefore all the cohomology groups of $\tilde{M}_0$ are also torsion-free.

Since $g = 3$, the complex codimension of $f^{-1}(k)$ in $\tilde{M}_0 = g - 1 \geq 2$. This implies $\text{Codim}_g f^{-1}(K) \subset \tilde{M}_0 \geq 4 = g - 1 \geq 2$.

Consider the homology sequence of the pair $(\tilde{M}_0, D)$

$$
\tilde{M}_0 : H_k(\tilde{M}_0, D, \mathbb{Z}) \to H_{k-1}(D, \mathbb{Z}) \to H_k(\tilde{M}_0, \mathbb{Z}) \to H_{k-1}(\tilde{M}_0, D, \mathbb{Z})
$$

By Alexander duality to the pair $(\tilde{M}_0, D)$ to get

$$
H_k(\tilde{M}_0, D, \mathbb{Z}) \cong H^{n-k}(\tilde{M}_0 - D, \mathbb{Z})
= H^{n-k}(f^{-1}(K), \mathbb{Z})
= n = \dim_{\mathbb{R}} \tilde{M}_0.
$$

Since $\dim_{\mathbb{R}} f^{-1}(K) \leq n - 4$, we therefore get

$$
H_2(\tilde{M}_0, D, \mathbb{Z}) = H^{n-2}(f^{-1}(K), \mathbb{Z}) = 0
$$

and similarly $H_3(\tilde{M}_0, D, \mathbb{Z}) = 0$.

$$
H_2(D, \mathbb{Z}) = H_2(\tilde{M}_0, \mathbb{Z}).
$$

By the ‘universal coefficient theorem’ one has torsion subgroup of $H_4(T, \mathbb{Z})$ to be that of $H^{k+1}(T, \mathbb{Z})$, $T$ any topological space, and therefore we conclude that

$$
H^3(D, \mathbb{Z})_{\text{tors}} = H^3(\tilde{M}_0, \mathbb{Z})_{\text{tors}} = (0).
$$

Note that $Z - Y = M_0^*$ and this completes the proof. Q.E.D.

The claim (b) is due to Ramanan (p. 52 [18]).

Theorem 3. $H^3(Z, \mathbb{Z})$ is torsion free.
Cohomology of certain moduli spaces of vector bundles

\textit{Proof.} Consider the Gysin sequence for \((Z, Z - Y)\),

\[ H^1(Y, Z) \to H^3(Z, Z) \to H^3(Z - Y, Z) \xrightarrow{g} H^2(Y, Z) \]

Now by Cor. 2, \(Y\) is a \(\mathbb{P}^{p-2} \times \mathbb{P}^{q-2}\) fibration over \(K - K_0\) and by (\((19)\) p.159) \(H^1(K - K_0, Z) = 0\) implying by standard arguments \(H^1(Y, Z) = 0\) (note that \(H^1(Y, Z)\) is torsion-free by the universal coefficient theorem).

Thus we have from the Gysin sequence an injection

\[ H^3(Z, Z) \hookrightarrow H^3(Z - Y, Z). \quad \text{(*)} \]

Now note that \(H^1(Z - Y, Z) = 0\). (This follows for example from the Gysin sequence. For, note that \(H^1(Z - Y, Z) \cong H^1(Z, Z)\). Also we will be seeing in §5 that the codimension of \(N_0 - Z\) in \(N_0\) is actually 6. But \(N_0\) is unirational and is therefore simply connected, being smooth projective (cf. Serre [16]). Hence \(H^1(N_0, Z) = 0\) implying \(H^1(Z, Z) = 0 = H^1(Z - Y, Z)\).

Thus we can now apply Prop. 5 and Prop. 10 to see that \(H^3(Z - Y, Z)_{\text{tors}}\) is generated by \(b_{p - p_1}\), the Brauer element coming from the conic bundle \(P - P_1\) over \(N_0 - N_1 = Z - Y\). By Prop. 4 the total space of \(P - P_1\) is smooth and hence the theorem due to Nitsure mentioned in §41 is applicable. Thus we have

\[ g(b_{p - p_1}) = \alpha \neq 0 \quad (\alpha \neq 0 \text{ by Prop. 9}). \]

This together with (*) and the exactness of the Gysin sequence gives \(H^3(Z, Z)_{\text{tors}} = 0\). Q.E.D.

\textbf{Lemma 3.} Pic\(Z\) is generated by Pic\((Z - Y)\) and the element \([Y]\) coming from the irreducible divisor \(Y \subset Z\).

\textit{Proof.} This follows from the following general fact:

If \(X\) is a smooth variety, \(U \subset X\) open with \(Y = X - U\) an irreducible divisor, then

\[ \text{Pic} X \to \text{Pic} U \]

is a surjection and the kernel of this homomorphism is generated by \([Y]\).

\textbf{Lemma 4.} Let \(N_1 \subset N_0\) be as in §3. Then Pic\(N_0\) is generated by Pic\(M_0\) and \([N_1]\) over \(\mathbb{Q}\)(*).

\textit{Proof.} Firstly, we remark that \(N_1\) is precisely \(\overline{Y}\) in \(N_0\). Actually, we will be showing in §5 that \(Y \subset N_1\) is precisely the set of non-singular points of \(N_1\). Let us assume this. Suppose \(N_1\) is not irreducible and let \(A, B\) be subvarieties such that \(N_1 = A \cup B\). Then \(A \cap B \subset N_1 - Y\) and hence \(A \cap Y\) and \(B \cap Y\) will disconnect \(Y\) which

\[ (*) \text{ In fact, over } \mathbb{Z} \text{ see Remark in Appendix 2).} \]
is false since $Y$ is connected. Thus $N_1$ is irreducible. Also since $Y$ is irreducible it follows that $\overline{Y} = N_1$.

An application of Lemma 4 and the result of Appendix 2 yields our result.

Remark 7. Thus by the above lemma, any $L \in \text{Pic } N_0$ can be expressed as $L = aL_1 + bL_2$, $L_1 = [N_1]$ and $L_2 \in \text{Pic } M_0$, $a, b \in \mathbb{Q}$.

In particular, let $L$ be chosen ample. Then if $F$ is the fibre of $Y \to K - K_0$, $L$ when restricted to $F$ is $(aL_1 + bL_2)|F$. But since $L_2 \in \text{Pic } M_0$, which is trivial on $F$, we have

$$L|F = (aL_1)|F$$

$F$ is $\mathbb{P}^2 \times \mathbb{P}^2$ and $L$ is ample, therefore we have the restriction of $L_1$ to each $\mathbb{P}^2$ to be either ample or negatively ample.

Let $e \in H^2(Y, \mathbb{R})$ be the Euler class of the irreducible divisor $Y$ in $Z$. Then by the 'adjunction formula', we have

$$e = [Y],$$

where $[Y]$ is the class of $Y \subset Z$. Now $L_1 = [N_1]$ and $N_1 = \overline{Y}$, hence it follows from the above reasoning that the Euler class $e$ when restricted to the factors of $F$ is ample or negatively ample.

PROPOSITION 11.

Let $E$ be the normal bundle of $Y$ in $Z$ and $E_0$ be the compliment of the zero section. Consider the Gysin sequence for the 2-plane bundle $(E, E_0)$

$$H^i(Y, \mathbb{R}) \to H^{k+2}(Y, \mathbb{R}) \to H^{k+2}(E_0, \mathbb{R}) \to H^{k+1}(Y, \mathbb{R}) \to H^{k+3}(Y, \mathbb{R}).$$

Then the Gysin homomorphism

$$h: H^k(Y, \mathbb{R}) \to H^{k+2}(Y, \mathbb{R}),$$

given by 'wedging' with the Euler class $e \in H^2(Y, \mathbb{R})$ is an injection for $k \leq \dim R \mathbb{P}^2 - 2 = 2g - 6$.

Proof. By Prop. 8 we have

$$H^{k}(Y) \simeq \sum_{i+m=k} H^i(K - K_0) \otimes H^m(F)$$

or using the subspace $W$ of $H^*(Y)$ as in Prop. 3.8, we have, any $u \in H^k(Y) u \neq 0$ and $k \leq \dim R F$, to be expressible as

$$v = \sum_i u_i \otimes w_i, \quad u_i \in H^*(K - K_0), \quad w_i \in W,$$

where not all $w_i = 0$ (this is so since $k \leq \dim R F$). Without loss of generality, the $u_i$'s can be chosen linearly independent.

Now consider $u \otimes e$, $e$ the Euler class in $H^2(Y, \mathbb{R})$

$$u \otimes e = \sum_i u_i \otimes (w_i \otimes e).$$
Cohomology of certain moduli spaces of vector bundles

Consider the class $w_i \otimes e$. This when restricted to the fibre $F$ is non-zero, since by Remark 7, the class $e$ restricted to the factors of $F$ is ample or negatively ample and $w_i$ by definition lies in $W$ and so $w_i \wedge e$ is non-zero on $F$ for $w_i \in H^4(F, \mathbb{R})$, $k \leq \dim \mathbb{P}^{n-2} - 2$. Hence by the linear independence of the $u_i$'s we get

$$u \otimes e = \sum_i u_i \otimes (w_i \otimes e) \neq 0$$

Thus $h: H^k(Y, \mathbb{R}) \to H^{k+2}(Y, \mathbb{R})$ is an injection for $k \leq \dim \mathbb{P}^{n-2} - 2 = 2g - 6$.

**Corollary 3**

The Gysin map considered in Theorem 3 i.e.

$$h^1: H^k(Y, \mathbb{R}) \to H^{k+2}(Z, \mathbb{R})$$

is also an injection for $k \leq 2g - 6$.

**Proof.** In fact, the Gysin sequences for $(E, E_0)$ and $(Z, Z - Y)$ are related as follows.

$$\xymatrix{ H^k(Y, \mathbb{R}) \ar[r]^k \ar[dr]^{k'} \ar[d]_{\text{Res}} & H^{k+2}(Y, \mathbb{R}) \ar[d]_{\text{Res}} \ar[dl]_{\text{Res}} \ar[r] & H^{k+2}(Z, \mathbb{R}) }$$

and therefore, since $h$ is an injection by Prop. 11, so is $h^1$...

**Corollary 4**

$H^k(Z, \mathbb{R}) = H^{k-2}(Y, \mathbb{R}) \oplus H^k(Z - Y, \mathbb{R})$ for $k \leq 2g - 4$.

**Proof.** Consider the Gysin sequence for $(Z, Z - Y)$.

$$\cdots \to H^{k-2}(Y, \mathbb{R}) \to H^k(Z, \mathbb{R}) \to H^k(Z - Y, \mathbb{R}) \to H^{k-1}(Y, \mathbb{R}) \to H^{k+1}(Z, \mathbb{R})$$

Since $k'$ is an injection for $k \leq 2g - 6$, we get

$$0 \to H^{k-2}(Y, \mathbb{R}) \to H^k(Z, \mathbb{R}) \to H^k(Z - Y, \mathbb{R}) \to 0$$

for $k \leq 2g - 4$ and this proves the corollary.

**Remark 8.** By Kirwan [5], the Betti numbers of $M_0$ are known if genus $g \geq 4$, for $i < 2g - 3$. This together with Prop. 8, Cor. 4 and Spanier [19], yields the Betti numbers of $Z$ for $i < 2g - 3$.

**Remark 9.** Let us assume $g \geq 4$ and recall from Prop. 10, we had a topological $\mathbb{P}^1$-bundle $D$ on $Z - Y$. By the proof of Prop. 10 we see that if $g \geq 4$, then $\text{codim}_n f^{-1}(K)$ in $M_0 \geq 6$ and hence

$$H_k(D, \mathbb{Z}) = H_k(\tilde{M}_0, \mathbb{Z})$$

for $k \leq 4$. 

The homology groups of $\tilde{M}_0$ are known by [10] or by using Atiyah–Bott [2] for $M_{-1,x}$. In particular, rank of $H_3(\tilde{M}_0, \mathbb{Z})$ is $2g$ and hence rank of $H_3(D, \mathbb{Z})$ is $2g$.

We have already seen that $H^1(Z - Y, \mathbb{R}) = 0$. Now $D$ is a $\mathbb{P}^1$-fibration over $Z - Y$ and $H^1(\mathbb{P}^1, \mathbb{R}) = 0$, $H^1(Z - Y, \mathbb{R}) = 0$. Therefore by the Serre sequence of this fibration (see for example Spanier Algebraic topology pp. 519) we get an exact sequence

$$H_3(\mathbb{P}^1, \mathbb{R}) \rightarrow H_3(D, \mathbb{R}) \rightarrow H_3(Z - Y, \mathbb{R}) \rightarrow H_2(\mathbb{P}^1, \mathbb{R}) \rightarrow H_2(D, \mathbb{R}) \rightarrow H_2(Z - Y, \mathbb{R}) \rightarrow H_1(\mathbb{P}^1, \mathbb{R}).$$

Now, $H_3(\mathbb{P}^1, \mathbb{R}) = H_1(\mathbb{P}^1, \mathbb{R}) = 0$, $H_2(\mathbb{P}^1, \mathbb{R}) = \mathbb{R}$ Thus we have

$$0 \rightarrow H_3(D, \mathbb{R}) \rightarrow H_3(Z - Y, \mathbb{R}) \rightarrow H_2(\mathbb{P}^1, \mathbb{R}) \rightarrow H_2(D, \mathbb{R})$$

$H_2(Z - Y, \mathbb{R}) = 0$. By the Picard group computations it follows that, $H_2(D, \mathbb{R}) = \mathbb{R}^2$ and $H_2(Z - Y, \mathbb{R}) = \mathbb{R}$, and therefore we have

rank of $H_3(Z - Y, \mathbb{R}) = \text{rank } H_3(D, \mathbb{R}) = 2g$.

Thus the rank of $H_3(Z - Y, \mathbb{R}) = 2g$ and hence the rank of $H^3(Z - Y, \mathbb{R})$ is $2g$.

**Theorem 4.** $H^3(Z, \mathbb{Z}) = \mathbb{Z}^{2g}$, when $g \geq 4$.

**Proof.** By Theorem 3 $H^3(Z, \mathbb{Z})$ is torsion-free. By Cor. 4

$$H^3(Z, \mathbb{R}) = H^1(Y, \mathbb{R}) \oplus H^3(Z - Y, \mathbb{R})$$

Since $H^1(Y, \mathbb{R}) = 0$, using Remark 9 we conclude that $H^3(Z, \mathbb{Z}) = \mathbb{Z}^{2g}$.

5. **The main theorem**

Consider the stratification of $N_0$ in terms of the degeneracy locus as in §3, $N_3 \subset N_2 \subset N_1 \subset N_0$.

**PROPOSITION 12**

The subvariety $N_2$ has codimension 3 in $N_0$.

**Proof.** Consider the local morphism

$$\varphi: N_0 \rightarrow \mathcal{A}$$

of §2. We have already seen that $\varphi: N_1 \rightarrow \mathcal{A}_1$ and $\varphi: N_2 \rightarrow \mathcal{A}_2$. Moreover, $\varphi$ being a smooth local morphism, its fibres are equidimensional. Hence the codimension of $N_2$ in $N_0$ equals the codimension of $\mathcal{A}_2$ in $\mathcal{A}$. We have also seen that $\mathcal{A}_1 \subset \mathcal{A}$ is a hypersurface given by $\Delta = 0$ and $\mathcal{A}_2 \subset \mathcal{A}_1$ is precisely the singular locus of $\mathcal{A}_1$. So we would like to show that

$$\text{codim of } \mathcal{A}_2 \text{ in } \mathcal{A}_1 = 2.$$

Consider the natural conic bundle $C$ on $\mathbb{A}^6$ as in Lemma 1. Let $S$ be the hypersurface of $\mathbb{A}^6$ given by $\Delta = 0$ and let $S' \subset S$ be its singular locus. Then by Remark 5, it is
Cohomology of certain moduli spaces of vector bundles

enough to show that

\[ \text{codim of } S^1 \text{ in } S = 2. \]

By definition, if

\[ q = aX^2 + bY^2 + cZ^2 + fYZ + gXZ + hXY, \]

then \( \Delta \) is given by

\[ \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}. \]

Thus, if \( \text{Sym}(\mathcal{A},3) \) is all \((3 \times 3)\)-symmetric matrices

\[ S = \{ A \in \text{Sym}(\mathcal{A},3) | \text{rank } A < 2 \}, \]

The conditions \( \partial \Delta/\partial a = \partial \Delta/\partial b = \partial \Delta/\partial c = \partial \Delta/\partial f = \partial \Delta/\partial g = \partial \Delta/\partial h = 0 \), gives

\[ bc = f^2, \quad ac = g^2, \quad ab = h^2, \quad af = hg, \quad fh = hg, \quad ch = fg. \]

\[ a/h = h/b = g/f \quad \text{and} \quad a/g = h/f = g/c \]

i.e.

\[ S^1 = \{ A \in \text{Sym}(\mathcal{A},3) | \text{rank } A < 1 \}. \]

From which we obtain the codim of \( S^1 \) in \( S \).

Q.E.D.

**COROLLARY 5**

\[ H_k(N_0, \mathbb{Z}) = H_k(Z, \mathbb{Z}), k \leq 4. \]

**Proof.** Consider the homology sequence of the pair \((N_0, Z)\)

\[ H_{k+1}(N_0, Z; \mathbb{Z}) \rightarrow H_k(Z, \mathbb{Z}) \rightarrow H_k(N_0, Z) \rightarrow H_k(N_0, Z; \mathbb{Z}). \]

Since \( N_0 \) is a compact complex manifold, the Alexander duality as in Theorem 3, gives

\[ H_k(N_0, Z, \mathbb{Z}) \cong H^{n-k}(N_0 - Z, \mathbb{Z}) = H^{n-k}(N_2, \mathbb{Z}). \]

\[ n = \dim_\mathbb{R} N_0. \]

By Prop. 12, \( \dim_\mathbb{R} N_2 = n - 6 \) since \( \text{codim}_C(N_2, N_0) = 3. \) Hence \( H^{n-k}(N_2, \mathbb{Z}) = 0 \) for \( k < 6. \)

\[ \Rightarrow H_k(N_0, Z, \mathbb{Z}) = 0, \quad k < 6 \]

\[ \Rightarrow H_k(N_0, Z) = H_k(Z, \mathbb{Z}), \quad k \leq 4. \]

**Theorem 5.** \( H^2(N_0, \mathbb{Z}) = \mathbb{Z}^{2g}. \)

**Proof.** Firstly, \( H^2(N_0, \mathbb{Z}) \) is torsion-free. For, by Cor. 5, \( H_2(N_0, \mathbb{Z}) = H_2(Z, \mathbb{Z}) \) and
therefore by the universal coefficient theorem, since
\[ H^3(N_0, \mathbb{Z})_{\text{tors}} = H^2(N_0, \mathbb{Z})_{\text{tors}}, \]
we have
\[ H^3(N_0, \mathbb{Z})_{\text{tors}} = H^3(\mathbb{Z}, \mathbb{Z})_{\text{tors}} = (0) \]
by Theorem 3.6.
Now using Theorem 4 and for Cor. 5 we get
\[ H^3(N_0, \mathbb{Z}) = \mathbb{Z}^{2g}. \] Q.E.D.

**Theorem 6.** The Betti number \( B_4 \) of \( N_0 \) is \( B_4(N_0) = \binom{2g}{2} + 4. \)

**Proof.** To see this, we use Prop. 5.9 and Remark 5.11 of Kirwan [5] to get the Betti numbers of \( M_0^\bullet \) as \( B_0 = 1, B_1 = 0, B_2 = 1, B_3 = 2g, B_4 = 2, \) etc.
By Cor. 4,
\[ B_4(Z) = B_2(Y) + B_4(Z - Y). \]
Now, by Prop. 8, \( B_2(Y) = B_2(K - K_0) + B_2(\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}). \) Hence, by Spanier [19]
\[ B_2(Y) = \binom{2g}{2} + 2. \]
Also, \( B_4(Y) = 0, \) since the odd Betti numbers of \( K - K_0 \) and \( \mathbb{P}^{g-2} \times \mathbb{P}^{g-2} \) are zero (cf. [19] again). Combining this with (*), we get
\[ B_4(Z) = \binom{2g}{2} + 4. \]
Hence by Cor. 5 we get
\[ B_4(N_0) = \binom{2g}{2} + 4. \] Q.E.D.

**Appendix 1**

We present here a proof due to Colliot–Thélène of Theorem (A) mentioned in the introduction. We shall make a few remarks before going into the proof.

Let \( X \) be a smooth variety over \( \mathbb{C} \). For the notations and properties of most of the facts mentioned below (cf. Grothendieck [4] and Saltman [13],[14]).

Define \( Br(X) \) to be the Brauer group of Azumaya algebras on \( X \). Let \( Br'(X) \) be the 'cohomological Brauer group' of \( X \) defined to be \( H^2_{\text{et}}(X, \mathbb{G}_{m, \text{et}}) \). Then the following facts are well known:
(i) \( Br X \) is contained in \( Br'(X) \).
(ii) If \( X \) is a unirational smooth proper variety, then \( Br'(X) = H^2(X(\mathbb{C}), \mathbb{Z})_{\text{tors}} \).
(iii) Define \( Br_{ur}(X) \), the unramified Brauer group of \( X \) to be \( Br_{ur}(X) = Br'(X), X \) any
Cohomology of certain moduli spaces of vector bundles

smooth compactification of \( X \). Then it is known that \( \text{Br}_m(X) \) is independent of the choice of \( \overline{X} \) since we are in characteristic 0.

(iv) Another way of defining \( \text{Br}_m(X) \) is as follows: Let \( C(X) \) be the function field of \( X \). Then for every discrete valuation ring \( A \), with \( C \subset A \subset C(X) \), and quotient field of \( A = C(X) \), there exists a natural homomorphism

\[
\delta_A: \text{Br} C(X) \to H^1(\mathbb{A}_A, \mathbb{Q}/\mathbb{Z}).
\]

\( \mathbb{A}_A \)-the residue class field of \( A \).

Define

\[
\text{Br}_m C(X) = \bigcap_{\text{all such } A} (\text{Ker } \delta_A) \text{ and } \text{Br}_m X = \text{Br}_m C(X).
\]

(v) Let \( k \) be a field and \( C \) a conic over \( k \), i.e. a conic bundle coming from a quaternion algebra over \( k \). Then there is a canonical homomorphism

\[
\text{Br}'(k) \to \text{Br}'(C)
\]

and the kernel of this homomorphism is the 2-torsion element coming from the quaternion algebra over \( k \) associated to \( C \).

Note that for a field \( k \), \( \text{Br}'(k) = \text{Br}(k) \).

**PROPOSITION 13**

Let \( C \) be a conic bundle on \( X \) with \( \text{Br}'(C) = 0 \), and let \( \eta \) be the generic point of \( X \). Let \( C_\eta \) be the restriction of \( C \) over \( C(\eta) \). To \( C_\eta \) we associate an element \( x_\eta \in \text{Br} C(\eta) \). Suppose that for the conic bundle \( C \) on \( X \), there exists a discrete valuation ring \( A \), with quotient field of \( A = C(X), C \subset A \subset C(X) \), such that \( \delta_A(x_\eta \neq 0 \). Then \( \text{Br}_m(X) = 0 \).

**Proof.** Suppose that \( \text{Br}_m(X) \neq 0 \) and let \( x \in \text{Br}_m(X) = \text{Br}_m(C(X)) \) be a non-zero element. Consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Br}_m X & \longrightarrow & \text{Br}_m C \\
\downarrow & & \downarrow \\
\text{Br}' C(\eta) & \longrightarrow & \text{Br}' C_\eta,
\end{array}
\]

where the map \( \text{Br}_m(X) \to \text{Br}_m(C) \) is the canonical map induced from \( C \to X \) and the vertical maps are

\[
\begin{align*}
\text{Br}_m C & \subset \text{Br} C \subset \text{Br}' C \to N \text{ Br}' C_\eta \\
\text{Br}_m X & \subset \text{Br} C(X) = \text{Br} C(\eta) = \text{Br}' C(\eta).
\end{align*}
\]

Consider the image of \( x \) in \( \text{Br}' C(\eta) \), call it \( x_\eta \). Then since \( \text{Br}' C = 0 \), the above diagram gives

\[
x_\eta \in \text{Ker } [\text{Br}' C(\eta) \to \text{Br}' C_\eta]
\]

and therefore by Remark (v), \( x_\eta \) is the element in \( \text{Br} C(\eta) \) associated to the conic \( C_\eta \). Now by the hypothesis of the proposition, there exists a discrete valuation ring
Let $A, \mathbb{C} < A < \mathbb{C}(X)$ with quotient field of $A = \mathbb{C}(X)$, such that
\[ \partial_A(\pi) \neq 0. \] (\ast)

But $x \in \text{Br}_\mathbb{C}(\mathbb{C}(X))$ and $\text{Br}_\mathbb{C}(\mathbb{C}(X))$ is by definition equal to
\[ \bigcap_{\text{all such } A} (\ker \partial_A), \]
implying
\[ \partial_A(\pi) = 0 \]
which contradicts (\ast). Hence the proposition.

Q.E.D.

Now let us consider the variety $M_0^s$, the moduli space of stable vector bundles of rank 2 and trivial determinant. Then by Prop. 3, there is a conic bundle $D$ on $M_0^s$ with $H^1(D, \mathcal{O}_D) = (0)$ and therefore $\text{Br}(D) = 0$.

The existence of an $A$ with the requisite properties of the Prop. 13 is precisely the theorem due to Nitsure [9]. Indeed, in the notation of §4, the irreducible divisor $Y \subset Z$ provides us with the discrete valuation ring $A$.

Hence by Prop. 13, $\text{Br}_\mathbb{C}(M_0^s) = 0$. This implies by Remark (3), that $\text{Br}^r(N_0) = 0$, since $N_0$ is a smooth compactification of $M_0^s$. Now $N_0$ is unirational, smooth-projective and therefore by Remark (ii), $\text{Br}^r(N_0) = H^3(N_0, \mathbb{Z}) = (0)$.

Appendix 2

**Theorem (C S Seshadri).** Let $M$ be the moduli space of semi-stable vector bundles of rank $2^n$ and degree $d$. Then
\[ \text{Pic } M^s \text{ (as well as Pic } M) \simeq \mathbb{Z}. \]

**Proof.** For simplicity we present the proof only for rank 2 and degree zero. Choose $m$ such that for all stable bundles $V$ of rank two and degree zero, $V(m)$ is generated by the global sections. Then if $E$ denotes the trivial vector bundle of rank $r = \dim H^0(V(m))$, $V(m)$ is canonically a quotient of $E$ and $V(m)$ represents a point of $Q = Q(E/P)$, the Quot scheme of quotients of $E$ with Hilbert polynomial equal to $P$.

We then have an open subscheme $Q'$ of $Q$ representing quotient vector bundles $W$ of $E$ such that $W$ is stable and the canonical homomorphism $H^0(E) \to H^0(W)$, is an isomorphism. Thus we have a canonical morphism
\[ p: Q' \to M_1^s, \]
where $M_1^s$ is the moduli space of stable vector bundles of rank 2 and $\det = \mathcal{O}_X(2m)$ and $p$ is a $G$-principal fibre space with $G = PGL(H^0(E))$. Note that $M_1^s \simeq M_0^s$ of §2.

Let $q:B \to M_1^s$ be the fibre space associated to $p$ with fibre the projective space of dimension $(r - 1)$. Hence if $W \in M_1^s$, the fibre $q^{-1}(W)$ can be canonically identified with $\mathbb{P}(H^0(W))$.

Let $A$ denote the projective space $\mathbb{P}(\text{Ext}(L, I))$, the ‘Atiyah family’ on the vector
space of all extensions of the form

\[ 0 \to I \to W \to L \to 0, \]

where \( I \) is the trivial vector bundle of rank one and \( L \) the line bundle \( \mathcal{O}_X(2m) \). Let \( A^e \) denote the subset of \( A \) defined by

\[ A^e = \{ 0 \to I \to W \to L \to 0 | W \text{ is stable} \}. \]

Then \( A^e \) is open and we have a canonical surjective morphism

\[ \lambda: A^e \to M_1^e \]

which associates to an extension as above the vector bundle \( W \). Observe that giving an extension as above is equivalent to giving a section \( s \in H^0(W) \) which is non-vanishing at every point \( x \in X \). From this observation we deduce easily that \( A^e \) can be identified canonically as an open subset of the projective bundle \( B \) over \( M_1^e \); in fact we have a commutative diagram

\[
\begin{array}{ccc}
A^e & \xrightarrow{\iota} & B \\
\downarrow{\lambda} & & \downarrow{p} \\
M_1^e & & W_x
\end{array}
\]

Note that \( p^{-1}(W) = \lambda^{-1}(W) \) is irreducible in \( \mathbb{P}(H^0(W)) \) for \( p^{-1}(W) = \lambda^{-1}(W) \) is the canonical image in \( \mathbb{P}(H^0(W)) \) of the set \( S = \{ s | s \in H^0(W), s \text{ vanishes at least at one point of } X \} \) i.e.

\[ S = \bigcup_{x \in X} \ker(H^0(W) \to W_x). \]

Since \( \lambda^{-1}(W) \) is the complement of an irreducible closed subset in a projective space, nonvanishing regular functions on \( \lambda^{-1}(W) \) reduce to constants. From this, we easily conclude that, if \( U \) is an open subset in \( M_1^e \) and \( f \) a regular nonvanishing function on \( \lambda^{-1}(U) \), then \( f \) is a pull-back of a regular nonvanishing function on \( U \).

From these properties, it follows easily that the canonical homomorphism

\[ \lambda^*: \text{Pic } M_1^e \to \text{Pic } A^e \]

is injective. To see this, let \( L \in \ker \lambda^* \). Then if \( L \) is given by transition functions \( \{ \theta_{ij} \} \) on \( V_i \cap V_j \), we have nonvanishing regular functions \( \varphi_i \) on \( \lambda^*(V_i) \) such that \( \lambda^*(\theta_{ij}) = \varphi_i \varphi_j^{-1} \).

Now the \( \varphi_i \) are pull-backs of functions \( \theta_i \) on \( V_i \) and the required assertion follows.

Now \( A^e \) is an open subset of \( \mathbb{P}(\text{Ext}(L, I)) \) and therefore \( \text{Pic } A^e \) is either \( \mathbb{Z} \) or \( \mathbb{Z}/(m) \), \( m \neq 0 \). Consider the normal projective variety \( M_1 \). The ample line bundle on \( M_1 \) restricted to \( M_1^e \) shows that \( \text{Pic } M_1^e \) is not torsion. But \( \lambda^*: \text{Pic } M_1^e \to \text{Pic } A^e \) is injective implying, \( \text{Pic } A^e = \mathbb{Z} \) and also \( \text{Pic } M_1^e = \mathbb{Z} \). Since \( M_1 \) is normal, it follows that \( \text{Pic } M_1 \subset \text{Pic } M_1^e \) and hence \( \text{Pic } M_1 = \mathbb{Z} \).

**Remark.** A priori, \( \text{Pic } M_1 \) is just a proper subgroup of \( \text{Pic } M_1^e \). But if \( M_1 \) is locally
factorial then $\text{cl} M_1 = \text{Pic} M_1$ and we would have $\text{Pic} M_1 = \text{Pic} M_1$.

In fact, this is so and has been recently proved by J M Drezet and M S Narasimhan.

References